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On null Killing vector fields (**)

1 - Introduction

When studying geometrical vector fields on a *pseudo-Riemannian Manifold* (M, g), a problem of current interest is to consider various types of null vector fields. Such vector fields intervene in different situations as for instance: Co-isotropic hypersurfaces in a Lorentzian manifold, isotropic and pseudo-isotropic submanifolds (M, g), null Sachs frames in a general space-time, etc. On the other hand, Killing vector fields play also an important role in the study of connected Lorentzian manifolds [2]. In the present paper we are concerned with a certain type of *null Killing vector fields* in the following cases:

1. (M, g) is a *para-Kählerian manifold*
2. (M, g) is a *pseudo-Sasakian manifold*
3. (M, g) is a *general space-time.*

1. Let $M(\mathcal{U}, \Omega, g)$ be a *para-Kählerian manifold* [12], where $\mathcal{U}$ and Ω are the (1, 1)-structure tensor field and the structure symplectic form of M respectively. Any null vector field X (i.e. $\|X\|^2 = 0$) whose covariant derivative is the 2-vector $X \wedge \mathcal{U}X$, that is such that

$$\nabla X = X \wedge \mathcal{U}X \qquad \|X\|^2 = 0\,, \tag{1}$$

is defined as a null structure vector field (abr. NSK). After having showed that

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the existence of such an X is determined by an exterior differential system in involution (in the sense of [4]) the following properties are proved:

a Any $M(\mathcal{U}, \Omega, g)$ which carries a NSK vector field is the local Riemannian product $M = M_X \times M_X^{\perp}$ where M_X is a totally geodesic surface tangent to X and $\mathcal{U}X$, whilst $M_X^{\perp}$ is a 2-codimensional totally isotropic submanifold

b X and $\mathcal{U}X$ commute and each of them is a strict geodesic [16]

c X is a global Hamiltonian vector field of Ω and an infinitesimal automorphism of all $(2q+1)$-forms $\alpha_q = \alpha \wedge \Omega^q$, where α is the dual form of X.

2. Let $M(\mathcal{U}, \eta, \xi, g)$ be a *pseudo-Sasakian manifold* [19] where $\mathcal{U}$ is as in 1 the paracomplex operator [12], whilst η and ξ are the contact structure 1-form and the structure vector field respectively. A NSK vector field X on $M(\mathcal{U}, \eta, \xi, g)$ satisfies as in 1. (1) and similarly its existence is proved by an exterior differential in involution. One has the following properties:

a X defines a relative contact transformation [1], i.e. $d(\mathcal{L}_X \eta) = 0$

b $\mathcal{U}X$ defines an infinitesimal conformal transformation of X, i.e. $\mathcal{L}_{\mathcal{U}X} X = (*)X$

c $\eta(X)$ is an isoparametric function [23]

d Any $M(\mathcal{U}, \eta, \xi, g)$ is foliated by co-isotropic hypersurfaces M_X having X as characteristic vector field.

3. Finally let (M, g) be a *general space-time manifold*. Then in terms of the complex vectorial formalism [3], let h_4 and h_1 be Debever's vector field and its associated null real vector field, respectively. It is proved that if both h_4 and h_1 are Killing vector fields, then the space-time (M, g) is of type D in Petrov's classification and is foliated by totally pseudo-isotropic space-like surfaces. We agree to say that such a space-time has the Killing property.

2 - Preliminaries

Let (M, g) be a *Riemannian* or *pseudo-Riemannian* C^∞ *manifold* and let ∇ be the covariant differential operator defined by the metric tensor g. Let $\Gamma(TM) = \Xi(M)$ and $b\colon TM \to T^*M$ be the set of sections of the tangent bundle TM and the musical isomorphism [15] defined by g respectively. Following W. A. Poor [15] we set

$$A^q(M, TM) = \Gamma \operatorname{Hom}(\wedge^q TM, TM)$$

and notice that the elements of $A^q(M, TM)$ are vector valued 1-forms. The vector 1-form $\mathrm{d}p \in A^q(M, TM)$, where $p \in M$, is called the *soldering form* of M ($\mathrm{d}p$: canonical vector valued 1-form of M [6]). Next the *operator*

