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A. K. MAHATO and N. K. AGRAWAL (*)

The n-dimensional distributional Gauss-hypergeometric transformation (**)

1 - Introduction

In an earlier paper [3], authors have extended the integral transform

$$F(s) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} \int_0^\infty {}_2F_1(\alpha, \beta;\ \gamma;\ -sx) f(x)\, \mathrm{d}x \tag{1.1}$$

where ${}_2F_1$ denotes the *Gauss-hypergeometric function*, to a class of generalized functions. In [4], a complex inversion formula has also been extended to a class of generalized functions for the above transform.

In the present paper we develop an n-dimensional distributional Gauss-hypergeometric transformation.

2 - Notation

For real and complex n-dimensional euclidean spaces we use the notations $\boldsymbol{R}^n$ and $\boldsymbol{C}^n$ respectively. An n-tuple is denoted by $z = \{z_1, \ldots, z_n\}$. We restrict x and y to the set

$$I = \{x \in \boldsymbol{R}^n \,|\, 0 < x_v,\ \ v = 1, \ldots, n\}.$$

We shall use the usual euclidean norm $|x| = (\sum x_v^2)^{\frac{1}{2}}$.

A function on a subset of $\boldsymbol{R}^n$ will be denoted by $f(x) = f(x_1, \ldots, x_n)$. A similar

(*) Dept. of Math., Marwari College, Ranchi 834001, India.
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notation is used for functions on C^n or $R^n \times C^n$. For example,

$$xt = \{x_1 t_1, \ldots, x_n t_n\} \qquad e^{-st} = \{e^{-s_1 t_1}, \ldots, e^{-s_n t_n}\}.$$

By $[x]$, we mean the product $x_1 x_2 \ldots x_n$. Thus

$$[e^{-st}] = \exp(-s_1 t_1 - s_2 t_2 - \ldots - s_n t_n).$$

We put $$P = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} \quad \text{and} \quad [P] = \frac{\prod_{j=1}^{n} \Gamma(\alpha_j) \prod_{j=1}^{n} \Gamma(\beta_j)}{\prod_{j=1}^{n} \Gamma(\gamma_j)}.$$

The notation $x < t$ means and $x_v < t_v$ ($v = 1, \ldots, n$). k will always denote a non-negative integer in R^n i.e. $k = \{k_1, \ldots, k_n\}$ where k_v is a non-negative integer ($v = 1, \ldots, n$). Symbols (k) and D_x^n stand for

$$(k) = k_1 + k_2 + \ldots + k_n \qquad D_x^n = \frac{\partial^{(k)}}{\partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \ldots \partial_{x_n}^{k_n}}.$$

3 - The testing function space $G_{a,b}$ and its dual $G'_{a,b}$

Let $a, b \in R^n$ and $t \in I$. Also let a_v, b_v, t_v be the components of a, b, t respectively. We consider the real positive smooth function

$$K_{a_v, b_v}(t_v) = \exp(a_v t_v) t_v^{b_v}$$

and the positive smooth function from I into R, given by

$$K_{a,b}(t) = \prod_{v=1}^{n} K_{a_v, b_v}(t_v).$$

$G_{a,b}$ will denote the space of all smooth functions $\phi(t)$ from I into C such that, for each non-negative integer k, we have

$$|K_{a,b}(t)(tD_t)^k \phi(t)| < \infty \tag{3.1}$$

where $(tD_t)^k = \prod_{v=1}^{n} (t_v \frac{\partial}{\partial t_v})^k$.

We assign a topology to $G_{a,b}$ by making use of the following separating system of seminorms $\{P_{a,b,k}\}_{k=1}^{\infty}$, where

$$P_{a,b,v}(\phi) = \max_{0 \leq |k| \leq v} \sup_t |K_{a,b}(t)(tD_t)^k \phi(t)| \qquad v = 0, 1, 2, \ldots. \tag{3.2}$$

A sequence $\{\phi_v\}_{v=1}^{\infty}$ is a Cauchy sequence in $G_{a,b}$ if and only if each ϕ_v is in $G_{a,b}$ and for each fixed k, the funxtions $K_{a,b}(t)(tD_t)^k \phi_v(t)$ converge uniformly on I as $v \to \infty$. It follows that the limit function of $\{\phi_v\}_{v=1}^{\infty}$ is also in $G_{a,b}$. Hence $G_{a,b}$ is *sequentially complete.*

$G'_{a,b}$ is the space of continuous linear functionals on $G_{a,b}$ (i.e. the dual space of $G_{a,b}$). The number that $f \in G'_{a,b}$ assigns to $\phi \in G_{a,b}$ will be denoted by $\langle f, \phi \rangle = \langle f(t), \phi(t) \rangle$.

