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L. MARIA ABATANGELO (*)

A study of submanifolds in Finsler spaces by principal bundle techniques (**)

1 - Introduction

Let (M, F) be an *n-dimensional Finsler space* and $O(E) \to V(M)$ the principal $O(n)$-bundle of all Finslerian frames of M.

Let $\theta \in \Gamma^\infty(T^*(O(E)) \otimes \boldsymbol{R}^{2n})$ be *the canonical 1-form* considered in [4] (i.e. the direct product of the h- and v-basic 1-forms of [9]). As observed in [4] (cf. also [6]) if H is a connection in $O(E)$ then $\theta_z \colon H_z \to \boldsymbol{R}^{2n}$ is an $\boldsymbol{R}$-linear isomorphism, for any $z \in O(E)$. Therefore θ may be thought of as the Finslerian analogue of the canonical 1-form in [8] (I, p. 118). It is noteworthy that, unlike the classical case, where the canonical 1-form does not depend on the Riemannian structure of M, the construction of its Finslerian analogue makes use of the Dombrowski map (cf. [2]) and therefore depends on the Langrangian function F.

In the present note we build on work in [4] and show that θ satisfies the *structure equation* (2.3), analogous to the first structure equation associated with a linear connection on M ([8], I, p. 120).

The applications we have in mind concern the geometry of submanifolds in Finsler spaces.

Let (N, F_0) be an $(n+p)$-dimensional Finsler space and $f\colon M \to N$ an immersion, so that $F(u) = F_0(f_* u)$ for any $u \in T(M)$. Let $\theta_0 \in \Gamma^\infty(T^*(O(E_0)) \otimes \boldsymbol{R}^{2(n+p)})$ be the canonical 1-form of (N, F_0). If (M, F) and (N, F_0) are Riemannian manifolds, then the pullback of θ to the principal

(*) Dip. di Matematica, Univ. Bari, Via E. Orabona 4, 70125 Bari, Italia.
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$O(n) \times O(p)$ bundle $O(E_0, E)$ of all adapted frames coincides with the restriction of θ_0 to $O(E_0, E)$ (cf. [8], II, Prop.1.1 p. 3). This fact no longer holds for Finslerian immersions and our Theorem 2 furnishes its Finslerian analogue.

The main tool in the proof of Theorem 2 is a formula of [7]

$$\beta X = \beta_0 X + \gamma_0 H(X, v) \tag{1.1}$$

for any Finslerian vector field X tangent to M (Sec. 4). Its meaning is that horizontal tangent vectors on $V(M)$ may fail to be horizontal with respect to the Cartan-Chern connection of the ambient space (N, F_0).

In Sec. 5 we deal with induced connection 1-forms on M and formulate an open problem.

2 - The canonical 1-form

Let M be an n-dimensional C^∞*-differentiable manifold*. Let $T(M) \to M$ be its tangent bundle and set $V(M) = T(M) \setminus 0$. Let π: $V(M) \to M$ be the natural projection and $E = \pi^{-1} TM \to V(M)$ the pullback of $T(M)$ by π. Let F: $T(M) \to [0, \infty)$ be a Lagrangian function on M, i.e. (M, F) is a *Finsler space*. Then E becomes a Riemannian bundle, in a natural way (cf. [5], p. 2). Let g be the Riemannian bundle metric associated with F. Then g is parallel with respect to the Cartan-Chern connection of (M, F).

Let $E \to X$ be a vector bundle of real rank r over a C^∞ manifold X. Let $g \in \Gamma^\infty (S^2 (E^*))$ be a Riemannian bundle metric in E. Then $O(E) \to X$ denotes the principal $O(r)$-bundle of all orthonormal frames in the fibres of (E, g). That is, if $z \in O(E)_u$, $u \in X$, then z: $\boldsymbol{R}^r \to \boldsymbol{E}_u$ is an $\boldsymbol{R}$-linear isomorphism, so that $g_u(\boldsymbol{Y}_i, \boldsymbol{Y}_j) = \delta_{ij}$ for $1 \leqslant i, j \leqslant r$, where $z(e_i) = Y_i$ and $\{e_1, \ldots, e_r\}$ denotes the canonical basis of $\boldsymbol{R}^r$.