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$$

denotes the exterior covariant derivative with respect to ∇ (see also [15]). Notice that generally

$$d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0 \quad \text{unlike} \quad d^2 = d \circ d .$$

A vector field $X \in \Xi(M)$ such that

(2) $$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge \mathrm{d}p \in A^2(M, TM)$$

for some 1-form π is said to be an *exterior concurrent vector field* [21], [13] (abr. EC). If X is a tangent vector field, then the 1-form π (which is called the *concurrence form*) is defined by

(3) $$\pi = fb(X) \qquad f \in A^0 M .$$

In this case if $\mathcal{R}$ denotes the Ricci tensor of ∇, then by (3) one has

(4) $$\mathcal{R}(X, Z) = -(n-1) fg(X, Z) \qquad Z \in \Xi M$$

where $n = \dim M$. A function $f: R^n \to R$ is *isoparametric* [23] if $\|\mathrm{grad} f\|^2$ and $\mathrm{div}(\mathrm{grad} f)$ can be expressed as functions of f. The operator

(5) $$d^\omega = d + e(\omega)$$

acting on $\wedge M$, where $e(\omega)$ means the exterior product by the closed 1-form ω, is called the *cohomology operator* [9]. One has

(6) $$d^\omega \circ d^\omega = 0$$

and any form $u \in \wedge M$, such that $d^\omega u = 0$, is said to be d^ω-closed. In particular if the cohomology form ω is exact, then u is said to be a d^ω-exact form.

Let $E = \mathrm{vect}\{e_A \mid A = 1, \ldots, n\}$ be a local field of adapted vectorial frames over M and let $E^* = \mathrm{covect}\{\omega^A\}$ be the associated coframe. Then *E. Cartan's structure equations*, written in indexless form are

(7) $$\nabla e = \theta \otimes e \qquad d\omega = -\theta \wedge \omega \qquad \mathrm{d}\theta = -\theta \wedge \theta + \Theta .$$

In the above equations, θ (resp. Θ) are the *local connection forms* in the tangent bundle TM (resp. the *curvature 2-form* on M). Further, let $x: M \to \widetilde{M}$ be the inclusion of a submanifold M in a pseudo-Riemannian manifold $\widetilde{M}$, and let N be a normal section associated with x. If the Gauss map corresponding to N satisfies

$$\langle \nabla N, \nabla N \rangle = 0 \tag{8}$$

then, following [**17**], N is called a *pseudoisotropic normal section* (see also [**7**]). If all normal sections associated with x are pseudo-isotropic, the M is called a *pseudo-isotropic submanifold* of $\widetilde{M}$.

If S denotes the *shape operator* of M, the above property is equivalent to write $\langle SX, SX \rangle = 0$, for all tangent vectors to M [**7**].

3 - Null Killing vector fields in a para-Kählerian manifold

Let $M(\mathcal{U}, \Omega, g)$ be a $2m$-dimensional *para-Kählerian manifold* [**12**] that is a neutral pseudo-Riemannian manifold endowed with a Kählerian structure (see also [**20**]). The triple $(\mathcal{U}, \Omega, g)$ of structure tensor fields, denotes the para-complex operator, a symplectic form and a para-Hermitian metric exchangeable with Ω, respectively.

Let $W = \mathrm{vect}\{h_a, h_{a^*} \mid a = 1, \ldots, m;\ a^* = a + m\}$ be a local field of Witt frames and let $W^* = \mathrm{covect}\{\omega^a, \omega^{a^*}\}$ be the associated coframe of W. One has

$$\mathcal{U}^2 = \mathrm{Id} \qquad \mathcal{U} h_a = h_a \qquad \mathcal{U} h_{a^*} = -h_{a^*} \qquad g(h_a, h_{b^*}) = \delta_{ab} \tag{9}$$

and the matrix connection $\mathcal{M}_\theta$ in the bundle $W(M)$ is a *Chern-Libermann matrix*; that is

$$\mathcal{M}_\theta = \begin{pmatrix} \theta^a_b & 0 \\ 0 & \theta^{a^*}_{b^*} \end{pmatrix}. \tag{10}$$

