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On some new sequence spaces (**)

1 - Introduction

Let l_∞ be the set of all real or complex sequences $x = (x_n)$ with the norm $\|x\| = \sup_n |x_n| < \infty$. A linear functional L on l_∞ is said to be a *Banach limit* [**1**], if it has the properties,

(i) $L(x) \geqslant 0$ if $x \geqslant 0$ (i.e., $x_n \geqslant 0$ for all n)
(ii) $L(e) = 1$, where $e = (1, 1, \ldots)$
(iii) $L(Sx) = L(x)$, where the shift operator S is defined by $(Sx)_n = x_{n+1}$.

If p is any sublinear functional on l_∞, then we write $\{l_\infty, p\}$ to denote the set of all linear functionals φ on l_∞, such that $p > \varphi$ i.e., $p(x) \geqslant \varphi(x)$, $\forall x \in l_\infty$. A sublinear functional p is said to generate Banach limits if $\varphi \in \{l_\infty, p\}$ implies that φ is a Banach limit; p is said to dominate Banach limits if φ is a Banach limit implies that $\varphi \in \{l_\infty, p\}$. Then, if p both generates and dominates Banach limits, then $\{l_\infty, p\}$ is the set of all Banach limits. It is known [**1**] that $\{l_\infty, q\}$ is the set of all Banach limits, where

$$q(x) = \inf_{n_1, n_2, \ldots, n_r} \overline{\lim_k} \frac{1}{r} \sum_{i=1}^{r} x_{k+n_i}.$$

It is well known that $q(x) = t(x)$, $x \in l_\infty$, where

$$t(x) = \overline{\lim_n} \sup_i \frac{1}{n} \sum_{k=0}^{n-1} x_{k+1}.$$

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Let B be the set of all Banach limits on l_∞. A sequence $x \in l_\infty$ is said to be *almost convergent* to a number s if $L(x) = s$ for all $L \in B$. Lorentz [6] has shown that x is almost convergent to s if and only if

$$t_{ki} = t_{ki}(x) = (x_i + x_{i+1} + \ldots + x_{i+k-1})\, k^{-1} \to s$$

as $k \to \infty$ uniformly in i. Let f denote the set of all almost convergent sequences.

Maddox [7], [8] has defined x to be *strongly almost convergent* to a number s if

$$t_{ki}(|x - s|) = \frac{1}{k} \sum_{j=0}^{k-1} |x_{i+j} - s| \to 0$$

as $k \to \infty$ uniformly in i. Throughout the paper we will write $x - s$ for $(x_k - s)$. Let $[f]$ denote the set of all strongly almost convergent sequences. It is easy to see that $[f] \subset f \subset l_\infty$.

The following sequences spaces have been introduced and examined for what concerns their relative strenghts by Das and Sahoo [5].

$$w = \{x \mid \lim_n \left(\frac{1}{n+1} \sum_{k=0}^{n} t_{ki}(x - s)\right) = 0 \text{ uniformly in } i, \text{ for some } s\}$$

$$[w] = \{x \mid \lim_n \left(\frac{1}{n+1} \sum_{k=0}^{n} |t_{ki}(x - s)|\right) = 0 \text{ uniformly in } i, \text{ for some } s\}$$

$$[w_1] = \{x \mid \lim_n \left(\frac{1}{n+1} \sum_{k=0}^{n} t_{ki}(|x - s|)\right) = 0 \text{ uniformly in } i, \text{ for some } s\}.$$

It may be noted that almost convergent sequences are nencessarily bounded but the sequence spaces $w, [w]$ may contain unbounded sequences. If $x \in w$ then we say that x is *w-convergent*. Similarly, we define *$[w]$-convergent sequences* and *$[w_1]$-convergent sequences.*

By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$. The ratio $k_r(k_{r-1})^{-1}$ will be denoted by q_r.

The object of the present paper is to determine a new sublinear functionals involving lacunary sequence that both dominates and generates Banach limits. Also we introduce a new concept of strong almost convergence through a lacunary sequence.

2 - Sublinear functionals and lacunary sequences

A sequences x of real or complex numbers is said to be *lacunary w-convergent, lacunary* $[w]$*-convergent* or *lacunary* $[w_1]$*-convergent* (with respect to the lacunary sequence θ) to the value s if

$$\lim_r \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(x-s) = 0 \qquad \lim_r \sup_i \frac{1}{h_r} \sum_{k \in I_r} |t_{ki}(x-s)| = 0$$

$$\lim_r \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(|x-s|) = 0$$

respectively. Let w_θ, $[w]_\theta$ and $[w_1]_\theta$ denote the set of the lacunary w-convergent sequences, the lacunary $[w]$-convergent and the lacunary $[w_1]$-convergent sequences, respectively.