Let (M, F) be an n-dimensional Finsler space and (E, g) its (induced) Riemannian bundle, as above. Let ρ: $O(E) \to V(M)$ be the corresponding principal $O(n)$-bundle. Each $z \in O(E)$ is referred to as a *Finslerian frame* on M. Consider $\theta^h \in \Gamma^\infty (T^*(O(E)) \otimes \boldsymbol{R}^n)$ given by $\theta^h_z = z^{-1} \circ L_u \circ (d_z \rho)$ for any $z \in O(E)$, where $u = \rho(z)$. Also L is the bundle epimorphism L: $T(V(M)) \to E$ given by $L_u X = (u, (d_u \pi) X)$, for any $X \in T_u(V(M))$, $u \in V(M)$. The $\boldsymbol{R}^n$-valued 1-form θ^h on $O(E)$ is, up to a bundle isomorphism, the h-*basic* 1-*form* ([10], p. 48).

Let ∇ be the *Cartan-Chern connection* of (M, F). Let N be its horizontal distribution (on $V(M)$) i.e. $X \in \Gamma^\infty(N)$, iff $\nabla_X v = 0$, where $v \in \Gamma^\infty(E)$ is the *Liouville vector*, $v(u) = (u, u)$, for $u \in V(M)$. Then (cf. e.g. [7]) N is a *nonlinear con-*

nection on $V(M)$, i.e.

$$T_u(V(M)) = N_u \oplus \operatorname{Ker}(d_u \pi) \tag{2.1}$$

for any $u \in V(M)$.

Let Q_u: $T_u(V(M)) \to \operatorname{Ker}(d_u \pi)$ be the natural projection associated with (2.1). The *Dombrowski map* of (M, F) is the bundle epimorphism K: $T(V(M)) \to E$ defined by $K = \gamma^{-1} \circ Q$ where γ: $E \to \operatorname{Ker}(d\pi)$ is the vertical lift. At this point we may recall the construction of the *v-basic* 1-*form* $\theta^v \in \Gamma^\infty(T^*(O(E)) \otimes \boldsymbol{R}^n)$, i.e.

$$\theta_z^v = z^{-1} \circ K_u \circ (d_z \rho)$$

for any $z \in O(E)$, where $u = \rho(z)$.

Let $\theta = \theta^h \oplus \theta^v$ be the direct product of the h- and v-basic 1-forms of (M, F). Then θ is an $\boldsymbol{R}^{2n}$-valued 1-form on $O(E)$ (called the *canonical* 1-*form* of (M, F)). Let H be the connection-distribution in $O(E)$ corresponding to the Cartan-Chern connection ∇ (cf. [5] for the construction of H). Then by a result in [4], θ_z: $H_z \to \boldsymbol{R}^{2n}$ is a $\boldsymbol{R}$-linear isomorphism, for any $z \in O(E)$.

We summarize our constructions so far in the following diagram

$$\begin{array}{ccccc}
\boldsymbol{R}^n & \xleftarrow{\theta_z^h} & T_z(O(E)) & \xrightarrow{\theta_z^v} & \boldsymbol{R}^n \\
\downarrow{\scriptstyle z} & & \downarrow{\scriptstyle d_z \rho} & & \downarrow{\scriptstyle z} \\
E_u & \xleftarrow[L_u]{} & T_u(V(M)) & \xrightarrow[K_u]{} & E_u .
\end{array}$$

Let $\xi \in \boldsymbol{R}^{2n}$. Let $H(\xi) \in \Gamma^\infty(H)$ denote the unique horizontal tangent vector field on $O(E)$, so that $\theta_z(H(\xi)_z) = \xi$ for any $z \in O(E)$. This turns out to possess properties, which are similar to those of the standard horizontal vector fields in [8] (I, p. 119). Precisely, the following result holds.

Proposition 1. *We have*

$$(d_z R_a) H(\xi)_z = H(a^{-1} \xi)_{za}$$

for any $a \in O(n)$, $z \in O(E)$

$$\xi \neq 0 \Rightarrow H(\xi)_z \neq 0$$

for any $z \in O(E)$.

In Proposition 1. $R_a\colon O(E) \to O(E)$ stands for *right translation* with $a \in O(n)$. Also $O(n)$ acts canonically on $\boldsymbol{R}^{2n} = \boldsymbol{R}^n \oplus \boldsymbol{R}^n$, i.e. $a\xi = a\xi_1 \oplus a\xi_2$ where $\xi = \xi_1 \oplus \xi_2$ and $\xi_i \in \boldsymbol{R}^n$, $i = 1, 2$. The proof is straightforward.