The structure equations are

$$\begin{aligned} &g(Z, \mathcal{U}Z) = 0 \qquad i_Z \Omega = b(\mathcal{U}Z) \\ &\Omega(Z, Z') = g(\mathcal{U}Z, Z') \qquad \mathcal{U}\nabla Z = \nabla \mathcal{U} Z \qquad Z, Z' \in \Xi M \end{aligned} \tag{11}$$

and in terms of the W-basis, Ω and g are expressed by

$$\Omega = \sum \omega^a \wedge \omega^{a^*} \qquad g = 2 \sum \omega^a \otimes \omega^{a^*}.$$

It should be noticed that, with respect to g, one has

$$Z = Z^A h_A \Rightarrow b(Z) = \sum Z^a \omega^{a^*} + Z^{a^*} \omega^a .$$

Now we give the following

Definition. Any null vector field X, such that

$$\nabla X = X \wedge \mathscr{U} X \tag{12}$$

is called a *null structure Killing vector field.*

Setting $\alpha = b(X)$, $\beta = b(\mathscr{U} X)$ then, as is well known, one may write (12) as

$$\nabla X = \beta \otimes X - \alpha \otimes \mathscr{U} X . \tag{13}$$

Therefore it follows at once from (13) that one has

$$\langle \nabla_Z X, Z' \rangle + \langle \nabla_{Z'} X, Z \rangle = 0 \Leftrightarrow \mathscr{L}_X g = 0 \qquad Z, Z' \in \Xi M$$

($\langle , \rangle$ instead of g), i.e. X satisfies the Killing equation $\mathscr{L}_Z g = 0$, $Z \in \Xi M$.
Setting $X = X^a h_A$ one has

$$\langle X, X \rangle = 0 \Rightarrow \sum X^a X^{a^*} = 0$$

and by reference to (6) the 1-forms α and β are expressible as

$$\alpha = \sum (X^a \omega^{a^*} + X^{a^*} \omega^a) \qquad \beta = \sum (X^a \omega^{a^*} - X^{a^*} \omega^a) = i_X \Omega .$$

Making now use of the structure equations $(7)_1$ and $(7)_2$ one derives from (13)

$$\mathrm{d}\beta = 0 \qquad \mathrm{d}\alpha = 2\beta \wedge \alpha \Leftrightarrow \mathrm{d}^{-2\beta} \alpha = 0 . \tag{14}$$

Hence, by the above equations we may say that the dual form α of X is $d^{-2\beta}$-closed (see (5)).
Next, making use of equations (11), one quickly derives from (13)

$$\nabla \mathscr{U} X = \beta \otimes \mathscr{U} X - \alpha \otimes X . \tag{15}$$

Since X and $\mathscr{U} X$ are both null vector fields one derives from (13) and (15)

$$\nabla_X X = 0 \qquad \nabla_{\mathscr{U} X} \mathscr{U} X = 0 \qquad [X, \mathscr{U} X] = 0 . \tag{16}$$

Hence X and $\mathscr{U} X$ are both null vector fields, and both are *strict geodesics* [**16**].

In addition by (15) one gets

$$\langle \nabla_Z \mathcal{U}X, Z' \rangle = \langle \nabla_{Z'} \mathcal{U}X, Z \rangle$$

which, as is known, proves that the vector field $\mathcal{U}X$ is a gradient. So one may write

$$\beta = b(\mathcal{U}X) = \mathrm{d}f. \tag{17}$$

We notice also the following fact. Since the connection ∇ is Riemannian, the Ricci identity

$$\mathcal{L}_U \langle Z, Z' \rangle = \langle \nabla_U Z, Z' \rangle + \langle Z, \nabla_U Z' \rangle \qquad U, Z, Z' \in \Xi M$$

holds good. Identifying in the above formulas Z and Z' with X and $\mathcal{U}X$ it is easily seen that (13) and (15) are matching the Ricci identity.

On the other hand since by (17) one has

$$i_X \Omega = \beta = \mathrm{d}f \Rightarrow \mathcal{L}_X \Omega = 0 \tag{18}$$

one may say that f is a Hamiltonian function and that X is a *global Hamiltonian vector field.* Since X and $\mathcal{U}X$ are null vector fields, one derives by $(14)_1$ and (16)

$$\mathcal{L}_X \alpha = 0 \qquad \mathcal{L}_{\mathcal{U}X} \beta = 0. \tag{19}$$

which proves that both X and $\mathcal{U}X$ are autoinvariant vector fields [22].

