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An orthogonal type property for the Bessel polynomials (**)

1 - Introduction

Bessel polynomials constitute an important class of polynomials with many applications ([3], p. 131-149). Grosswald ([3], p. 25-33) discussed only the orthogonality of Bessel polynomials on the unit circle and the corresponding moments. Exton ([2], p. 215, (14)), gave the *orthogonality relation*

$$\int_0^\infty x^{a-2}\, e^{-\frac{1}{x}}\, y_m(x;\, a,\, 1)\, y_n(x;\, a,\, 1)\, dx = \frac{(-1)^n\, n!(n+a-2)\,\pi}{\Gamma(a+n)(2n+a-1)\, \sin(\pi a)}\, \delta_{m,\, n} \tag{1.1}$$

where $\operatorname{Re} a < 1-m-n$ or $\operatorname{Re} a > 1-m-n$.

Srivastava [5] noted that the orthogonality property of Bessel polynomials

$$\int_0^\infty x^{1-a}\, e^{-x}\, y_m(1;\, a,\, x)\, y_n(1;\, a,\, x)\, dx = n!\,\Gamma(2-a-n)\,\delta_{m,\, n} \tag{1.2}$$

obtained by Hamza in [4] was not correct.

The author [1], in view of the remark of Srivastava, using (1.1) derived the following *orthogonality relation*

$$\int_0^\infty x^{-a}\, e^{-x}\, y_m(1;\, a,\, x)\, y_n(1;\, a,\, x)\, dx = \frac{n!\,(2-a-n)}{1-a-2n}\,\delta_{m,\, n} \tag{1.3}$$

where $\operatorname{Re} a < 1-(m-n)$ or $\operatorname{Re} a > 1-(m-n)$.

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Remark. Exton [2] did not consider condition $\operatorname{Re} a > 1 - m - n$ and so did the author in [1]. This fact kept the author in confusion for a long time and delayed the work of the author himself in the field of Fourier Bessel expansions based on (1.1) and (1.3).

Bessel polynomials are defined by the relation ([3], p. 38, (1))

$$y_n(x;\ a,\ b) = {}_2F_0(-n,\ n + a - 1;\ -;\ -\frac{x}{b}). \tag{1.4}$$

In view of the interest shown in Bessel polynomials [3], it appears worthwhile to investigate further the matter of the orthogonality of the Bessel polynomials and the functions related to them.

In this paper, we establish a new orthogonal type property for the Bessel polynomials with respect to an elementary multiplier function.

2 - Orthogonal type property

The orthogonal type property we establish is given below. Let us consider the integral

$$I_{mn} = \int_0^\infty x^{-a-1}\, e^{-\frac{x}{b}}\, y_m(b;\ a,\ x)\, y_n(b;\ a,\ x)\, dx$$

where $\operatorname{Re} a < -(m + n)$ or $\operatorname{Re} a > 2 - (m + n)$. Then we have

(2.1) $I_{mn} = 0$ if $n-1 > m$ or $m > n+1$

(2.2) $$I_{mn} = \frac{n!\,\Gamma(1-a-n)(a+n-2)_{n-1}}{b^a(a+n)_n} \quad \text{if } m = n-1$$

(2.3) $$I_{mn} = \frac{n!}{b^a}\Big[\frac{n\Gamma(1-a-n)(a+n-1)_{n-1}}{(a+n)_n} + \frac{(n+1)\Gamma(-a-n)(a+n-1)_n}{(1+a+n)_n}\Big] \quad \text{if } m = n$$

(2.4) $$I_{mn} = \frac{(n+1)!}{b^a}\,\frac{(n+1)\Gamma(-a-n)(a+n)_n}{(1+a+n)_n}$$

$$+ \frac{(n+1)!}{b^a}\Big[\frac{n\Gamma(1-a-n)(a+n)_{n-1}}{2(a+n)_n} + \frac{(n+2)\Gamma(-1-a-n)(a+n)_{n+1}}{2(2+a+n)_n}\Big] \quad \text{if } m = n+1$$

where $(p)_q = p(p+1)\dots(p+n-1)$.

Proof. In view of (1.4), the integral I_{mn} can be written as

$$\int_0^\infty x^{-a-1} e^{-\frac{x}{b}} {}_2F_0(-m, m+a-1; -; -\frac{b}{x}) {}_2F_0(-n, n+a-1; -; -\frac{b}{x}) dx. \tag{2.5}$$

We next express the hypergeometric functions as infinite series, interchange the order of integration and summation, which, incidentally, is justified by the absolute convergence of the integral and summations involved, and write (2.5) as

$$\sum_{r=0}^{\infty} \frac{(-m)_r(m+a-1)_r}{r!}(-b)^r \sum_{u=0}^{\infty} \frac{(-n)_u(n+a-1)_u}{u!}(-b)^u \int_0^\infty x^{-a-1-r-u} e^{-\frac{x}{b}} dx. \tag{2.6}$$

