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On the invariance of conjugation in cyclic homology (**)

Introduction

Let A be an *algebra* over a field of characteristic zero. We consider $A^{\otimes n}$, the n-th fold tensor product of A with itself, and the map b

$$b\colon A^{\otimes(n+1)} \to A^{\otimes n}$$

defined by

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1} .$$

Then, the composition

$$b^2 : A^{\otimes(n+1)} \to A^{\otimes(n-1)}$$

is zero, and hence the pair $(A^{\otimes n}; b)$ give rise to a *chain complex*

$$\ldots \xrightarrow{b} A^{\otimes(n+1)} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} A^{\otimes(n-1)} \xrightarrow{b} \ldots .$$

For $n \geq 1$ we define the n-th *Hochschild Homology group* of A by

$$H_n(A; A) = \frac{\mathrm{Ker}[A^{\otimes(n+1)} \xrightarrow{b} A^{\otimes n}]}{\mathrm{Im}[A^{\otimes(n+2)} \xrightarrow{b} A^{\otimes(n+1)}]}$$

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and for $n = 0$

$$H_0(A; A) = \frac{A}{[A, A]}.$$

If we delete the last term of the sum in the definition of the map b, then the resulting map, commonly denoted b', would also give rise to a *chain complex*. However, in most cases this latter complex turns not to be very interesting. For example, if the algebra A has a unit 1, then the $(A^{\otimes n}; b')$ complex in *acyclic*. Indeed if $b'(a_0 \otimes \ldots \otimes a_n) = 0$, then we can write $a_0 \otimes \ldots \otimes a_n = b'(1 \otimes a_0 \otimes \ldots \otimes a_n)$.

Let Λ denote the *cyclic group* of order $n + 1$ generated by t. Then Λ acts on $A^{\otimes(n+1)}$ by cyclicly permuting the entries, i.e.

$$t(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}.$$

For $n = 0$ we simply take $t =$ identity. Then, it is readily verified that

$$b \circ (1 - t) = (1 - t) \circ b'$$

and hence b factors to a map

$$b\colon A^{\otimes(n+1)}/(1 - t) \to A^{\otimes n}/(1 - t).$$

The homology groups of the corresponding chain complex are called the *cyclic homology groups* of A and will be denoted $H_n^\lambda(A)$.

Let A be endowed with a unit 1 and let A^* denote the set of all invertible elements of A. Then each f in A^*, defines an *action* on $A^{\otimes(n+1)}$ or on $A^{\otimes(n+1)}/(1 - t)$ by *conjugation by f*

$$a_0 \otimes \ldots \otimes a_n \mapsto fa_0 f^{-1} \otimes \ldots \otimes fa_n f^{-1}.$$

We shall denote both actions by $Ad(f)$.

Since $Ad(f)$ commutes with b, it induces an action on both the Hochschild and cyclic homology of A. In the next two sections, we shall derive some formulae concerning this action and establish that *the induced action on the homology is trivial.*

Although the triviality of the action in cyclic homology has already been established for differential graded algebras, the proof is based on the fact that the *infinitesimal counterpart* of Ad, namely ad defined by

$$a_0 \otimes \ldots \otimes a_n \mapsto \sum_{i=0}^{n} a_0 \otimes \ldots \otimes [X, a_i] \otimes \ldots \otimes a_n$$

for $X \in A$, is zero on $H_n^\lambda(A)$ and so, in the spirit of Newton, this infinitesimal picture can be integrated up to show that *Ad is constant on* $H_n^\lambda(A)$.

However, the proof fails to provide formulae for the chain homotopies which can be useful in the theory of Chern classes in algebraic K-theory.

In the final section, we use the triviality of the action to construct maps from K_0 of a ring into its cyclic homology. In dualized form, this map is due to A. Connes.

1 - A formula in Hochschild homology

Theorem 1. *For each* $f \in A^*$, *define the mapping*

$$h_f\colon A^{\otimes(n+1)} \to A^{\otimes(n+2)}$$

by the formula

$$h_f(a_0 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n+1} (-1)^{i+1} fa_0 \otimes a_1 \otimes \ldots \otimes a_{i-1} \otimes f^{-1} \otimes fa_i f^{-1} \otimes \ldots \otimes fa_n f^{-1}$$

then

$$bh_f + h_f b = Ad(f) - 1.$$

In particular, $Ad(f)$ *acts trivially on* $H(A, A)$.

