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On the relaxation processes
in a mixture of Maxwell gases (**)

1 - Introduction

Several papers published in the last 10-15 years about the Boltzmann equation were devoted to the investigation of nonlinear relaxation problems in the case of Maxwell-molecules (for a review, see Ernst [3], Bobylev [2], and Spiga [5]). The choice of Maxwell-type interactions was motivated by the fact that it allows by far the deepest analytical investigation, and that it is the only case where one might hope to devise something like exact solutions. On the other hand, it is well known that all macroscopic applications relevant to kinetic theory are to a great extent independent of the particular microscopic collision model. One of the main achievement in the previous papers is the considerable simplification provided by the use of Fourier transform technique, since the five-fold collision integral reduces, in the Fourier representation, to a two-fold integral over the unit sphere. Such simplification remains valid also for mixtures of gases with different masses [2], and even in the presence of other types of interactions of interest for applications (extended kinetic theory [5]). Nevertheless, only few of the articles above were actually dealing with gas mixtures [4], [1]. In particular, [1] was concerned with the Fourier transformed Boltz-

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mann equations for a binary mixture, and with their asymptotic solution for weakly interacting components. Gas mixtures are indeed of practical interest in many physical problems. These problems are in general very complicated, but can often be handled in an appropriate and convenient way by taking into account the natural occurrence of small parameters. Typical examples are the problems of neutron physics (small density ratio) and plasma physics (small mass ratio). The first case is described by linear transport theory. The second case, which will be addressed to in the present work, retains its nonlinear features even in the asymptotic limit. We restrict here our attention to relaxation problems in a two component mixture, with masses m_1 and m_2, $\varepsilon = \dfrac{m_1}{m_2} \ll 1$, and distribution functions f_1 and f_2. We will show that it is possible to reduce the system of two Boltzmann equations (at least for isotropic distributions) to a single Boltzmann equation, so that all results available for it are simply generalized to a binary mixture when $\varepsilon \to 0$. We will also consider a typical problem from extended kinetic theory, namely the case when the light component constitutes a fixed background of field particles with given equilibrium distribution, and removal interactions can take place together with elastic scattering. It is shown that in the asymptotic limit the solution can be obtained from the knowledge of the solution for a non-scattering background (for which some results are available [5]). The problem of the effects of an externally applied force field can be analyzed in a similar way, taking again advantage of the simplifications due to the Fourier representation. However, this will be the subject of a subsequent paper.

2 - Fourier transformed kinetic equations

Our starting point is the set of space homogeneous *Boltzmann equations* for a mixture of N Maxwell-gases

$$(1) \qquad \frac{\partial_i}{\partial t} = \sum_{j=1}^{N} \int_{R_3 \times S_2} g_{ij} \left(\frac{v - w}{|v - w|} \cdot \widehat{n} \right) [f_i(v', t)\, f_j(w', t) - f_i(v, t)\, f_j(w, t]\, dw\, d\widehat{n}$$

$i = 1, \ldots, N$, where v and w are velocity variables, $\widehat{n}$ denotes a unit vector spanning the unit sphere S_2, g stands for the differential microscopic collision frequency for elastic scattering, and v' and w' represent the postcollisional velocities of particles i and j associated to an encounter between the same pair at ve-

locities v and w. They are given by

(2) $$v' = \frac{m_i v + m_j (w + V\widehat{n})}{m_i + m_j} \qquad w' = \frac{m_i (v - V\widehat{n}) + m_j w}{m_i + m_j}$$

with $V = |v - w|$.

The main macroscopic fields are *number density*

(3) $$\rho_i(t) = \int\limits_{R_3} f_i(v, t)\, dv$$

drift velocity

(4) $$u_i(t) = \frac{1}{\rho_i} \int\limits_{R_3} v f_i(v, t)\, dv$$

and *temperature* (in energy units)

(5) $$\Theta_i(t) = \frac{1}{3} m_i \Big[\frac{1}{\rho_i} \int\limits_{R_3} v^2 f_i(v, t)\, dv - u_i^2 \Big].$$

Equilibrium distributions are the *Maxwellians*

(6) $$f(v) = \rho (\frac{m}{2\pi\Theta})^{\frac{3}{2}} \exp\big[-\frac{m}{2\Theta} (v - u)^2 \big].$$

The Fourier transform approach has proved very effective in the investigation of (1); in that frame one has to determine the unknown *characteristic functions*

(7) $$\widetilde{f}_i(k, t) = \int\limits_{R_3} f_i(v, t) \exp(-ik \cdot v)\, dv.$$