For a lacunary sequence θ, we define sublinear functionals on l_∞ by

$$\phi_\theta(x) = \overline{\lim_r} \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(x)$$

$$\psi_\theta(x) = \overline{\lim_r} \sup_i \frac{1}{h_r} \sum_{k \in I_r} |t_{ki}(x)|$$

$$\zeta_\theta(x) = \overline{\lim_r} \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(|x|).$$

It can be easily seen that each of the above functionals are finite, well defined and sublinear on l_∞.

In the following theorem, we demonstrate that $\{l_\infty, \phi_\theta\}$ is the set of all Banach limits on l_∞.

Theorem 1. *The sublinear functional* ϕ_θ *on* l_∞ *both dominates and generates Banach limits for every lacunary sequence; in other words*

$$\phi_\theta(x) = t(x) = q(x) \qquad x \in l_\infty .$$

Proof. It is easy to verify that $\phi_\theta(x) \leq t(x)$ for all $x \in l_\infty$. Hence, ϕ_θ generates Banach limits. Using the properties of $L \in B$, we obtain

$$L(x) = \frac{1}{h_r} L\Big(\sum_{k \in I_r} t_{ki}(x)\Big) \leq \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(x).$$

This implies that $L(x) \leqslant \phi_\theta(x)$ for all $x \in l_\infty$ and then proves that $B \subset \{l_\infty, \phi_\theta\}$, that is ϕ_θ dominates Banach limits. This completes the proof.

Corollary 1. *We have*

$$f = \{x \in l_\infty \mid \varphi(x) = s \text{ for all } \varphi \in \{l_\infty, \phi_\theta\}\}$$

$$= \{x \in l_\infty \mid \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(x) \to s \quad \text{uniformly in } i\} = l_\infty \cap w_\theta .$$

Proof. This follows from the fact that $\varphi(x) = s$ for all $\varphi \in \{l_\infty, \phi_\theta\}$ if and only if ([2] Theorem 6)

$$\phi_\theta(x) = -\phi_\theta(-x). \tag{1}$$

But this condition holds if and only if

$$\frac{1}{h_r} \sum_{k \in I_r} t_{ki}(x) \to s \qquad (r \to \infty, \text{ uniformly in } i),$$

i.e. $x \in w_\theta \cap l_\infty$. But condition (1) also is equivalent (by Theorem 1) to $t(x) = -t(-x)$, i.e. $x \in f$.

Corollary 2. *For every* θ, $l_\infty \cap w_\theta = l_\infty \cap w = f$.

Proof. This follows from Corollary 1 and Theorem 2 **c** of [3].

If $f(x - se) = 0$ for all $f \in \{l_\infty, \psi_\theta\}$, then we say that x is ψ_θ*-convergent* to s. Similarly we define the ζ_θ*-convergent* sequences. In the following theorem, we charactirize the ψ_θ-convergent sequences and the ζ_θ-convergent sequences.

Theorem 2. *We have*

a $[w]_\theta \cap l_\infty = \{x \mid \psi_\theta(x - se) = 0 \text{ for some } s\} = \{x \mid f(x - se) = 0, \ \forall f \in \{l_\infty, \psi_\theta\}\}$
b $[w_1]_\theta \cap l_\infty = \{x \mid \zeta_\theta(x - se) = 0 \text{ for some } s\} = \{x \mid f(x - se) = 0, \ \forall f \in \{l_\infty, \zeta_\theta\}\}$.

Proof. By Hahn-Banach theorem, $\{l_\infty, \psi_\theta\}$ is non-empty. If $f \in \{l_\infty, \psi_\theta\}$, then we have

$$-\psi_\theta(-x) \leqslant f(x) \leqslant \psi_\theta(x) \qquad x \in l_\infty$$

or equivalently $\qquad -\psi_\theta(-x + se) \leqslant f(x - se) \leqslant \psi_\theta(x - se).$

Now $f(x-se)=0$ if and only if $\psi_\theta(x-se)=-\psi_\theta(-x+se)=0$ ([2], Theorem 6). But since by definition $\psi_\theta(x)=\psi_\theta(-x)$, it follows that $f(x)=s$ for $f\in\{l_\infty, \psi_\theta\}$ if and only if $\psi_\theta(x-se)=0$. It is easy to verify that $\psi_\theta(x-se)=0$ is equivalent to the fact that

$$\frac{1}{h_r}\sum_{k\in I_r}|t_{ki}(x-s)|\to 0 \quad (r\to\infty, \text{ uniformly in } i).$$

This completes the proof of **a**.

The proof of **b** is similar.

It is evident from Theorem 2 that $[w]_\theta\cap l_\infty$ and $[w_1]_\theta\cap l_\infty$ are the sets of all ψ_θ-convergent sequences and all ζ_θ-convergent sequences, respectively.