Consider then the (1, 1)-type operator L of S. Goldberg [8], defined by the symplectic form Ω, that is L: $u \to u \wedge \Omega$ and set

$$L^q \alpha = \alpha \wedge \Omega^q = \alpha_q .$$

Then by (18) and (19) one gets at once $\mathcal{L}_X \alpha_q = 0$. Hence one may say that X defines an *infinitesimal automorphism* of all $(2q+1)$-forms α_q.

Let now $D_X = \{X, \mathcal{U}X\}$ be the 2-distribution defined by X and $\mathcal{U}X$ and let X', X'' be any vector fields of D_X. Then on behalf of (13) and (15) one finds $\nabla_{X''} X' \in D_X$. This, as is known, proves that D_X is an autoparallel foliation and that the leaves M_X of D_X are totally geodesic surfaces (see also [11]). Furthermore since

$$\langle X, X \rangle = 0 \qquad \langle X, \mathcal{U}X \rangle = 0 \qquad \langle \mathcal{U}X, \mathcal{U}X \rangle = 0 \tag{20}$$

it is easily seen that any vector field of D_X is a null vector field. From the above considerations it follows by (8) and (14), (20) that M is the local Riemannian product $M = M_X \times M_X^\perp$ where M_X is a *totally geodesic and null surface* tangent to X and $\mathcal{U}X$, whilst $M_X^\perp$ is a 2-codimensional and *totally pseudo-isotropic submanifold* of M.

If we denote by Σ the exterior differential system which defines X, it is seen by (14) that the characteristic numbers of Σ are $r = 2$, $s_0 = 0$, $s_1 = 2$. Since $r = s_0 + s_1$, it follows by E. Cartan's test [4] that Σ is in involution and depends on two arbitrary functions of one argument.

Theorem. *Let X be a null structure Killing vector field on a para-Kählerian manifold $M(\mathcal{U}, \Omega, g)$. Then any $M(\mathcal{U}, \Omega, g)$, which carries such an X, is the local Riemannian product $M = M_X \times M_X^\perp$ where*

1. *M_X is a totally geodesic and null surface, tangent to X and $\mathcal{U}X$*
2. *$M_X^\perp$ is a 2-codimensional and totally pseudo-isotropic submanifold of M*

and the existence of X is determined by an exterior differential system in involution. In addition

3. *X and $\mathcal{U}X$ commute and both of them is a strict geodesic*
4. *X is a globally Hamiltonian vector field of the symplectic form Ω and an infinitesimal automorphism of all $(2q+1)$-forms $\alpha_q = \alpha \wedge \Omega^q$, where α is the dual form of X.*

4 - Null Killing vector fields in a pseudo-Sasakian manifold

Pseudo-Sasakian manifolds $M(\mathcal{U}, \eta, \xi, g)$ have been defined in [**19**] and, roughly speaking, as a pseudo-Riemannian version of a Sasakian manifold. One may prove that any $M(\mathcal{U}, \eta, \xi, g)$ is derived from a para-Kählerian manifold in a similar manner as a Sasakian manifold is derived from a Kählerian manifold.

In the present case $\mathcal{U}$ denotes, as in the previous section, the paracomplex operator, whilst η and ξ are the structure 1-form and the structure vector field respectively.

Since $M(\mathcal{U}, \eta, \xi, g)$ is endowed with a contact structure one has $\eta \wedge d\eta \neq 0$, i.e. η is of maximal rank ($\dim M = 2m + 1$).

If Z, Z' are any vector fields on M one has the following *structure equations.*

$$(21)\qquad \begin{aligned} &\mathcal{U}^2 = \mathrm{Id} - \eta \otimes \xi \qquad \mathcal{U}\xi = 0 \qquad \eta(\xi) = 1 \\ &g(\mathcal{U}Z, \mathcal{U}Z') = -g(Z, Z') + \eta(Z)\,\eta(Z') \qquad g(Z, \xi) = \eta(Z) \\ &\mathrm{d}\eta(Z, Z') = -2g(Z, \mathcal{U}Z') \qquad \nabla_Z \xi = \mathcal{U}Z \\ &(\nabla\mathcal{U})\,Z = -\eta(Z)\,\mathrm{d}p + b(Z) \otimes \xi \qquad (\nabla\mathcal{U})\,Z = \nabla\mathcal{U}Z - \mathcal{U}\nabla Z. \end{aligned}$$