Proof. We shall first compute $bh_f(a_0 \otimes \ldots \otimes a_n)$. This will yield $(n+2)(n+1)$ homogeneous terms, which we group in $(n+1)$ groups each having $(n+2)$ terms. We use the notation (i, j) to designate the i-th term in the j-th group, that is the i-th term of b applied to the j-th term of $h_f(a_0 \otimes \ldots \otimes a_n)$. For example

$$(1, 1) = Ad(f)(a_0 \otimes \ldots \otimes a_n) = fa_0 f^{-1} \otimes \ldots \otimes fa_n f^{-1}$$

$$(n+2, n+1) = -a_0 \otimes \ldots \otimes a_n.$$

Subclaim. *For each* $k = 2, 3, \ldots, n+1$, *we have that* $(k, k-1) = -(k, k)$, *giving rise to* n *cancellations each involving two terms.*

Proof. $(k, k-1)$ equals the k-th term of

$$b((-1)^k fa_0 \otimes \ldots \otimes a_{k-2} \otimes f^{-1} \otimes fa_{k-1} f^{-1} \otimes \ldots \otimes fa_n f^{-1})$$

which equals

$$(-1)^k(-1)^{k+1}fa_0 \otimes a_1 \otimes \ldots \otimes a_{k-2} \otimes a_{k-1} f^{-1} \otimes fa_k f^{-1} \otimes \ldots \otimes fa_n f^{-1}.$$

While (k, k) equals the k-th term of

$$b((-1)^{k+1}fa_0 \otimes a_1 \otimes \ldots \otimes a_{k-1} \otimes f^{-1} \otimes fa_k f^{-1} \otimes \ldots \otimes fa_n f^{-1})$$

which equals

$$(-1)^{k+1}(-1)^{k+1}fa_0 \otimes a_1 \otimes \ldots \otimes a_{k-2} \otimes a_{k-1} f^{-1} \otimes fa_k f^{-1} \otimes \ldots \otimes fa_n f^{-1}.$$

Hence, we obtain that

$$bh_f(a_0 \otimes \ldots \otimes a_n) = (Ad(f) - 1)(a_0 \otimes \ldots \otimes a_n) + I$$

where
$$I = \sum_{k=3}^{n+2} [\sum_{j=1}^{k-2} (j, k-1) + \sum_{i=k}^{n+2} (i, k-2)].$$

But for $j = 1, 2, \ldots, k-2$, $(j, k-1)$ equals the j-th term of

$$b((-1)^k fa_0 \otimes a_1 \otimes \ldots \otimes a_{k-2} \otimes f^{-1} \otimes fa_{k-1} f^{-1} \otimes \ldots \otimes fa_n f^{-1})$$

which equals

$$(-1)^k(-1)^{j+1}fa_0 \otimes \ldots \otimes a_{j-1} \otimes a_j \otimes \ldots \otimes a_{k-2} \otimes f^{-1} \otimes fa_{k-1} f^{-1} \otimes \ldots \otimes fa_n f^{-1}$$

which equals (-1) times the $(k-2)$-nd term of

$$h_f((-1)^{j+1} a_0 \otimes \ldots \otimes a_{j-1} a_j \otimes \ldots \otimes a_n)).$$

So $\sum_{j=1}^{k-2} (j, k-1)$ is equal to minus the $\sum_{j=1}^{k-2}$ of the $(k-2)$-nd terms of the last expression.

Similarly, $\sum_{i=k}^{n+2} (r, k-2)$ is equal to minus the $\sum_{j=k-1}^{n}$ of the $(k-2)$-nd terms of

$$h_f((-1)^{j+1} a_0 \otimes \ldots \otimes a_{j-1} a_j \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1})).$$

So finally,

$$I = \sum_{k=3}^{m+2} (-1)((k-2)\text{-nd–term of } h_f(b(a_0 \otimes \ldots \otimes a_n))) = (-1)\, h_f b(a_0 \otimes \ldots \otimes a_n).$$

Hence, $bh_f(a_0 \otimes \ldots \otimes a_n) = (Ad(f) - 1)(a_0 \otimes \ldots \otimes a_n) - h_f b(a_0 \otimes \ldots \otimes a_n)$.

2 - A formula in cyclic homology

Given an invertible element $f \in A^*$ and $a_0 \otimes \ldots \otimes a_n \in A^{\otimes(n+1)}$, we consider the set $H_f(a_0 \otimes \ldots \otimes a_n)$ consisting of all terms in $A^{\otimes(n+2)}$ satisfying the following *conditions*:

1) *The* 0*-th entry is* f.

2) *For* $1 \leq i \leq n$, *the i-th entry is either* a_{i-1} *or* fa_{i-1} *or* $fa_{i-1}f^{-1}$ *with the convention that the i-th entry begins with an* f *iff the* $(i-1)$*-st entry ends with an* f^{-1}. *In particular the* 1*-st entry must either be* a_0 *or* $a_0 f^{-1}$.