Let D be the space of all smooth functions having *compact supports* in I. S denotes the space of all smooth functions *of rapid descent* ([7], p. 3).

Theorem 1. *If* $[T] = [P]\, {}_2F_1(\alpha, \beta; \gamma; -st)$, *then* $[T] \in G_{a,b}$ *provided that* $a < 0$ *and* $b > 0$.

Proof. In view of the differential property of ${}_2F_1$ function (Erdelyi [2], p. 58)

$$\frac{d^n}{dx^n}\, {}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n}\, {}_2F_1(\alpha + n, \beta + n; \gamma + n; x)$$

and of the asymptotic properties of ${}_2F_1$ function (see [2], p. 63) ${}_2F_1(\alpha, \beta; \gamma; -x) = o(x^{-\beta})$ as $x \to \infty$, we can prove that $|K_{a,b}(t)(tD_t)^k [T]| < \infty$ as $t \to \infty$, provided that $a < 0$, and $|K_{a,b}(t)(tD_t)^k [T]| \to 0$ as $t \to 0$, provided that $b > 0$.

Hence $[T] \in G_{a,b}$ for $a < 0$ and $b > 0$.

Theorem 2. *Let* $a, b, \sigma \in \boldsymbol{R}^n$ *with* $a < \sigma < b$ *and* $a < 0 < b$, *where* σ *is the real part of* $s \in \boldsymbol{C}^n$. *If*

(i) $\theta \in S$, *then* $\theta[T] \in G_{a,b}$

(ii) $\{\theta_v\}_{v=1}^{\infty}$ *converges to zero in S, then* $\{[T]\theta_v\}_{v=1}^{\infty}$ *also converges in* $G_{a,b}$ *to zero.*

Proof. (ii) will be proved first. We have

$$K_{a,b}(t)(tD_t)^k \{\theta_v(t)[T]\} = [P] K_{a,b}(t)\, t^k \sum_{r=0}^{k} [{}_2F_1(\alpha, \beta; \gamma; -\sigma t)]_r [\theta_v(t)]_{k-r} \tag{3.3}$$

where the suffixes r and $k - r$ indicate orders of differentiation and $[P]$ does not depend on t.

A typical term on the right hand side of (3.3) is

$$A = K_{a,\,b}(t)\, Q[\theta_v(t)]_{k-r}\ {}_2F_1(\alpha + r, \beta + r;\ \gamma + r;\ -\sigma t)$$

Q being a quantity independent of t. Since $a < 0 < b$, A is bounded. Since $\{\theta_v\}$ converges to zero in S, the left hand side of (3.3) converges to zero uniformly for all t. Hence (ii) is proved. The proof of (i) is similar.

4 - Properties of the testing function space $G_{a,\,b}$ and its dual $G'_{a,\,b}$

Property 1. *The space $D(I)$ of the smooth functions with compact support on I, is a subspace of $G_{a,\,b}$. Thus, the restriction of any $f \in G'_{a,\,b}$ to $D(I)$ is in $D'(I)$.*

Property 2. *$G_{a,\,b}$ is a subspace of the space $E(I)$ of the smooth functions on I. $G_{a,\,b}$ is dense in $E(I)$. Moreover, the topology of $G_{a,\,b}$ is stronger than the topology induced on it by $E(I)$. It follows that $E'(I)$ is a subspace of $G'_{a,\,b}$.*

Property 3. *For each $f \in G'_{a,\,b}$, there exist a positive constant $C \in \boldsymbol{R}^n$ and a non-negative integer r such that for all $\phi \in G_{a,\,b}$,*

$$|\langle f, \phi\rangle| \leqslant C P_{a,\,b,\,r}(\phi)\,.$$

5 - The distributional n-dimentional Gauss-hypergeometric transformation

A distribution f is ${}_2F_1$*-transformable* if there exist two points $a, b \in \boldsymbol{R}^n$ such that $f \in G'_{a,\,b}$.

A point $s \in \boldsymbol{C}^n$ is said to be in Ω_f, if and only if there exist $a, b \in \boldsymbol{R}^n$ such that $a < 0 < \operatorname{Re} s < b$ and $f \in G'_{a,\,b}$.

The Gauss-hypergeometric transform Lf of a ${}_2F_1$-transformable distribution f is defined as the function F, from the subset Ω_f of $\boldsymbol{C}^n$ into $\boldsymbol{C}$, given by

$$(Lf)(s) = F(s) = \langle f(t), [T]\rangle. \tag{5.1}$$

The right hand side of (5.1) has a sense as application of $f \in G'_{a,\,b}$ to $[T] \in G_{a,\,b}$ for any fixed value of $s \in \Omega_f$.