Let $$\mathrm{d}p = \omega^\alpha \otimes h_\alpha + \eta \otimes \xi \qquad \alpha = 1, \ldots, 2m$$

be the *soldering form* of M. Then one may write

$$(22)\qquad \nabla\xi = \mathcal{U}\,\mathrm{d}p = \omega^a \otimes h_a - \omega^{a^*} \otimes h_{a^*}$$

where $a = 1, \ldots, m$; $a^* = a + m$; and from (22) one quickly gets

$$\nabla_Z \xi = \mathcal{U}Z \Rightarrow \mathcal{L}_\xi g = 0$$

which shows that ξ is a Killing vector field and

$$\nabla^2 \xi = \eta \wedge \mathrm{d}p$$

which shows that ξ is an EC vector field (see (2)). We recall that similar results hold for Sasakian manifold [14].

Let then $W = \mathrm{vect}\{h_a, h_{a^*}, h_0 = \xi \mid a = 1, \ldots, m;\ a^* = a + m\}$ be a local frame of Witt frames on $M(\mathcal{U}, \eta, \xi, g)$ [12]. Then in addition of equation (9) one has

$$(23)\qquad g(h_\alpha, \xi) = 0 \qquad g(h_a, h_{b^*}) = \delta_{ab} \qquad g(\xi, \xi) = 1$$

and g is expressed by $\quad g = 2\sum \omega^a \otimes \omega^{a^*} + \eta \otimes \eta.$

In consequence of the contact structure defined by η one has in addition of equation (10)

$$\theta_a^0 + \theta_0^{a^*} = 0 \qquad \theta_0^a + \theta_{a^*}^0 = 0 \qquad \theta_a^0 = \omega^{a^*} \qquad \theta_{a^*} = -\omega^a.$$

If $$X = X^\alpha h_\alpha + X^0 \xi \qquad \alpha = 1, \ldots, 2m$$

is a null vector field, then by (9) and (23) one has

$$(24)\qquad \|X\|^2 = 2\sum X^a X^{a^*} + \|X^0\|^2 = 0.$$

If, in addition, X is a Killing structure vector field, then as in (12), $(7)_2$ one may write

$$\nabla X = X \wedge \mathcal{U}X = \beta \otimes X - \alpha \otimes \mathcal{U}X \tag{25}$$

where $\qquad \alpha = b(X) \qquad \beta = b(\mathcal{U}X)$.

Setting $\qquad X^0 = \eta(X) = f$

we agree to call f the *distinguished scalar associated with* X. From (24), (25) one derives

$$d\alpha = 2\beta \wedge \alpha \qquad df = (f-1)\beta. \tag{26}$$

Let now Σ be the exterior differential system which defines the vector field X on $M(\mathcal{U}, \eta, \xi, g)$. By (26) we see that the characteristic numbers of Σ are the same as in the para-Kählerian case.

Take now the Lie derivative of the structure 1-form η with respect to X. By (21) and $(22)_2$ one finds

$$\mathcal{L}_X\eta = (f+1)\beta \Rightarrow d(\mathcal{L}_X\eta) = 0$$

which shows that X *defines a relative contact transformation on* $M(\mathcal{U}, \eta, \xi, g)$. Further, by reference to the expression for the covariant differential of $\mathcal{U}$ given by (21), one finds by (25)

$$\nabla\mathcal{U}Z = -f\,dp + \beta \otimes \mathcal{U}X - \alpha \otimes X + (f+1)\alpha \otimes \xi. \tag{27}$$

From the above equation, a short calculation gives

$$[\mathcal{U}X, X] = \mathcal{L}_{\mathcal{U}X}X = (f+1)fX$$

which shows that $\mathcal{U}X$ *defines an infinitesimal conformal transformation of* X.