Now, the integral in (2.6) can be evaluated with the help of definition of the gamma function

$$\int_0^\infty x^v e^{-\frac{x}{z}} dx = \Gamma(v+1) z^{v+1} \qquad \text{Re}\, v > -1.$$

Thus (2.6) is equivalent to

$$\frac{1}{b^a} \sum_{r=0}^{\infty} \frac{(-m)_r(m+a-1)_r(-1)^r}{r!} \sum_{u=0}^{\infty} \frac{(-n)_u(n+a-1)_u(-1)^u}{u!} \Gamma(-a-r-u). \tag{2.7}$$

Using the relation $\Gamma(1+a-n) = \dfrac{(-1)^n \Gamma(1+a)}{(a)_n}$ and simplifying, (2.7) reduces to

$$\frac{1}{b^a} \sum_{r=0}^{\infty} \frac{(-m)_r(m+a-1)_r(-1)^r}{r!} \Gamma(-a-r) {}_2F_1(-n, n+a-1; 1+a+r; 1). \tag{2.8}$$

Applying Vandermonde's theorem ${}_2F_1(-n, b; c; 1) = \dfrac{(c-b)_n}{(c)_n}$ and the relation $(r-n+2)_n = (-1)^n(-r-1)_n$, (2.8) reduces to the form

$$\frac{1}{b^a} \sum_{r=0}^{\infty} \frac{(-m)_r(-r-1)_n(m+a-1)_r\Gamma(-a-r)}{(1+a+r)_n r!}(-1)^{r+n}. \tag{2.9}$$

If $r < n-1$, the numerator of (2.9) vanishes, and since r runs from 0 to m, it follows that (2.9) also vanishes, when $m < n-1$. Since $I_{mn} = I_{nm}$, it is evident that

(2.9) also vanishes when $m > n+1$. Now, it is clear that for $m \neq n-1, n, n+1$ all terms of (2.9) vanish, which proves (2.1).

When $m = n-1$, using the standard result $(-n)_n = (-1)^n n!$, (2.8) yields

$$(2.10)\quad \int_0^\infty x^{-a-1} e^{-\frac{x}{b}} y_{n-1}(b;\, a, x)\, y_n(b;\, a, x)\, dx = \frac{n!\Gamma(1-a-n)(a+n-2)_{n-1}}{b^a (a+n)_n}$$

which proves (2.2).

For $m = n$, we employ the standard results $(-n)_{n-1} = (-1)^{n-1} n!$ and $(-n-1)_n = (-1)^n (n+1)!$ and adding the resulting two terms $(r = n-1, n)$, we have

$$(2.11)\quad \begin{aligned} &\int_0^\infty x^{-a-1} e^{-\frac{x}{b}} (y_n(b;\, a, x))^2\, dx \\ &= \frac{n!}{b^a}\left[\frac{n\Gamma(1-a-n)(a+n-1)_{n-1}}{(a+n)_n} + \frac{(n+1)\Gamma(-a-n)(a+n-1)_n}{(1+a+n)_n}\right] \end{aligned}$$

which proves the relation (2.3).

Finally, when $m = n+1$, using standard results like

$$(-n-1)_{n-1} = \frac{(-1)^{n-1}(n+1)!}{2!},\quad (-n)_{n-1} = (-1)^{n-1} n!,\quad (-n-1)_n = (-1)^n (n+1)!,$$

$$(-n-1)_{n+1} = (-1)^{n+1}(n+1)!,\qquad (-n-2)_n = \frac{(-1)^n (n+2)!}{2!}$$

and adding the resulting terms $(r = n-1, n, n+1)$, we get

$$(2.12)\quad \begin{aligned} &\int_0^\infty x^{-a-1} e^{-\frac{x}{b}} y_{n+1}(b;a,x)\, y_n(b;a,x)\, dx = \frac{(n+1)!}{b^a}\,\frac{(n+1)\Gamma(-a-n)(a+n)_n}{(1+a+n)_n} \\ &+ \frac{(n+1)!}{b^a}\left[\frac{n\Gamma(1-a-n)(a+n)_{n-1}}{2(a+n)_n} + \frac{(n+2)\Gamma(-1-a-n)(a+n)_{n+1}}{2(2+a+n)_n}\right] \end{aligned}$$

which proves (3.4).

Since, without loss of generality, we have

$$y_n(b;\, a, x) = y_n\left(\frac{1}{x};\, a, \frac{1}{b}\right) = y_n\left(1;\, a, \frac{x}{b}\right) = y_n\left(\frac{b}{x};\, a, 1\right)$$

we can obtain alternative forms of our results.

References

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Summary

We briefly discuss the matter of the orthogonality of Bessel polynomials and establish an orthogonal type property for Bessel polynomials with an elementary multiplier function.

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