Since almost all interesting features of the relaxation processes in gas mixtures can be clarified by studying the technically simpler case of a binary mixture, we shall take $N = 2$, and with $j \neq i$, write the Fourier transformed kinetic equations as [2]

(8) $$\frac{\partial \widetilde{f}_i}{\partial t} = I_i(\widetilde{f}_i, \widetilde{f}_i) + \int\limits_{S_2} g_* (\widehat{k} \cdot \widehat{n}) [\widetilde{f}_i(\frac{\varepsilon_i k + k\widehat{n}}{1 + \varepsilon_i}) \widetilde{f}_j(\frac{k - k\widehat{n}}{1 + \varepsilon_i}) - \widetilde{f}_i(k) \widetilde{f}_j(0)]\, d\widehat{n} \qquad \varepsilon_i = \varepsilon^{3-2i}$$

where t-dependence has been dropped for brevity, $\widehat{k}$ stands for k/k, $k = |k|$, g_* is the common value of g_{12} and g_{21}, while g_{ii} will be labeled by g_i. The bilinear

operator I_i is defined by

(9) $$I_i(\widetilde{f}, \widetilde{f}) = \int_{S_2} g_i(\widehat{k} \cdot \widehat{n})[\widetilde{f}(\frac{k + k\widehat{n}}{2})\widetilde{f}(\frac{k - k\widehat{n}}{2}) - \widetilde{f}(k)\widetilde{f}(0)]\, d\widehat{n}.$$

At this point we remark that it is convenient to resort to dimensionless variables. Let the initial value (relevant to $t = 0$) of any quantity be labeled by a subscript 0, and let T_0 denote a characteristic constant temperature (e.g., the average of the initial temperatures weighted on the initial densities).

Initial conditions for (8) read as $\widetilde{f}_i(k, 0) = \widetilde{f}_{0i}(k)$, and one may introduce new dependent variables $\widetilde{\varphi}_i$ by

(10) $$\widetilde{f}_i(k, t) = \widetilde{\varphi}_i[(\frac{T_0}{m_i})^{\frac{1}{2}} k, t]$$

and rename $(\frac{T_0}{m_i})^{\frac{1}{2}} k$ by k in the i-th equation, to get the symmetric form

(11) $$\frac{\partial \widetilde{\varphi}_i}{\partial t} = I_i(\widetilde{\varphi}_i, \widetilde{\varphi}_i) + \int g_*(\widehat{k} \cdot \widehat{n})[\widetilde{\varphi}_i(\frac{\varepsilon_i k + k\widehat{n}}{1 + \varepsilon_i})\widetilde{\varphi}_j(\frac{\varepsilon_i^{\frac{1}{2}} k - \varepsilon_i^{\frac{1}{2}} k\widehat{n}}{1 + \varepsilon_i}) - \widetilde{\varphi}_i(k)\widetilde{\varphi}_j(0)]\, d\widehat{n}.$$

Notice that in the new setting we have

(12) $$\rho_i = \widetilde{\varphi}_i(0, t) \qquad u_i = i(\frac{T_0}{m_i})^{\frac{1}{2}} \frac{1}{\rho_i}(\overline{\nabla}_k \widetilde{\varphi}_i)(0, t)$$
$$\Theta_i = -\frac{T_0}{3}\frac{1}{\rho_i}(\nabla_k^2 \widetilde{\varphi}_i)(0, t) - \frac{1}{3} m_i u_i^2$$

and Maxwellians take the form $\rho \exp[-i(\frac{m}{T_0})^{\frac{1}{2}} k \cdot u - \Theta k^2/2T_0]$. We shall proceed now to consider the problem of solving the approximate form taken by (11) in the asymptotic limit when $\varepsilon \to 0$ and $0(\varepsilon^2)$ terms are neglected. It is worth remarking that specialization at $k = 0$ yields at once the first integrals $\rho_i = $ constant. Moreover, if even $0(\varepsilon)$ terms were disregarded in the limiting procedure, the second equation would decouple and become

$$\frac{\partial \widetilde{\varphi}_2}{\partial t} = I_2(\widetilde{\varphi}_2, \widetilde{\varphi}_2)$$

whereas the first, reading as

(13) $$\frac{\partial \widetilde{\varphi}_1}{\partial t} = I_1(\widetilde{\varphi}_1, \widetilde{\varphi}_1) + \rho_2 \int g_*(\widehat{k} \cdot \widehat{n})[\widetilde{\varphi}_1(k\widehat{n}) - \widetilde{\varphi}_1(k)]\, d\widehat{n}$$

would collapse to the single species Boltzmann equation only in isotropic condi-

tions. This is clear from a physical point of view, since for $\varepsilon = 0$ heavy particles are not deflected by collision, and light ones change their direction but not their speed.