Let $\mathfrak{o}(n)$ be the Lie algebra of $O(n)$. Let $A \in \mathfrak{o}(n)$, $\xi \in \boldsymbol{R}^{2n}$. Then

$$[A^*, H(\xi)] = H(A\xi) \tag{2.2}$$

where $A\xi = A\xi_1 \oplus A\xi_2$. Here $A^* \in \Gamma^\infty(\operatorname{Ker}(\mathrm{d}\rho))$ denotes the fundamental vector field associated with the left invariant vector field A. The proof of (2.2) is similar to the proof of Prop. 2.3 in [8] (I, p. 120), and therefore is left as an exercise for the reader.

Let $\Theta = \mathrm{D}\theta$ be the *covariant derivative* of θ, i.e. $\Theta(X, Y) = (\mathrm{d}\theta)\,(hX, hY)$ for any $X, Y \in T(O(E))$. Here $h_z\colon T_z(O(E)) \to H_z$ denotes the natural projection associated with $T_z(O(E)) = H_z \oplus \operatorname{Ker}(d_z\rho)$ for $z \in O(E)$. We have

Theorem 1. *Let (M, F) be a Finsler space and $\theta \in \Gamma^\infty(T^*(O(E)) \otimes \boldsymbol{R}^{2n})$, its canonical 1-form. Let Θ be the covariant derivative (with respect to the connection H in $O(E) \to V(M)$ induced by the Cartan-Chern connection of (M, F)) of θ. Then*

$$(\mathrm{d}\theta)(X, Y) = -\frac{1}{2}(\omega(X)\,\theta(Y) - \omega(Y)\,\theta(X)) + \Theta(X, Y) \tag{2.3}$$

for any $X, Y \in T(O(E))$. Here $\omega \in \Gamma^\infty(T^*(O(E)) \otimes \mathfrak{o}(n))$ is the connection 1-form associated with H.

Proof. It is sufficient to check (2.3) for $X = A^*$ and $Y = H(\xi)$, where $A \in \mathfrak{o}(n)$, $\xi \in \boldsymbol{R}^{2n}$. Both sides in (2.3) may be shown to be equal to $-\frac{1}{2}A\xi$ (one should use (2.2) to evaluate the left hand member of (2.3)).

3 - Adapted Finslerian frames

Let (M, F), (N, F_0) be two Finslerian spaces, $\dim_{\boldsymbol{R}} M = n$, $\dim_{\boldsymbol{R}} N = n + p$ and $f\colon M \to N$ a C^∞-immersion which is *isometric*, i.e. $F(u) = F_0((d_x f)u)$ for any $u \in T_x(M)$, $x \in M$. Let $\pi_0\colon V(N) \to N$ be the natural projection and set $E_0 = \pi_0^{-1}TN$. In the sequel, an index 0 attached to a symbol indicates a geometric object (Lagrangian function, induced bundle, Cartan-Chern connection, etc.) associated with the ambient space N.

Corresponding to (N, F_0) one may consider a Riemannian vector bundle

(E_0, g_0) and the principal $O(n+p)$-bundle $\rho_0\colon O(E_0) \to V(N)$. Set

$$O(E_0)|_{V(M)} = \{z \in O(E_0) \mid \rho_0(z) \in V(M)\}.$$

As usual, since all our considerations are local, we do not distinguish notationally between x and $f(x)$, u and $(d_x f)u$, etc., for $x \in M$, $u \in T_x(M)$.

A Finslerian frame $z \in O(E_0)|_{V(M)}$ is called *adapted* if

$$z = (u\{Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+p}\})$$

with $\{Y_1, \ldots, Y_n\} \subseteq E_u$ and $\{Y_{n+1}, \ldots, Y_{n+p}\} \subseteq \nu(f)_u$, $u \in V(M)$. Here $\nu(f) \to V(M)$ denotes the normal bundle of the given immersion f (cf. also [5], [7]). That is, if z is adapted then $z(\boldsymbol{R}^n) = E_u$ and $z(\boldsymbol{R}^p) = \nu(f)_u$, where $\boldsymbol{R}^{n+p} = \boldsymbol{R}^n \otimes \boldsymbol{R}^p$ canonically.