Following the general definition one has

$$\operatorname{div} \mathcal{U}X = \operatorname{trace} \nabla\mathcal{U}X = \sum \omega^{a^*}(\nabla_{h_a}\mathcal{U}X) + \sum \omega^a(\nabla_{h_{a^*}}\mathcal{U}X) + \eta(\nabla_\xi\mathcal{U}X)$$

and on behalf of (27) one gets

$$\operatorname{div}\mathcal{U}X = 1 - f^2 - 2mf. \tag{28}$$

On the other hand by $(26)_2$ one may write

$$\operatorname{grad} f = (f-1)\mathcal{U}X \tag{29}$$

and since by (24) one has $\|\mathcal{U}X\|^2 = f^2$, it follows by (29)

$$\|\operatorname{grad} f\|^2 = f^2(f-1)^2. \tag{30}$$

On the other hand by $(26)_2$ and (28) one derives

$$\operatorname{div}(\operatorname{grad} f) = -(f-1)(2mf-1). \tag{31}$$

Since by (30) and (31) it is seen that $\|\operatorname{grad} f\|^2$ and $\operatorname{div}(\operatorname{grad} f)$ are functions of f, then by reference to [23] (see also Preliminaries) it follows that the distinguished scalar f associated with X is an *isoparametric function.*

On the other hand, equation $(26)_1$ shows that *$M(\mathcal{U}, \eta, \xi, g)$ is foliated by hypersurfaces M_X normal to X.* But X being a null vector field it follows by (25) that M_X is a coisotropic hypersurface having X as characteristic vector field [18] that is

$$X \subset T_{p_X}(M_X) \cap T^{\perp}_{p_X}(M_X)$$

where $T_{p_X}(M_X)$ and $T^{\perp}_{p_X}(M_X)$ is the tangent space and the normal space to M_X at $p_X \in M_X$ respectively.

In order to simplify, we agree to denote the induced elements on M_X by the same latters. It follows then by (27) that $\mathcal{U}X$ on M_X satisfies

$$\nabla \mathcal{U}X = -f\,\mathrm{d}p + \beta \otimes \mathcal{U}X$$

and with the help of $(26)_2$ the second covariant derivative of $\mathcal{U}X$ is expressed by

$$\nabla^2 \mathcal{U}X = \beta \wedge \mathrm{d}p .$$

Therefore one may say that on M_X, the vector field $\mathcal{U}X$ is EC and has, as ξ, $+1$ as conformal scalar. In consequence of this fact, the Ricci curvature of $\mathcal{U}X$ is expressed by

$$\operatorname{Ric}(\mathcal{U}X) = -2mf^2 .$$

Theorem. *Let X be a null structure Killing vector field on a $(2m+1)$-dimensional pseudo-Sasakian manifold $M(\mathcal{U}, \eta, \xi, g)$ and let $f = \eta(X)$ be the distinguished scalar associated with X. The existence of such a vector field on $M(\mathcal{U}, \eta, \xi, g)$ is, as in the para-Kählerian case, determined by an exterior differential system in involution, which depends on two functions of one argument. The following properties are proved:*

1. *X defines a relative contact transformation $M(\mathcal{U}, \eta, \xi, g)$,* i.e. $\mathcal{U}(\mathcal{L}_X \eta) = 0$

2. *$\mathcal{U}X$ defines an infinitesimal conformal transformation of X,* i.e. $\mathcal{L}_{\mathcal{U}X} = \tau X$, $\tau = f(f+1)$

3. *f is an isoparametric function*

4. *Any $M(\mathcal{U}, \eta, \xi, g)$ is foliated by coisotropic hypersurfaces M_X having X as characteristic vector field and, on M_X, the vector field $\mathcal{U}X$ is* EC *and* $\mathrm{Ric}(\mathcal{U}X) = -2mf^2$.

5 - Null Killing vector fields on space-time manifolds

Let (M, g) be a *general space-time* satisfying the usual integrability condition and let $S = \mathrm{vect}\{h_A \,|\, A = 1, \dots, 4\}$ be a local field of Sachs frames over M [3]. The normalization conditions for the vector fields h_A are

(32) $\langle h_1, h_4 \rangle = 1 \quad \langle h_2, h_3 \rangle = -1$ where $\langle , \rangle$ stands for g

and all the other scalar products are zero. Therefore h_1, h_4 are real null vectors, whilst h_2, h_3 are complex conjugate.