In the following theorem we examine the relationship between $[w_1]$-convergence and lacunary $[w_1]$-convergence. We need a Lemma.

Lemma 1. *Suppose, for given* $\varepsilon>0$, *there exist* n_0 *and* i_0 *such that*

$$\frac{1}{n}\sum_{k=0}^{n-1}t_{ki}(|x-s|)<\varepsilon$$

for all $n\geqslant n_0$, $i\geqslant i_0$. *Then* $x\in[w_1]$.

Proof. Let $\varepsilon>0$ be given. Choose n_0', i_0 such that

$$\frac{1}{n}\sum_{k=0}^{n-1}t_{ki}(|x-s|)<\frac{\varepsilon}{6} \tag{2}$$

for all $n\geqslant n_0'$ and $i\geqslant i_0$. It is enough to prove that there exists n_0'' such that for $n\geqslant n_0''$, $0\leqslant i\leqslant i_0$

$$\frac{1}{n}\sum_{k=0}^{n-1}t_{ki}(|x-s|)<\varepsilon. \tag{3}$$

Since, taking $n_0=\max(n_0', n_0'')$, (3) will hold for $n\geqslant n_0$ and for all i, we obtain the result.

Once i_0 has been chosen, i_0 is fixed, so

$$\sum_{k=0}^{i_0-1}\left(\frac{1}{k}\sum_{j=0}^{i_0-1}|x_j-s|\right)=M \quad (\text{constant}). \tag{4}$$

Now, taking $0 \leqslant i \leqslant i_0$ and $n > i_0$, we have (from (4) and (2))

$$\frac{1}{n}\sum_{k=0}^{n-1} t_{ki}(|x-s|) = \frac{1}{n}\Big(\sum_{k=0}^{i_0-1} + \sum_{k=i_0}^{n-1}\Big)\Big[\frac{1}{k}\Big(\sum_{j=i}^{i_0-1} + \sum_{j=i_0}^{i+k-1}\Big)|x_j - s|\Big] \leqslant \frac{M}{n} + \frac{\varepsilon}{2}.$$

Taking, n sufficiently large, we can make $\frac{M}{n} + \frac{\varepsilon}{2} < \varepsilon$ which gives (3) and hence the result.

Theorem 3. *We have* $[w_1]_\theta = [w_1]$ *for every* θ.

Proof. Let $x \in [w_1]_\theta$. Given $\varepsilon > 0$, there exist r_0 and s such that

$$\frac{1}{h_r}\sum_{k=0}^{h_r-1} t_{kq}(|x-s|) < \varepsilon$$

for $r \geqslant r_0$ and $q = k_{r-1} + 1 + i$, $i \geqslant 0$.

Let $n \geqslant h_r$. Write $n = m h_r + \theta$ where $0 \leqslant \theta \leqslant h_r$, m is an integer. Since $h \geqslant h_r$, $m \geqslant 1$. Now

$$\frac{1}{n}\sum_{k=0}^{n-1} t_{kq}(|x-s|) \leqslant \frac{1}{n}\sum_{k=0}^{(m+1)h_r-1} t_{kq}(|x-s|) = \frac{1}{n}\sum_{\mu=0}^{m}\sum_{k=\mu h_r}^{(\mu+1)h_r-1} t_{kq}(|x-s|)$$

$$\leqslant \frac{m+1}{n} h_r \varepsilon \leqslant \frac{2m h_r}{n}\varepsilon \qquad (m \geqslant 1).$$

For $\frac{h_r}{n} \leqslant 1$, since $m\,\frac{h_r}{n} \leqslant 1$, we get

$$\frac{1}{n}\sum_{k=0}^{n-1} t_{kq}(|x-s|) \leqslant 2\varepsilon.$$

Then by Lemma 1, $[w_1] \subset [w_1]_\theta$. It is trivial that $[w_1]_\theta \subset [w_1]$ for every θ. Hence we have the result.

In order to prove Theorem 4, we require the following Lemma.

Lemma 2. *Suppose, for a given* $\varepsilon > 0$, *that there exist* n_0 *and* i_0 *such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} |t_{ki}(x-s)| < \varepsilon$$

for all $n \geqslant n_0$, $i \geqslant i_0$. *Then* $x \in [w]$.