Let $O(E_0, E)$ consist of all adapted Finslerian frames and $\rho\colon O(E_0, E) \to V(M)$ the natural projection. Then $O(E_0, E)$ is a principal $O(n) \times O(p)$ bundle over $V(M)$. Define the principal bundle morphism $h'\colon O(E_0, E) \to O(E)$ by $h'(z) = (u, \{Y_1, \ldots, Y_n\})$ for any adapted frame

$$z = (u, \{Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+p}\})$$

as above. That is $h'(z) = z|_{\boldsymbol{R}^n}$ where $z\colon \boldsymbol{R}^{n+p} \to E_{0,u}$. Thus $O(E) \cong O(E_0, E)/O(p)$ (a principal bundle isomorphism). Note also that $O(\nu(f)) \cong O(E_0, E)/O(n)$. Finally, there is a natural principal bundle morphism $h''\colon O(E_0, E) \to O(\nu(f))$ and $O(\nu(f)) \times_{O(p)} \boldsymbol{R}^p \cong \nu(f)$ (a vector bundle isomorphism).

We summarize our constructions so far in the following commutative diagram

$$\begin{array}{ccccc}
O(E) & \xleftarrow{\ h'\ } & O(E_0, E) & \xrightarrow{\ h''\ } & O(\nu(f)) \\
{\scriptstyle O(n)}\big\downarrow & & {\scriptstyle O(n)\times O(p)}\big\downarrow{\scriptstyle \rho} & & {\scriptstyle O(p)}\big\downarrow{\scriptstyle \rho''} \\
V(M) & \xleftrightarrow{\ 1\ } & V(M) & \xleftrightarrow{\ 1\ } & V(M).
\end{array}$$

4 - Immersions and canonical 1-forms

Let $\theta \in \Gamma^\infty(T^*(O(E)) \otimes \boldsymbol{R}^{2n})$ and $\theta_0 \in \Gamma^\infty(T^*(O(E)) \otimes \boldsymbol{R}^{2(n+p)})$ be the canonical 1-forms of (M, F) and (N, F_0), respectively. We wish to relate $(h')^*\theta$ and (the restriction to $O(E_0, E)$ of) θ_0.

To this end we shall need the *Gauss formula* (cf. e.g. [1], p. 276)

$\nabla^0_x Y = \nabla_x Y + \widehat{H}(X, Y)$ for any $X \in \Gamma^\infty(T(V(M)))$, $Y \in \Gamma^\infty(E)$. Here ∇^0, ∇ and $\widehat{H}$ are respectively the Cartan-Chern connection of (N, F_0), the induced connection, and the second fundamental form (of f). Let N be the nonlinear connection of ∇ and β: $E \to N$ the corresponding horizontal lift. The *horizontal second fundamental form* H is given by $H(X, Y) = \widehat{H}(\beta X, Y)$ for any $X, Y \in \Gamma^\infty(E)$.

To state our main result we shall need the 1-form $\varphi \in \Gamma^\infty(T^*(O(E_0, E)) \otimes \boldsymbol{R}^p)$ given by $\varphi_z X = z^{-1} H(L_u d_z(\rho' h')X, v)$ for any $X \in T_z(O(E_0, E))$, $z \in O(E_0, E)$, where $u = \rho(z)$. Here $\boldsymbol{R}^p \cong \{0\} \times \boldsymbol{R}^p \subset \boldsymbol{R}^{n+p}$.

Theorem 2. *The following identities hold*:

$$i^* j^* \theta_0^h = (h')^* \theta^h \oplus 0 \qquad i^* j^* \theta_0^v = (h')^* \theta^v \oplus \varphi \tag{4.1}$$

where i: $O(E_0, E) \to O(E_0)|_{V(M)}$ and j: $O(E_0)|_{V(M)} \to O(E_0)$ *are canonical inclusions. In particular, the restriction to* $O(E_0, E)$ *of the* $\boldsymbol{R}^{n+p}$*-valued* 1*-form* θ_0^h *is* $\boldsymbol{R}^n$*-valued.*

We summarize our constructions in the following commutative diagram

$$\begin{array}{ccccc}
O(E_0, E) & \xrightarrow{\ i\ } & O(E_0)|_{V(M)} & \xrightarrow{\ j\ } & O(E_0) \\
{\scriptstyle O(n)\times O(p)}\big\downarrow{\scriptstyle \rho} & & {\scriptstyle O(n+p)}\big\downarrow & & {\scriptstyle O(n+p)}\big\downarrow{\scriptstyle \rho_0} \\
V(M) & \xleftrightarrow[\ 1\ ]{} & V(M) & \xrightarrow[\ f_*\ ]{} & V(N).
\end{array}$$