LEMMA 1. *Let $s \in \boldsymbol{C}^n$, $a, b \in \boldsymbol{R}^n$, and $a < 0 < \operatorname{Re} s < b$, s being fixed. Let Δs_v be an increment in the v-th component of s, such that $|\Delta s_v| < r$ and $a_v < \operatorname{Re} s_v - r_1 < \operatorname{Re} s_v - r < \operatorname{Re} s_v$, where r and r_1 are real positive numbers. Finally, for $\Delta s_v \neq 0$ let*

$$\phi_{\Delta s_v}(t) = \frac{g((s_v + \Delta s_v)t) - g(s_v t)}{\Delta s_v}$$

where $g(st) = P_2F_1(\alpha, \beta; \gamma; -st)$. Then as $|\Delta s_v| \to 0$, $\phi_{\Delta s_v}(t)$ converges in $G_{a,b}$ to $\frac{\partial}{\partial s} g(s_v t)$.

PROOF. Let

$$\psi_{\Delta s_v}(t) = \frac{g((s_v + \Delta s_v)t) - g(s_v t)}{\Delta s_v} - \frac{\partial}{\partial s} g(s_v t).$$

Now it is sufficient to prove that $\psi_{\Delta s_v}(t)$ converges to zero in $G_{a,b}$.

Let C denote the circle, having centre at s_v and radius r_1. Applying Cauchy's integral formula, we have

$$\frac{\partial^n}{\partial t^n} \psi_{\Delta s_v} = \frac{1}{\Delta s_v}[g_{n_t}((s_v + \Delta s_v)t) - g_{n_t}(s_v t)] - \frac{\partial}{\partial s} g_{n_t}(s_v t)$$

where the suffix n_t in g indicates n times partial derivative with respect to t, and then

$$\frac{\partial^n}{\partial t^n} \psi_{\Delta s_v} = \frac{1}{\Delta s_v}\Big[\frac{1}{2\pi i}\int_C \Big(\frac{1}{z - s_v - \Delta s_v} - \frac{1}{z - s_v}\Big) g_{n_t}(zt)\Big] dz - \frac{1}{2\pi i}\int_C \frac{g_{n_t}(zt)}{(z - s_v)^2} dz$$

$$= \frac{\Delta s_v}{2\pi i}\int_C \frac{g_{n_t}(zt)}{(z - s_v - \Delta s_v)(z - s_v)^2} dz.$$

Now,

$$|K_{a,b}(t)(tD_t)^n \psi_{\Delta s_v}(t)| = \Big|\frac{\Delta s_v}{2\pi i}\int_C \frac{e^{at} t^{b+n} g_{n_t}(zt)}{(z - s_v - \Delta s_v)(z - s_v)^2} dz\Big|$$

$$\leqslant \frac{|\Delta s_v|}{2\pi} M \frac{2\pi r_1}{(r - r_1) r_1^2} = \frac{|\Delta s_v| M}{(r - r_1) r_1}$$

because $|e^{at} t^{b+n} g_{n_t}(zt)|$ is bounded and so less or equal to M on any compact subset of Ω_f. So

$$\sup_{0<t} |K_{a,b}(t)(tD_t)^n \psi_{\Delta s_v}(t)| \to 0 \qquad \text{as } |\Delta s_v| \to 0\,.$$

Hence the lemma.

Theorem 3 (analyticity theorem). *The function $Lf(s) = F(s)$ is analytic in Ω_f and*

$$\frac{\partial F}{\partial s_v} = \left\langle f(t),\ \frac{\partial}{\partial s_v}[T]\right\rangle.$$

Proof. Let $s \in \Omega_f$ and $a < 0 < \operatorname{Re} s < b$. Let us restrict Δs_v as in Lemma 1, then by the linearity of f we have

$$\frac{1}{\Delta s_v}[F(s_1, \ldots, s_v + \Delta s_v, \ldots, s_n) - F(s_1, \ldots, s_v, \ldots, s_n)] = \langle f(t),\ \phi_{\Delta s_v}(t)\rangle.$$

In view of Lemma 1, as $|\Delta s_v| \to 0$ the right hand side converges to $\left\langle f(t),\ \frac{\partial}{\partial s}[T]\right\rangle$ and the theorem is proved.

Theorem 4 (continuity theorem). *If $\{f_v\}_{v=1}^{\infty}$ converges in $G'_{a,b}$ to f for some $a, b \in \boldsymbol{R}^n$ and if $Lf_v = F_v(s)$, then $Lf(s) = F(s)$ exists for at least $a < 0 < \operatorname{Re} s < b$, and $\{F_v(s)\}_{v=1}^{\infty}$ converges pointwise to $F(s)$.*

The proof is similar to Zemanian [**6**], p. 51.

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Summary

The n-dimensional distributional Gauss-hypergeometric transformation is developed using the testing function space $G_{a,b}$ and its dual $G'_{a,b}$. The standard theorems on analyticity and continuity are proved.

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