3) *The* $(n+1)$*st entry is either* $a_n f^{-1}$ *or* $fa_n f^{-1}$ *where the latter occurs iff the n-th entry ends with an* $f-1$.

These conditions ensure that in b of such a term, there are no build ups of either f of f^{-1} to any power greater than 1. For example

$$f \otimes a_0 \otimes a_1 f^{-1} \otimes fa_2 \otimes a_3 f^{-1} \otimes fa_4 f^{-1}.$$

Next, for $0 \leq j \leq n$, let $T_j(a_0 \otimes \ldots \otimes a_n)$ be the subset of $H_f(a_0 \otimes \ldots \otimes a_n)$ consisting of those terms having j of the n-middle entries ending with f^{-1}. For example

$$f \otimes a_0 \otimes a_1 f^{-1} \otimes fa_2 \otimes a_3 f^{-1} \otimes fa_4 f^{-1} \in T_2(a_0 \otimes \ldots \otimes a_4).$$

Also
$$T_0(a_0 \otimes \ldots \otimes a_n) = \{f \otimes a_0 \otimes \ldots \otimes a_n f^{-1}\}$$

while
$$T_n(a_0 \otimes \ldots \otimes a_n) = \{f \otimes a_0 f^{-1} \otimes fa_1 f^{-1} \otimes \ldots \otimes fa_n f^{-1}\}.$$

In general, $T_j(a_0 \otimes \ldots \otimes a_n)$ has $\binom{n}{j}$ elements. Moreover, the sets $T_j(a_0 \otimes \ldots \otimes a_n)$ $j = 0, \ldots, n$ determine a *partition* of the set $H_f(a_0 \otimes \ldots \otimes a_n)$. If σ is in $T_j(a_0 \otimes \ldots \otimes a_n)$, then we will call σ a *term of type j*.

Theorem 2. *For each* $f \in A^*$, *let*

$$h_f^n : A^{\otimes(n+1)}/(1-t) \to A^{\otimes(n+2)}/(1-t)$$

be defined by
$$h_f^n(a_0 \otimes \ldots \otimes a_n) = \frac{1}{n+1} \sum_{\lambda \in \Lambda} \sum_{j=0}^{n} \binom{n}{j}^{-1} \sum_{\sigma \in T_j(\lambda(a_0 \otimes \ldots \otimes a_n))} \sigma.$$

Then for $n \geq 1$
$$b h_f^h + h_f^{n-1} b = Ad(f) - 1 \mod (1-t).$$

In particular $Ad(f)$ acts trivially on $H^{\lambda}(A)$.

Proof. First of all, since we are summing over all cyclic permutations λ of $a_0 \otimes \ldots \otimes a_n$, it follows that h_f^n is well defined modulo $1-t$.

For $a_0 \otimes \ldots \otimes a_n$ and $1 \leqslant i \leqslant n+1$, we write $b_i(a_0 \otimes \ldots \otimes a_n)$ to denote the i-th term of $b(a_0 \otimes \ldots \otimes a_n)$. For example $b_4(a_0 \otimes \ldots \otimes a_3) = -a_3 a_0 \otimes a_1 \otimes a_2$. In other words

$$b = b_1 + \ldots + b_{n+1}.$$

Lemma 1. *We have*

$$(b_1 + b_n) \circ h_f^n (a_0 \otimes \ldots \otimes a_n) = (Ad(f) - 1)(a_0 \otimes \ldots \otimes a_n).$$

Proof. We first observe that for $\lambda \in \Lambda$ and $\sigma \in T_n(\lambda(a_0 \otimes \ldots \otimes a_n))$

$$b_1(\sigma) = Ad(f)(\lambda(a_0 \otimes \ldots \otimes a_n)) = Ad(f)(a_0 \otimes \ldots \otimes a_n)$$

modulo $1-t$.