Let $z \in O(E_0, E)$, $u = \rho(z)$, and set $z' = h'(z) \in O(E)$. We wish to compute $(h')_z^* \theta_{z'}^h$, where $(h')_z^*$: $T_{z'}^*(O(E)) \otimes \boldsymbol{R}^n \to T_z^*(O(E_0, E)) \otimes \boldsymbol{R}^n$. By the chain rule

$$(h')_z^* \theta_{z'}^h = (z')^{-1} L_u d_z(\rho' h'). \tag{4.2}$$

We shall need the following commutative diagram

$$\begin{array}{ccccc}
 & \overset{h'}{\nearrow} & O(E) & \xrightarrow{\ \rho'\ } & V(M) \\
O(E_0, E) & & & & \big\downarrow{\scriptstyle f_*} \\
 & \underset{j\circ i}{\searrow} & O(E_0) & \xrightarrow[\ \rho_0\ ]{} & V(N)
\end{array}$$

that is, the identity

$$\rho_0 \circ j \circ i = f_* \circ \rho' \circ h' . \tag{4.3}$$

Let $(\mathrm{D}f)_u \colon E_u \to (E_0)_u$ be the restriction of $d_x f \times d_x f$ to E_u, where $x = \pi(u)$. The following diagram is commutative

$$\begin{array}{ccc}
\boldsymbol{R}^n & \xrightarrow{\ \alpha\ } & \boldsymbol{R}^{n+p} \\
{\scriptstyle (z')^{-1}}\uparrow & & \uparrow{\scriptstyle z^{-1}} \\
E_u & \xrightarrow{\ (Df)_u\ } & (E_0)_{\overline{u}} \\
{\scriptstyle L_u}\uparrow & & \uparrow{\scriptstyle (L_0)_{\overline{u}}} \\
T_u(V(M)) & \xrightarrow[\ d_u f_*\ ]{} & T_{\overline{u}}(V(N))
\end{array}$$

where $\alpha(\xi) = (\xi, 0)$ and $\overline{u} = f_*(u)$, $\xi \in \boldsymbol{R}^n$, $u \in V(M)$. Let us check the commutativity of the lower square. Let $Z \in T_u(V(M))$. Since $\pi_0 \circ f_* = f \circ \pi$ we may perform the following calculation

$$(L_0)_{\overline{u}}(d_u f_*) Z = (\overline{u}, (d_{\overline{u}} \pi_0)(d_u f_*) Z) = (\overline{u}, d_u(\pi_0 f_*) Z)$$

$$= ((d_{\pi(u)} f) u, (d_{\pi(u)} f)(d_u \pi) Z) = (\mathrm{D}f)_u (u, (d_u \pi) Z) = (\mathrm{D}f)_u L_u Z.$$

At this point we may prove $(4.1)_1$. To this end we use (4.2), (4.3) and the identity

$$(L_0)_u (d_u f_*) = (\mathrm{D}f)_u L_u \tag{4.4}$$

where $f_* u$ is identified with u. We may perform the following calculation

$$\alpha(h')^*_z \theta^h_{z'} = \alpha(z')^{-1} L_u d_z(\rho' h') = z^{-1} (\mathrm{D}f)_u L_u d_z(\rho' h')$$

$$= z^{-1} (L_0)_u (d_u f_*) d_z(\rho' h') = z^{-1} (L_0)_u d_z(f_* \rho' h')$$

$$= z^{-1} (L_0)_u\, d_z(\rho_0 ji) = (ji)^*_z z^{-1} (L_0)_u d_{ji(z)} \rho_0 = (ji)^*_z\, \theta^h_{0,z} .$$

The proof of $(4.1)_2$ is somewhat trickier. Firstly, note that

$$f_{**}(\mathrm{Ker}\,(\mathrm{d}\pi)) \subset \mathrm{Ker}\,(\mathrm{d}\pi_0) .$$

Moreover, the following diagram is commutative

$$\begin{array}{ccc} E_u & \xrightarrow{\ \gamma_u\ } & \mathrm{Ker}\,(d_u \pi) \\ {\scriptstyle (Df)_u}\Big\downarrow & & \Big\downarrow{\scriptstyle d_u f_*} \\ (E_0)_{f_* u} & \xrightarrow[\ \gamma_{0,\, f_* u}\ ]{} & \mathrm{Ker}\,(d_{f_* u} \pi_0) \end{array}$$

for any $u \in V(M)$.