Proof. Let $\varepsilon > 0$ be given. Choose n_0', i_0 such that

$$\frac{1}{n}\sum_{k=0}^{n-1}|t_{ki}(x-s)| < \frac{\varepsilon}{4} \quad \text{for } n \geqslant n_0', \quad i \geqslant i_0 . \tag{5}$$

As in Lemma 1, it is enough to show, there exist n_0'' such that for $n \geqslant n_0''$, $0 \leqslant i \leqslant i_0$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1}|t_{ki}(x-s)| < \varepsilon .$$

Since i_0 is fixed, put

$$\sum_{k=0}^{i_0-1}\frac{1}{k}\sum_{j=0}^{i_0-1}|x_j - s| = M.$$

Now, let $0 \leqslant i \leqslant i_0$ and $n > i_0$, then

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1}|t_{ki}(x-s)| &\leqslant \frac{1}{n}\sum_{k=0}^{i_0-1}\frac{1}{k}\sum_{j=0}^{i_0-1}|x_j-s| + \frac{1}{n}\sum_{k=0}^{i_0-1}\Big|\frac{1}{k}\sum_{j=i_0}^{i+k-1}(x_j-s)\Big| \\ &\quad + \frac{1}{n}\sum_{k=i_0}^{n-1}\Big|\frac{1}{k}\sum_{j=i}^{i+k-1}(x_j-s)\Big| \\ &\leqslant \frac{M}{n} + \frac{1}{n}\sum_{k=0}^{i_0-1}\Big|\frac{1}{k}\sum_{j=i_0}^{i_0+(k+i-i_0)-1}(x_j-s)\Big| + \frac{1}{n}\sum_{k=i_0}^{n-1}\Big|\frac{1}{k}\sum_{j=i}^{i+k-1}(x_j-s)\Big|.\end{aligned} \tag{6}$$

Let $k - i_0 > n_0'$. Then for $0 \leqslant i < i_0$, we have $k + i - i_0 \geqslant n_0'$. From (5)

$$\frac{1}{i_0}\sum_{k=0}^{i_0-1}\Big|\frac{1}{k+i-i_0}\sum_{j=i_0}^{i_0+(k+i-i_0)-1}(x_j-s)\Big| < \frac{\varepsilon}{4}. \tag{7}$$

From (6) and (7) $\displaystyle\frac{1}{n}\sum_{k=0}^{n-1}|t_{ki}(x-s)| \leqslant \frac{M}{n} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$

for sufficiently large n. Hence the result.

Theorem 4. *For every θ, we have $[w]_\theta \cap l_\infty = [w]$.*

Proof. Let $x \in [w]_\theta \cap l_\infty$. For $\varepsilon > 0$, there exist r_0 and q_0 such that

$$\frac{1}{h_r}\sum_{k=0}^{h_r-1}|t_{kq}(x-s)| < \frac{\varepsilon}{2} \tag{8}$$

for $r \geqslant r_0$ and $q \geqslant q_0$, $q = k_{r-1} + 1 + i$, $i \geqslant 0$.

Now, let $n \geqslant h_r$, m is an integer greater than equal to 1. Then

$$\frac{1}{n}\sum_{k=0}^{n-1}|t_{kq}(x-s)| \leqslant \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k}\sum_{\mu=0}^{m-1}\Big|\sum_{j=q+\mu h_r}^{q+(\mu+1)h_r-1}(x_j-s)\Big| + \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{k}\sum_{j=q+mh_r}^{q+k-1}|x_j-s|$$

$$(9) \qquad \leqslant \frac{1}{n}\sum_{\mu=0}^{m-1}\sum_{k=\mu h_r}^{(\mu+1)h_r-1}\frac{1}{k}\Big|\sum_{j=q}^{q+k-1}(x_j-s)\Big| + \frac{1}{n}\sum_{k=mh_r}^{n-1}\frac{1}{k}\sum_{j=q}^{q+k-1}|x_j-s|.$$

Since $x \in l_\infty$, for all j, $|x_j - s| < M$. So, from (8) and (9)

$$\frac{1}{n}\sum_{k=0}^{n-1}|t_{kq}(x-s)| \leqslant \frac{1}{n}\, m h_r \frac{\varepsilon}{2} + \frac{M h_r}{n}.$$

For, $\frac{h_r}{n} \leqslant 1$, $M\frac{h_r}{n}$ can be made less than $\frac{\varepsilon}{2}$ by taking n sufficiently large and since $m\frac{h_r}{n} \leqslant 1$, then

$$\frac{1}{n}\sum_{k=0}^{n-1}|t_{kq}(x-s)| < \varepsilon$$

for $r \geqslant r_0$, $q \geqslant q_0$. Hence, by Lemma 2, $[w]_\theta \cap l_\infty \subset [w]$.

It is trivial that $[w] \subset [w]_\theta \cap l_\infty$. The proof is completed.

We have the following corollary if we consider together with Theorem 4 of [3] Theorem 3 and Theorem 4.

Corollary 3. $[f] \subset [w_1]_\theta \subset l_\infty \cap [w]_\theta \subset l_\infty \cap w_\theta = f$.

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Summary

The object of this paper is to introduce some new sequence spaces related with the concept of lacunary strong almost convergence, [3] [4].

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