Similarly, for $\sigma \in T_0(\lambda(a_0 \otimes \ldots \otimes a_n))$ we have

$$\sigma = \operatorname{sgn} \lambda f \otimes a_{\lambda(0)} \otimes \ldots \otimes a_{\lambda(n)} f^{-1}.$$

So

$$\begin{aligned} b_{n+2}(\sigma) &= (-1)^{n+1} \operatorname{sgn} \lambda a_{\lambda(n)} \otimes a_{\lambda(0)} \otimes \ldots \otimes a_{\lambda(n-1)} \\ &= (-1)^{n+1}(-1)^n \operatorname{sgn} \lambda t(a_{\lambda(0)} \otimes \ldots \otimes a_{\lambda(n)}) \\ &= -t \circ \lambda(a_0 \otimes \ldots \otimes a_n) = -a_0 \otimes \ldots \otimes a_n. \end{aligned}$$

To finish proving the lemma, we show that b_1 of terms of type j will cancel with b_{n+2} of terms of type $j+1$. Recall that a term σ of type j has j of the n middle entries ending with f^{-1}. Therefore $b_1(\sigma)$ will have $(j+1)$ entries which start with f. By applying cyclic permutations to $b_1(\sigma)$ so that each of the $(j+1)$ entries which start with f appear in the 0-th entry, we can express $b_1(\sigma)$ as b_1 of $(j+1)$ distinct terms of type j.

For example if

$$\sigma = f \otimes a_0 f^{-1} \otimes fa_1 f^{-1} \otimes fa_2 \otimes a_3 f^{-1} \in T_2(a_0 \otimes \ldots \otimes a_3)$$

then
$$b_1(\sigma) = fa_0 f^{-1} \otimes fa_1 f^{-1} \otimes fa_2 \otimes a_3 f^{-1}$$
$$= b_1(-f \otimes a_1 f^{-1} \otimes fa_2 \otimes a_3 f^{-1} \otimes fa_0 f^{-1})$$
$$= b_1(f \otimes a_2 \otimes a_3 f^{-1} \otimes fa_0 f^{-1} \otimes fa_1 f^{-1}).$$

In other words, if we apply b_1 to all terms of type j, each of the resulting terms will have coefficient $\binom{n}{j}^{-1}(j+1)$.

On the other hand, each of these terms arises in opposite sign and with coefficient $\binom{n}{j+1}^{-1}(n-j)$ from b_{n+2} of therms of type $j+1$. In fact, since $b_1(\sigma)$ has $(n-j)$ entries not ending with f^{-1}, it can be expressed as b_{n+2} of $(n-j)$ terms of type $j+1$. Again this is done by applying cyclic permutations until the entries not ending in f^{-1} appear in the 0-th entry, and then use the fact that $a_i = a_i f^{-1}(f)$ and $fa_i = fa_i f^{-1}(f)$ to espress the resulting term as b_{n+2} of a term of type $j+1$.

For instance, in the previous example, $b_1(\sigma)$ has $3-2=1$ entry not ending in f^{-1} and we can write

$$b_1(\sigma) = -b_5(-f \otimes a_3 f^{-1} \otimes fa_0 f^{-1} \otimes fa_1 f^{-1} \otimes fa_2 f^{-1}).$$

Finally, since $\binom{n}{j}^{-1}(j+1) = \binom{n}{j+1}^{-1}(n-j)$ we see that b_1 of terms of type j cancel with b_{n+2} of terms of type $j+1$.

Lemma 2. *We have*

$$(b_2 + \ldots + b_{n+1}) \circ h_f^n(a_0 \otimes \ldots \otimes a_n) + h_f^{n-1} \circ b(a_0 \otimes \ldots \otimes a_n) = 0.$$

Proof. If σ is a term of type j, then for $2 \leqslant i \leqslant n+1$, $b_i(\sigma)$ is either of type j or of type $j-1$. The latter occurs whenever the $(i-1)$-st entry ends with f^{-1}.

For example, if $\sigma = f \otimes a_0 f^{-1} \otimes fa_1 \otimes a_2 f^{-1} \otimes fa_3 \otimes a_4 f^{-1}$ is of type 2, then $b_3(\sigma)$ and $b_5(\sigma)$ are of type 2 while $b_2(\sigma)$ and $b_4(\sigma)$ are of type 1.

Assuming that $b_i(\sigma)$ is of type j, then $b_i(\sigma)$ also arises as $b_i(\sigma')$ for some σ' of type $j+1$. In fact, σ' is obtained from σ by inserting a f^{-1} at the end of the $(i-1)$-st entry and an f at the start of the i-th entry.

For example, in the previous example $b_3(\sigma) = b_3(\sigma')$, where

$$\sigma' = f \otimes a_0 f^{-1} \otimes f a_1 f^{-1} \otimes f a_2 f^{-1} \otimes f a_3 \otimes a_4 f^{-1}.$$

Therefore $b_i(\sigma)$ occurs with a coefficient $\frac{1}{n+1}\left(\binom{n}{j}^{-1} + \binom{n}{j+1}^{-1}\right)$.