The proof is in local coordinates. Let (U, u^α), (V, x^i) be local coordinate neighborhoods on M, N respectively (with $f(U) \subset V$). Let $(\pi^{-1}(U), u^\alpha, v^\alpha)$, $(\pi_0^{-1}(V), x^i, y^j)$ be the naturally induced local coordinates on $V(M)$, $V(N)$ respectively. We adopt the following convention for the indices: $1 \leqslant \alpha, \beta, \ldots \leqslant n$ and $1 \leqslant i, j, \ldots \leqslant n + p$. Set $X_\alpha(u) = (u, \frac{\partial}{\partial u^\alpha}|_x)$ for any $u \in \pi^{-1}(u)$, $x = \pi(u)$. Then $\{X_1, \ldots, X_n\}$ is a (local) frame field in E over $\pi^{-1}(U)$. Finally set $\partial_\alpha = \frac{\partial}{\partial u^\alpha}$, $\dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha}$. We shall need the following

Lemma 1. *We have*

$$(4.5) \qquad (\mathrm{D}f)_u X_\alpha(u) = B^i_\alpha(\pi(u)) X_i(f_* u) \qquad f_{**} \dot{\partial}_\alpha = B^i_\alpha \dot{\partial}_i$$

for any $u \in \pi^{-1}(U)$, *where* $B^i_\alpha = \dfrac{\partial f^i}{\partial u^\alpha}$ *and* $f^i = x \circ f$.

Let us firstly show how the commutativity of the diagram above follows from Lemma 1. Indeed

$$(d_u f_*) \gamma_u X_\alpha(u) = (d_u f_*) \dot{\partial}_\alpha(u) = B^i_\alpha \dot{\partial}_i (f_* u)$$

$$= B^i_\alpha \gamma_{0,\, f_* u} X_i(f_* u) = \gamma_{0,\, f_* u} (\mathrm{D}f)_u X_\alpha(u) .$$

It remains to establish Lemma 1. The proof of $(4.5)_1$ is a straighforward consequence of definitions. To check $(4.5)_2$ one may write $f_{**} \dot{\partial}_\alpha = A^i_\alpha \dot{\partial}_i + C^i_\alpha \partial_i$ for some $A^i_\alpha, C^i_\alpha \in C^\infty(\pi^{-1}(U))$. Applying $(\pi_0)_*$ furnishes $C^i_\alpha = 0$.

To compute the remaining functions A^i_α we need

Lemma 2. *We have*

$$(4.6) \qquad y^i(f_* u) = B^i_\alpha(\pi(u))\, v^\alpha(u)$$

for any $u \in \pi^{-1}(U)$.

The identity (4.6) may be written succintly $y^i = B^i_\alpha v^\alpha$ and is of interest in itself. In classical language, the submanifold M is tangent to the *supporting element* of the ambient space. Let us apply $f_{**}\dot\partial_\alpha = A^i_\alpha \dot\partial_i$ to y^i (thought of as a function $y^i: \pi_0^{-1}(V) \to \boldsymbol{R}$). We have

$$A^i_\alpha = (f_{**}\dot\partial_\alpha)\, y^i = \dot\partial_\alpha (y^i \circ f_*) = \dot\partial_\alpha (B^i_\alpha v^\beta) = B^i_\alpha \circ \pi .$$

Finally, it remains to check $(4.5)_2$. To this end, let $u \in \pi^{-1}(U)$ be written as $u = u^\alpha \dfrac{\partial}{\partial u^\alpha}\big|_x$, where $x = \pi(u)$, $u^\alpha = v^\alpha(u)$. Then $f(x) = f(\pi(u)) = \pi_0\, f_* u$ so that we may conduct the following calculation

$$f_* u = u^\alpha f_* \frac{\partial}{\partial u^\alpha}\Big|_x = u^\alpha B^i_\alpha(x) \frac{\partial}{\partial x^i}\Big|_{f(x)} = v^\alpha(u) B^i_\alpha(\pi(u)) \frac{\partial}{\partial x^i}\Big|_{\pi_0(f_* u)} .$$

So far we have obtained the identity

$$(\gamma_{0,\, f_* u})^{-1} \circ (d_u f_*) = (\mathrm{D}f)_u \circ \gamma_u^{-1} \tag{4.7}$$

for any $u \in V(M)$. Unlike the case of (4.4), the following diagram is *not commutative* in general (it only collects the arrows we need)

$$\begin{array}{ccc} E_u & \xrightarrow{(Df)_u} & (E_0)_{f_* u} \\ {\scriptstyle K_u}\uparrow & & \uparrow{\scriptstyle K_{0, f_* u}} \\ T_u(V(M)) & \xrightarrow[d_u f_*]{} & T_{f_* u}(V(N)) . \end{array}$$

Nevertheless, we may show that the Dombrowski maps K, K_0 of (M, F), (N, F_0) are related. More explicitly

Lemma 3. *We have*

$$(\mathrm{D}f)\, KZ = K_0\, f_{**} Z - H(LZ, v) \tag{4.8}$$

for any $Z \in T(V(M))$.