3 - Isotropic binary mixture

In isotropic conditions the characteristic functions depend only on k and t, drift velocities vanish, and new variables, defined by

(14)
$$\widetilde{\varphi}_i(k, t) = \varphi_i(\frac{1}{2}k^2, t) \qquad x = \frac{1}{2}k^2$$

are more suitable for the analysis.

Now

(15)
$$\rho_i = \varphi_i(0, t) = \text{constant} \qquad \Theta_i = -T_0 \frac{1}{\rho_i}(\frac{\partial \varphi_i}{\partial x})(0, t)$$

and Maxwellians are $\rho \exp(-x\frac{\Theta}{T_0})$. In (11) one has

$$|\frac{1}{2}(\mathbf{k} \pm k\widehat{n})| = k[\frac{1}{2}(1 \pm \mu)]^{\frac{1}{2}}$$

with $\mu = \widehat{k}\cdot\widehat{n}$, and similarly for the other arguments. By defining further

(16)
$$s = \frac{1 - \mu}{2} \qquad b_\alpha(s) = 4\pi g_\alpha(1 - 2s) \qquad \delta = \frac{4\varepsilon}{(1 + \varepsilon)^2}$$

it is matter of only little algebra to cast the kinetic equations as

(17)
$$\frac{\partial \varphi_i}{\partial t} = J_i(\varphi_i, \varphi_i) + \int_0^1 b_*(s)\{\varphi_i[(1 - \delta s)x]\varphi_j(\delta sx) - \varphi_i(x)\varphi_j(0)\}\,ds$$

where $i = 1, 2$, $j = 1, 2$, $j \neq i$, and

$$J_i(\varphi_i, \varphi_i) = \int_0^1 b_i(s)\{\varphi_i[(1 - s)x]\varphi_i(sx) - \varphi_i(x)\varphi_i(0)\}\,ds.$$

If we derive (17) with respect to x and specialize at $x = 0$, we get macroscopic equations for temperatures

(19)
$$\frac{d\Theta_i}{dt} = \lambda\rho_j(\Theta_j - \Theta_i) \qquad\qquad \text{with}$$

(20)
$$\lambda = \frac{1}{2}\delta g_*^{\text{tr}} \qquad g_\alpha^{\text{tr}} = 2\int_0^1 sb_\alpha(s)\,ds = 2\pi\int_{-1}^1 (1 - \mu)g_\alpha(\mu)\,d\mu.$$

Thus $\Sigma_i \rho_i \Theta_i$ is also a first integral (so that the choice $T_0 = (\Sigma_i \rho_{0i} \Theta_{0i})(\Sigma_i \rho_{0i})^{-1}$ is appropriate), and temperatures are analytically expressed by

$$(21) \qquad \Theta_i(t) = T_0 + \frac{\rho_j}{\rho_i + \rho_j}(\Theta_{0i} - \Theta_{0j}) \exp[-\lambda(\rho_i + \rho_j)t]$$

with exponential relaxation to T_0.

We expand now with respect to the small parameter δ in (17) and keep up to $0(\delta)$ terms, in order to get first order asymptotic kinetic equations. After some algebra, one ends up with

$$(22) \qquad \frac{\partial \varphi_i}{\partial t} = J_i(\varphi_i, \varphi_i) - \lambda \rho_j x \left[\frac{\partial \varphi_i}{\partial x} + \frac{\Theta_j(t)}{T_0} \varphi_i(x) \right],$$

coupled by densities and temperatures only, which are known a priori. It is worth noticing that (22) preserves the macroscopic equations. The final decoupling is achieved by the further ansatz

$$(23) \qquad \varphi_i(x, t) = \exp\left[-x\frac{\Theta_i(t)}{T_0}\right] \Psi_i(x, t)$$

leading to

$$(24) \qquad \frac{\partial \Psi_i}{\partial t} + \lambda \rho_j x \frac{\partial \Psi_i}{\partial x} = J_i(\Psi_i, \Psi_i) \qquad \Psi_i(x, 0) = \varphi_{0i}(x) \exp\left(x\frac{\Theta_{0i}}{T_0}\right)$$

where the standard method of characteristics allows reduction to the *single* Boltzmann equations

$$(25) \qquad \frac{\partial \Psi_i}{\partial \tau} = J_i(\Psi_i, \Psi_i)$$

in terms of the new independent variables $y = x \exp(-\lambda \rho_j t)$, $\tau = t$, $j \neq i$, in the i-th equation.