On the other hand, it also occurs in opposite sign with a coefficient $\frac{1}{n}\binom{n-1}{j}^{-1}$ from $h_f^{n-1} \circ b(a_0 \otimes \ldots \otimes a_n)$. But

$$\frac{1}{n+1}\left(\binom{n}{j}^{-1} + \binom{n}{j+1}^{-1}\right) = \frac{1}{n+1}\left(\frac{j!(n-j)!}{n!} + \frac{(j+1)!(n-(j+1))!}{n!}\right.$$

$$= \frac{1}{n+1}\left(\frac{j!(n-j)!}{n!}\right)\left(1 + \frac{j+1}{n-j}\right) = \frac{j!(n-j-1)!}{n!(n-j)!}$$

$$= \frac{1}{n}\left(\frac{j!(n-j-1)!}{(n-1)!}\right) = \frac{1}{n}\binom{n-1}{j}^{-1}.$$

Similarly, in the event that $b_i(\sigma)$ is of type $j-1$, then it also arises as $b_i(\sigma')$ for some σ' of type $j-1$. In fact σ' is obtained from σ by removing the f^{-1} in the $(i-1)$-st entry and the f in the i-th entry. So, $b_i(\sigma)$ occurs from $b_i \circ h_f^n$ with coefficient

$$\frac{1}{n+1}\left(\binom{n}{j}^{-1} + \binom{n}{j-1}^{-1}\right).$$

But it also occurs in opposite sign from $h_f^{n-1} \circ b$ with coefficient $\frac{1}{n}\binom{n-1}{j-1}^{-1}$. Once again be two coefficients are equal.

3 - Application to $K_0(A)$

There is a *natural map* S, that occurs in cyclic homology which decreases the degree by two

$$S\colon H_n^\lambda(A) \to H_{n-2}^\lambda(A).$$

The definition of S is as follows: If $\sigma \in H_n^\lambda(A)$, then $b\sigma \in \operatorname{Im}(1-t)$. Say $b\sigma = (1-t)\tau$ for some $\tau \in A^{\otimes n}$. But then $-b'\tau = N\alpha$ for some $\alpha \in A^{\otimes n-1}$, where $N = 1 + t + t^2 + \ldots + t^{n-2}$. Moreover as $b\alpha \in \operatorname{Im}(1-t)$, we define $S\sigma$ to be the class of α in H_{n-2}^λ. It is readily verified that S is well defined.

We now apply the previous theorem to obtain *homomorphisms*

$$Ch_n\colon K_0(A) \to H^{\lambda}_{2n}(A)$$

compatible with the map S. $K_0(A)$ denotes the group of isomorphism classes of finitely generated projective A-modules. For this purpose we shall now assume that A is a *ring*.

Let P be a *finitely generated projective A-module*. Then P is the image in A^k of an idempotent matrix $p \in M_k(a)$. Of course p is only well defined up to conjugation by an invertible matrix.

Theorem 3. *Define*

$$Ch_n\colon K_0(A) \to H^{\lambda}_{2n}(A)$$

by

$$Ch_n(p) = \mathrm{Tr}\left(\frac{(-1)^n (2n)!}{n!} p^{\otimes(2n+1)}\right)$$

where $p^{\otimes(2n+1)}$ denotes the $(2n+1)$-th tensor power of p.

Then Ch_n is a well defined group homomorphism compatible with the map S.

Proof. Tr denotes the *generalized trace map*

$$\mathrm{Tr}(a^0 \otimes a^1 \otimes \ldots \otimes a^n) = \sum a^0_{i_1, i_2} \otimes a^1_{i_2, i_3} \otimes \ldots \otimes a^n_{i_{n+1}, i_1}.$$

To see that b is well defined, we first note that for the tensor powers of p we have

$$b(p^{\otimes(2n+1)}) = p^{\otimes 2n}$$

and the last expression equals $(1-t)(\frac{1}{2} p^{\otimes 2n})$.

Moreover, since Ad acts trivially on $H^{\lambda}(A)$, it follows that the map is independent on the choice of representative of the idempotent p.

Finally, the compatibility with S is established as follows. Let σ denote $Ch_n(p)$, then $b\sigma = (1-t)\tau$ where

$$\tau = \frac{(-1)^n (2n)!}{n!\,2} p^{\otimes 2n}$$

so that

$$-b'\tau = \frac{(-1)^{n-1} (2n)!}{n!\,2} p^{\otimes(2n-1)}.$$

Finally, $-b'\tau = N\alpha$ where $\alpha = \dfrac{(-1)^{n-1}(2n)!}{n!\,2(2n-1)} p^{\otimes(2n-1)}$ which is equal to $Ch_{n-1}(p)$.

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Summary

In this paper we derive some combinatorial formulae concerning the action of conjugation on both the Hochschild and cyclic homology of an algebra A.

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