As a consequence of (4.8) the diagram above is commutative if and only if f is totally-geodesic. Our Lemma 3 may be used to end the proof of $(4.1)_2$. Indeed, we may conduct the following calculation

$$\alpha(h')^*_z\,\theta^v_{z'}Z = \alpha\theta^v_{z'}(d_z h')Z = \alpha(z')^{-1}K_u(d_{z'}\rho')(d_z h')Z = z^{-1}(\mathrm{D}f)_u K_u d_z(\rho' h')Z$$

$$= z^{-1}K_{0,\,f_*u}(d_u f_*)\,d_z(\rho' h')Z - z^{-1}H(L_u d_z(\rho' h')Z,\, v) = (ji)^*_z\,\theta^v_{0,\,z}Z - \varphi_z Z\,.$$

It remains to prove (4.8). The proof is in local coordinates. Let β be the horizontal lift associated with the induced connection ∇ and set $\delta_\alpha = \beta X_\alpha$, $1 \leq \alpha \leq n$. Then $\delta_\alpha = \partial_\alpha - N^\beta_\alpha \dot\partial_\beta$ where N^β_α are the coefficients of the nonlinear connection of ∇. We may write $f_{**}\,\partial_\alpha = A^i_\alpha \partial_i + C^i_\alpha \dot\partial_i$ for some A^i_α, $C^i_\alpha \in C^\infty(\pi^{-1}(U))$. Applying $(\pi_0)_*$ yields at once $A^i_\alpha = B^i_\alpha \circ \pi$. Next, we may use (4.6) to compute C^i_α, that is

$$C^i_\alpha = \partial_\alpha(y^i \circ f_*) = \partial_\alpha(B^i_\beta v^\beta) = B^i_{\alpha\beta}v^\beta \qquad \text{with} \qquad B^i_{\alpha\beta} = \frac{\partial^2 f^i}{\partial u^\alpha \partial u^\beta}\,.$$

We obtain

$$f_{**}\,\partial_\alpha = B^i_\alpha \partial_i + B^i_{\alpha\beta}v^\beta \dot\partial_i\,. \tag{4.9}$$

Using (4.9) and $(4.5)_2$ of Lemma 1 we may derive

$$f_{**}\,\delta_\alpha = B^i_\alpha \delta_i + (B^j_{\alpha\beta}v^\beta + N^j_i B^i_\alpha - N^\mu_\alpha B^j_\mu)\,\dot\partial_j \tag{4.10}$$

where $\delta_i = \partial_i - N^j_i \dot\partial_j$ and N^j_i are the coefficients of the non linear connection of the Cartan-Chern connection ∇^0 of (N, F_0). The Gauss formula

$$\nabla^0_{f^{**}\delta_\alpha}(\mathrm{D}f)X_\beta = (\mathrm{D}f)\,\nabla_{\delta_\alpha}X_\beta + H(X_\alpha, X_\beta)$$

may be written

$$F^\lambda_{\alpha\beta}B^k_\lambda + H^k_{\alpha\beta} = B^k_{\alpha\beta} + B^i_\beta\{B^i_\alpha F^k_{ji} + (B^j_{\alpha\lambda}v^\lambda + N^j_m B^m_\alpha - N^\lambda_\alpha B^j_\lambda)\,C^k_{ji}\}\,. \tag{4.11}$$

Contraction with v^β in (4.11) leads (as $F^\mu_{\alpha\beta}v^\beta = N^\mu_\alpha$) to

$$H^k_{\alpha 0} = B^k_{\alpha\beta}v^\beta + B^j_\alpha N^k_j - N^\lambda_\alpha B^k_\lambda \tag{4.12}$$

where $H^i_{\alpha 0} = H^i_{\alpha\beta}v^\beta$. Finally (4.10) may be written (by (4.12)) as

$$f_{**}\,\delta_\alpha = B^i_\alpha \delta_i + H^i_{\alpha 0}\,\dot\partial_i\,. \tag{4.13}$$

Consequently one has the identities

$$(\mathrm{D}f)K\dot{\partial}_a = B^i_\alpha X_i \qquad (\mathrm{D}f)K\partial_\alpha = 0$$

$$K_0 f_{**}\dot{\partial}_\alpha = B^i_\alpha X_i \qquad K_0 f_{**}\partial_\alpha = H^i_{\alpha 0} X_i$$

which yield (4.8).