The solution to the approximate kinetic equations for a binary Maxwell mixture with small mass ratio is thus amenable to the solution of standard Boltzmann equations for single gases. So, for instance, if initial conditions are Maxwellians at temperature Θ_{0i} (i.e. $\Psi_i(x, 0) = \rho_i$), then (25) is solved by $\Psi_i = \rho_i$, so that (22) provides again two Maxwellians, at temperatures $\Theta_i(t)$. All recently discovered exact solutions of the Boltzmann equation apply however to the two by two system (22). The solution associated to the BKW mode [2] corresponds to the initial condition $\Psi_{0i}(x) = \rho_i(1 - \beta_{0i}x) \exp(\beta_{0i}x)$, with $0 \leq \beta_{0i} < \frac{2}{5}$

for positivity. The actual distribution functions can be obtained by going back through all previous steps, and read as

(26)

$$f_i(v, t) = \rho_i \left[\frac{m_i}{2\pi\alpha_i(t)} \right]^{\frac{3}{2}}$$

$$\cdot \left\{ 1 + \frac{T_0 \beta_i(t) \exp(-\lambda \rho_j t)}{\alpha_i(t)} \left[\frac{m_i}{2\alpha_i(t)} v^2 - \frac{3}{2} \right] \right\} \exp \left[-\frac{m_i}{2\alpha_i(t)} v^2 \right]$$

where

(27)

$$\alpha_i(t) = \Theta_i(t) - T_0 \beta_i(t) \exp(-\lambda \rho_j t) \qquad \beta_i(t) = \beta_{0i} \exp(-g_i^* \rho_i t)$$

$$g_i^* = \frac{\pi}{2} \int_{-1}^{1} (1 - \mu^2) g_i(\mu) d\mu . .$$

Positivity of the solution (26) can always be guaranteed by choosing

$$\beta_{0i} < \min_{t \geq 0} \left[\frac{2}{5} \frac{\Theta_i}{T_0} \exp(\lambda \rho_j t + g_i^* \rho_i t) \right].$$

4 - Single gas in the presence of background and removal

Let us suppose now that, in the physical situation described in Sec. 2, the light species is a fixed background, whose given distribution function is an isotropic Maxwellian at an assigned temperature T_0 (which will be the temperature chosen for the adimensionalization (10)). In addition, let heavy particles may undergo removal collisions between themselves (*annihilation*) as well as with field particles (*absorption*), and let G_2 and G_* denote the relevant differential microscopic collision frequencies. Dropping the unnecessary index 2, (11) must be replaced by [5]

(28) $$\frac{\partial \widetilde{\varphi}}{\partial t} = I(\widetilde{\varphi}, \widetilde{\varphi}) + \int g_* (\widehat{k} \cdot \widehat{n}) [\widetilde{\varphi}(\frac{k + \varepsilon k\widehat{n}}{1 + \varepsilon}) \widetilde{\varphi}_1(\frac{\varepsilon^{\frac{1}{2}} k - \varepsilon^{\frac{1}{2}} k\widehat{n}}{1 + \varepsilon}) - \widetilde{\varphi}(k) \widetilde{\varphi}_1(0)] d\widehat{n}$$

$$- \int G(\widehat{k} \cdot \widehat{n}) \widetilde{\varphi}(k) \widetilde{\varphi}(0) d\widehat{n} - \int G_* (\widehat{k} \cdot \widehat{n}) \widetilde{\varphi}(k) \widetilde{\varphi}_1(0) d\widehat{n} .$$

Here $\widetilde{\varphi}_1(k) = \rho_1 \exp(-\frac{1}{2} k^2)$, and it is convenient to resort the group symmetries of the Boltzmann equation, setting $\widetilde{\varphi}(k, t) = \varphi(k, t) \exp(-\frac{1}{2} k^2)$. It is

readily obtained

(29) $\quad \dfrac{\partial \varphi}{\partial t} = I(\varphi, \varphi) + \rho_1 \int g_* (\widehat{k} \cdot \widehat{n})[\varphi(\dfrac{k + \varepsilon k\widehat{n}}{1 + \varepsilon}) - \varphi(k)] \, d\widehat{n} - (\widehat{G}\rho + \widehat{G}_* \rho_1) \, \varphi(k, t)$

with $\varphi(k, 0) = \widetilde{\varphi}_0(k) \exp(\dfrac{1}{2} k^2)$ and

(30) $\qquad\qquad\qquad \widehat{G}_\alpha = 2\pi \int\limits_{-1}^{1} G_\alpha(\mu) \, d\mu \, .$

Notice that now $\rho = \varphi(0, t)$ is not constant; (29) prescribes for it the continuity equation

(31) $\qquad\qquad\qquad \dfrac{d\rho}{dt} + \widehat{G}_* \rho_1 \rho + \widehat{G}\rho^2 = 0$

that can be easily solved independently [5], and implies of course decay to zero for nonvanishing $\widehat{G}_*$ and $\widehat{G}$.