If $S^* = \mathrm{covect}\{\theta^A\}$ is the associated coframe of S, then the soldering form $\mathrm{d}p$ of M is expressed in terms of S^* by

$$\mathrm{d}p = \theta^A \otimes h_A$$

which by (32) implies $\quad g = 2(\theta^1 \otimes \theta^4 - \theta^2 \otimes \theta^3)$.

If $T_p(M)$ is the tangent space at $p \in M$, then it may be split as

$$T_p(M) = D_h \oplus D_S \tag{33}$$

where $D_h = \{h_1, h_4\}$ and $D_s = \{h_2, h_3\}$ are the hyperbolic and the spatial distribution respectively.

In the following we will make use of the complex vectorial formalism (abr. CVF) constructed in [3]. This formalism is based on the local isomorphism $\mathcal{A}$: $L(4) \to SO^3(C)$ where $L(4)$ is a 4-dimensional Lorentz group and $SO^3(C)$ is the 3-dimensional complex rotation group. With such a formalism are associated the six 1-forms

$$\sigma_\alpha = \sigma_{\alpha A}\,\theta^A \qquad \bar{\sigma}_\alpha = \bar{\sigma}_{\alpha A}\,\bar{\theta}^A \qquad A = 1, 2, 3, 4;\ \alpha = 1, 2, 3 \tag{34}$$

(the bar denotes complex conjugate, i.e. $\theta^2 = \bar{\theta}^3$, $\theta^1 = \bar{\theta}^1$, $\theta^4 = \bar{\theta}^4$) where the coefficients $\sigma_{\alpha A}$, $\bar{\sigma}_{\alpha A}$ correspond to the 12 spinorial coefficients of Neumann and Penrose [10].

In terms of $\sigma_{\alpha A}$, $\bar{\sigma}_{\alpha A}$, the covariant derivatives of h are expressed by

$$\begin{aligned}
\nabla h_1 &= -\frac{1}{4}(\sigma_3 + \bar{\sigma}_3) \otimes h_1 + \frac{1}{2}\bar{\sigma}_2 \otimes h_2 + \frac{1}{2}\sigma_2 \otimes h_3 \\
\nabla h_2 &= -\frac{1}{2}\bar{\sigma}_1 \otimes h_1 + \frac{1}{4}(\bar{\sigma}_3 - \sigma_3) \otimes h_2 + \frac{1}{2}\sigma_2 \otimes h_4 \\
\nabla h_3 &= -\frac{1}{2}\bar{\sigma}_1 \otimes h_1 - \frac{1}{4}(\bar{\sigma}_3 - \sigma_3) \otimes h_3 + \frac{1}{2}\bar{\sigma}_2 \otimes h_4 \\
\nabla h_4 &= -\frac{1}{2}\bar{\sigma}_1 \otimes h_2 - \frac{1}{2}\bar{\sigma}_1 \otimes h_3 + \frac{1}{2}(\sigma_3 + \bar{\sigma}_3) \otimes h_4
\end{aligned} \tag{35}$$

and the first group of structure equations (EC) by

$$\begin{aligned}
d\theta^1 &= +\frac{1}{4}(\sigma_3 + \bar{\sigma}_3) \wedge \theta^1 + \frac{1}{2}\bar{\sigma}_1 \wedge \theta^2 + \frac{1}{2}\sigma_1 \wedge \theta^3 \\
d\theta^2 &= -\frac{1}{2}\bar{\sigma}_1 \wedge \theta^1 + \frac{1}{4}(\sigma_3 - \bar{\sigma}_3) \wedge \theta^2 + \frac{1}{2}\sigma_1 \wedge \theta^4 \\
d\theta^3 &= -\frac{1}{2}\sigma_2 \wedge \theta^1 - \frac{1}{4}(\sigma_3 - \bar{\sigma}_3) \wedge \theta^3 + \frac{1}{2}\bar{\sigma}_1 \wedge \theta^4 \\
d\theta^4 &= -\frac{1}{2}\sigma_2 \wedge \theta^2 - \frac{1}{2}\bar{\sigma}_2 \wedge \theta^3 - \frac{1}{4}(\sigma_3 + \bar{\sigma}_3) \wedge \theta^4 .
\end{aligned} \tag{36}$$