5 - Immersions and connection 1-forms

Let $\omega_0 \in \Gamma^\infty(T^*(O(E_0)) \otimes \mathfrak{o}(n+p))$ be the connection 1-form on $O(E_0)$ corresponding to the Cartan-Chern connection ∇^0 in (E_0, g_0). Then $j^*\omega_0$ is a connection 1-form on $O(E_0)|_{V(M)}$. Next, let $\mathfrak{g}(n, p)$ be the ortogonal complement (with respect to the Killing-Cartan form of $\mathfrak{o}(n+p)$) of $\mathfrak{o}(n) + \mathfrak{o}(p)$ in $\mathfrak{o}(n+p)$. Let ω be the $(\mathfrak{o}(n) + \mathfrak{o}(p))$-component of $i^*j^*\omega_0$ (with respect to the decomposition $\mathfrak{o}(n+p) = (\mathfrak{o}(n) + \mathfrak{o}(p)) \otimes \mathfrak{g}(n, p)$). Then ω is a connection 1-form for $O(E_0, E) \to V(M)$. Let $\omega_{\mathfrak{o}(n)}$ be the $\mathfrak{o}(n)$-component of ω. The following diagram describes our construction

$$\begin{array}{ccc} T_z(O(E_0, E)) & \xrightarrow{\ \omega_z\ } & \mathfrak{o}(n) + \mathfrak{o}(p) \\ & \searrow^{(\omega_{\mathfrak{o}(n)})_z} & \Big\downarrow \text{proj.} \\ & & \mathfrak{o}(n) \end{array}$$

for any $z \in O(E_0, E)$. By Prop. 6.1 in [8], vol. I, ch. II, there is a unique connection 1-form $\omega' \in \Gamma^\infty((T^*(O(E)) \otimes \mathfrak{o}(n))$ such that $(h')^*\omega' = \omega_{\mathfrak{o}(n)}$.

We may formulate the following

Problem. *Show that ω' is the connection 1-form in $O(E) \to V(M)$ corresponding to the induced connection ∇ in (E, g).*

If (M, F), (N, F_0) are Riemannian manifolds the problem above may be solved by showing that θ has zero torsion. This in turn follows by restriction of the first structure equation satisfied by θ_0

$$\mathrm{d}\theta_0 = -\omega_0 \wedge \theta_0 + \Theta_0 \tag{5.1}$$

to the bundle of adapted frames and making use of Prop. 1.1 of [8], vol. II, p. 3. As to our case, the Finslerian analogue of Prop. 1.1 in [8], vol. II, is Theorem 2.

One may apply i^*j^* to (5.1) and use (4.1) to derive

$$\begin{aligned}&[(h')^* \mathrm{d}\theta^h \oplus 0] \oplus [(h')^* \mathrm{d}\theta^v \oplus \varphi] \\ &= \{-i^*j^*\omega_0 \wedge [(h')^*\theta^h \oplus 0]\} \oplus \{-i^*j^*\omega_0 \wedge [(h')^*\theta^v \oplus \varphi]\} + i^*j^*\Theta_0 .\end{aligned} \tag{5.2}$$

While the structure equation (5.2) possesses a highly complicated character (in comparison with its Riemannian counterpart, where $\Theta_0 = 0$), it is reasonable to expect that it may yield (via Theor. 4.4 in [4], p. 82) the torsions T and S^1 of ω'. It is known (cf. e.g. [**1**], p. 277) that the induced connection ∇ is characterized by $\nabla g = 0$, $S^1 = 0$ and

$$T(X, Y) = \tan\{C^*(N(X), Y) - C^*(N(Y), X)\} \qquad \text{for any } X, Y \in \Gamma^\infty(E).$$

The author hopes to address these questions in a further paper.

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Summary

In this work the submanifolds geometry in Finslerian manifolds is studied by using principal bundles techniques.

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