The first order approximation to the kinetic equation (29) in the asymptotic limit $\varepsilon \to 0$ is obtained as in the previous section, and reads

(32) $\qquad \dfrac{\partial \varphi}{\partial t} + \varepsilon g_*^{\text{tr}} \rho_1 k \cdot \overline{\nabla}_k \varphi = I(\varphi, \varphi) - [\widehat{G}\rho(t) + \widehat{G}_* \rho_1] \varphi(k, t) \qquad\qquad$ since

(33) $\qquad\qquad\qquad \int(k - k\widehat{n}) \, g_* (\widehat{k} \cdot \widehat{n}) \, d\widehat{n} = g_*^{\text{tr}} k \, .$

We can proceed again by the method of characteristics, introduce $\omega = k \exp(-\varepsilon g_*^{\text{tr}} \rho_1 t)$ and $\tau = t$, and get the equation for a nonscattering background

(34) $\qquad\qquad\qquad \dfrac{\partial \varphi}{\partial \tau} = I(\varphi, \varphi) - [\widehat{G}\rho(\tau) + \widehat{G}_* \rho_1] \varphi(\omega, \tau) \, .$

This equation is also amenable to a standard Boltzmann equation. It is easy to verify, in fact, that, if $\Psi(\omega, \tau)$ is the solution to $\dfrac{\partial \Psi}{\partial \tau} = I(\Psi, \Psi)$ corresponding to the initial condition in (29), then the solution to the Cauchy problem (34) is given by

(35) $\qquad \varphi(\omega, \tau) = \exp[-\int\limits_{0}^{\tau} \Lambda(\tau') \, d\tau'] \, \Psi(\omega, \int\limits_{0}^{\tau} \exp[-\int\limits_{0}^{\tau'} \Lambda(u) \, du] \, d\tau')$

where $\Lambda(\tau) = \widehat{G}\rho(\tau) + \widehat{G}_* \rho_1$. Since $\rho_0 \exp[-\int\limits_{0}^{\tau} \Lambda(\tau') \, d\tau'] = \rho(\tau)$ in force of (31), an

example of exact solution to (34) is provided by the generalized BKW mode

(36)
$$\rho(\tau)[1 - \beta(\tau)\frac{\omega^2}{2}] \exp\left\{-\left[\frac{\Theta_0 - T_0}{T_0} - \beta(\tau)\right]\frac{\omega^2}{2}\right\}$$

$$\beta(\tau) = \beta_0 \exp\left[-g^*\int_0^{\tilde{\tau}}\rho(\tau')\,d\tau'\right]$$

corresponding to an initial test particle temperature Θ_0, and with a free parameter β_0, chosen in order to guarantee positivity of the distribution function. The choice $\beta_0 = 0$ yields a Maxwellian with density $\rho(\tau)$ and temperature

(37) $\Theta(\tau) = \Theta_0 \exp(-\lambda\rho_1 t) + T_0[1 - \exp(-\lambda\rho_1 t)] \qquad \lambda = 2\varepsilon g_*^{\text{tr}}$.

It must be noted that in general the second argument of Ψ in (35) does not tend to infinty when $\tau \to \infty$ (for instance, it tends to $1/\widehat{G}_*\rho_1$ for $\widehat{G} = 0$). Therefore, relaxation processes described in this section do not mean approach to a Maxwellian.

References

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Riassunto

L'articolo è dedicato alla descrizione dei processi di rilassamento in miscele di gas Maxwelliani per mezzo delle trasformate di Fourier delle relative equazioni di Boltzmann nonlineari. In particolare vengono considerate miscele binarie con piccolo rap-

porto tra le masse, e viene mostrato che il sistema di due equazioni di Boltzmann può essere ridotto ad un'unica equazione di Boltzmann. Il problema esteso di un gas in un assegnato mezzo ospite in equilibrio ed in presenza di rimozioni viene anch'esso studiato secondo lo stesso schema.

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