It should be noticed that θ^1 (resp. θ^4) is the dual form of h_4 (resp. h_1). We recall that the null vector field h_4, which is called *Debever's vector* [10], plays a distinguished role in the frame of the CVF. In consequence of this fact, it is natural to assume that at each point p of M, h_4 is a covariant skew symmetric (abr. CSS) Killing vector field. Therefore we will write

$$\nabla h_4 = h_4 \wedge U = u \otimes h_4 - \theta^1 \otimes U$$

for some vector field U having $u = b(U)$ as dual frame. If we set

$$U = \sum u_s h_s \qquad s = 2, 3, 4$$

then by comparison to (35) one may write $u = u_4 \theta^1 - u_2 \theta^3 - u_3 \theta^2$ and

$$\sigma_1 = 2u_2 \theta^1 \qquad \bar{\sigma}_1 = 2u_3 \theta^1 . \tag{37}$$

Hence, by (34) one has

$$\sigma_3 + \bar{\sigma}_3 = -4(u_2 \theta^3 + u_3 \theta^2) = -2(\sigma_{11} \theta^3 + \bar{\sigma}_{11} \theta^2)$$

and $\sigma_{11} = 2u_2 \quad \sigma_{13} = 0 \quad \sigma_{14} = 0 \quad \bar{\sigma}_{11} = 2u_2 \quad \bar{\sigma}_{13} = 0 \quad \bar{\sigma}_{14} = 0\,.$

On behalf of the CVF dictionnary [**10**] we recall that equations $\sigma_{14} = 0$, $\sigma_{13} = 0$ express that the congruence $\mathscr{C}(h_4)$ is geodetic and shear-free and that

(38) $$\sigma_{14} = 0 \qquad \sigma_{13} = 0 \qquad \sigma_3 + \bar{\sigma}_3 = -2(\sigma_{11}\,\theta^3 + \bar{\sigma}_{11}\,\theta^2)$$

are the general conditions in order that h_4 be a Killing vector field. These results are a criterion for the exactness of our calculations. By (37) one also quickly finds

(39) $$d\theta^1 = -2u \wedge \theta^1$$

which expresses that θ^1 is an exterior recurrent form [5]. Next, assume that the associated null real vector field h_1 of h_4 is, as h_4, a CSS *Killing vector field.* Therefore, one must write for some V

$$\nabla h_1 = h_1 \wedge V = v \otimes h_1 - \theta^4 \otimes V$$

where $v = b(V)$. By the same reasoning as for h_4 one finds by (35), (36)

(40) $$\sigma_2 = \sigma_{24}\,\theta^4 \qquad \bar{\sigma}_2 = \bar{\sigma}_{24}\,\theta^4$$

(41) $$d\theta^4 = 2v \wedge \theta^4$$

i.e.; θ^4 is as θ^1 an ER-form. Now since equation (40) implies

$$\sigma_{21} = 0 \qquad \sigma_{22} = 0$$

the above equations together with

$$\sigma_{13} = 0 \qquad \sigma_{14} = 0$$

(see equations (38)) are as is known in terms of CVF, characterizing *a space-time of type D* in Petrov's classification [3], [**10**]. Further, equations (39) and (41), show by Frobenius theorem that the space-time distribution (see the splitting (33)) is involutive. Let then M_S be the leaf (surface) of D_S. Since M_S is determined by $\theta^1 = 0$, $\theta^4 = 0$, then clearly h_4 and h_1 are normal to M_S, and on M_S one has

$$\nabla h_4 = u \otimes h_4 \qquad \nabla h_1 = v \otimes h_1$$

(we denote the induced elements by the same letters). From above it follows at once

$$\langle \nabla h_4, \nabla h_4 \rangle = 0 \qquad \langle \nabla h_1, \nabla h_1 \rangle = 0$$

and this reveals the significant fact, according to which M_S is a pseudo-isotropic surface of M. We agree to denominate any space-time having the above properties, *a space-time having the Killing property.*

Theorem. *Let (M, g) be a general space-time and let h_4 and h_1 be the Debever's vector field and its associated null real vector field, respectively at each point $p \in M$. If h_4 and h_1 are both skew symmetric Killing vector fields, then (M, g) is of type D in Petrov's classification, and M is foliated by pseudo-isotropic space-like surfaces, normal to h_4 and h_1.*

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Summary

See Introduction.

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