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On near-rings in which the ideals are annihilators (**)

A Bianca Manfredi con amicizia e stima

Introduction

Rings R in which all (left or right) ideals are (left or right) annihilators of subsets of R are studied in [4], [5], [6], [8]. The purpose of this paper is to extend the above situation to near-rings and to establish the structure theory for R-near-rings: namely the near-rings N in which every non-trivial ideal is a right annihilator of a subset of N.

Using [2], we characterize the R-near-rings which contain an ideal I such that its left annihilator is without nonzero nilpotent elements, and we prove, in particular, that such near-rings are subdirectly reducible and thus they have the right annihilator that equals $\{0\}$.

Moreover, we show that, by adding some little stronger conditions, we find integral near-rings in which all ideals are prime, linearly ordered and with integral factors. Thus such near-rings result special cases of near-rings studied in [3].

1 - Definitions and preliminary results

In this paper N stands for a left near-ring and in particular the additive group and the multiplicative semigroup of N are denoted by N^+ and $N^\bullet$ respectively.

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Moreover, if $M \subseteq N$ we put

$R(M) = \{x \in N\colon Mx = 0\}$, $L(M) = \{x \in N\colon xM = 0\}$ and $A(M) = R(M) \cap L(M)$.

Finally if S, Q are ideals of $N^\bullet$ we set $S^2 = \{xy\colon x, y \in S\}$ and $SQ = \{sq\colon s \in S,\ q \in Q\}$.

For fundamental notations and elementary results, we refer to [8].

The present section is devoted to establishing some results which will be of importance later. Most of these are obtained in an analogous manner to the corresponding results in ring theory.

Proposition 1. *Let N be a near-ring and let S, T be not-empty subsets of N. Then:*

a) *$S \subseteq T$ implies $R(T) \subseteq R(S)$ and $L(T) \subseteq L(S)$*

b) *$L(S) = L(R(L(S)))$ and $R(S) = R(L(R(S)))$*

c) *$L(R(S)) = L(R(T))$ implies $R(S) = R(T)$ and $R(L(S)) = R(L(T))$ implies $L(S) = L(T)$*

d) *$R(S \cup T) = R(S) \cap R(T)$ and $L(S \cup T) = L(S) \cap L(T)$*

e) *If S, T are normal subgroups of N^+, then $L(S + T) = L(S) \cap L(T)$.*

Immediate consequences of Proposition 1 are

Proposition 2. *Let N be a zero-symmetric near-ring. If N has a non-trivial ideal I such that $I = L(R(I))$ and $I \cap L(I) = 0$, then $L(I + R(I)) = 0$ and $L(N) = 0$.*

Proposition 3. *Let N be a zero-symmetric near-ring. If S is an ideal of $N^\bullet$ without nonzero nilpotent elements, then $R(S) = R(S^2)$.*

In fact $S^2 \subseteq S$, so immediately we have $R(S) \subseteq R(S^2)$. Now if $a \in R(S^2)$, then $s^2 a = 0$, for all $s \in S$, and by Oss. 2 of [2] $sasa = 0$ (because S is an ideal of $N^\bullet$). It follows that $sa = 0$, because S has no nonzero nilpotent elements and hence $a \in R(S)$.

We start to study near-rings satisfying the following condition.

Definition A. A near-ring N is called *R-near-ring* if N satisfies the right annihilator condition: i.e. for each ideal I of N, $(I \neq 0)$, there exists a non-empty subset S of N such that $I = R(S)$.

Using Definition A and Proposition 1, we now have

Proposition 4. *Let N be an R-near-ring. Then*

1) *N is zero-symmetric*
2) *If I, J are non-trivial ideals of N such that $L(I) \subseteq L(J)$, then $J \subseteq I$*
3) *$R(N) \subseteq I$, for any non-trivial ideal I of N*
4) *If $R(N) \neq 0$, then N is subdirectly irreducible.*

Proposition 5. *Let N be a near-ring.*

1) *If N is simple, N is an R-near-ring iff N is zero-symmetric.*
2) *If N is a zero-near-ring, N is an R-near-ring iff N is simple.*
3) *If N is integral, N is an R-near-ring iff N is simple and zero-symmetric.*

Proposition 6. *Let N be an R-near-ring. If I is a non-trivial ideal of N, then $I = R(L(I))$ and $L(I)$ is a non-trivial ideal of $N^\bullet$, if $I \neq N$.*

By Proposition 1 and Definition A, $I = R(L(I))$. Moreover, if $L(I) = 0$ then $I = R(L(I)) = R(0) = N$, hence $I = N$, which is absurd.

The above propositions suggest a first characterization of R-near-rings.

Proposition 7. *A near-ring N is an R-near-ring if and only if the proper ideals of N and the right annihilators of proper ideals of $N^\bullet$ coincide.*

By Definition A it follows that every nonzero element of an R-near-ring is a right zero divisor. Now we give an example showing that this property does not characterize the R-near-rings.

We consider the Klein's four group $(G, +)$ with the following multiplication table

$*$	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	b	a

Now $N = (G, +, *)$ is a near-ring and it can be verified that every element of N is a right zero divisor, but for the ideal $I = \{0, b\}$ there exists no non-empty subset S of N such that $R(S) = I$. Therefore N is not an R-near-ring.

2 - R-near-rings with a reduced ideal

We note that in general the R-near-rings are not closed under homomorphisms, but we can show the following theorem.

Theorem 1. *Let N be an R-near-ring such that $R(S) = R(S^2)$, for all ideals S of $N^{\bullet}$. Then each homomorphic image of N is an R-near-ring.*

Let I be a proper ideal of N and $N' = N/I$. If J' is a proper ideal of N', then there exists a proper ideal J of N such that $I \subset J$ and $J' = J/I$.

Firstly we observe that $L(I)$ and $L(J)$ are not trivial by Proposition 6. Then we prove $J' = R(L(J'))$. Obviously, $J' \subseteq R(L(J'))$. Let $[z] \in R(L(J'))$: thus $L(J')[z] = I$ and hence $tz \in I$, for all $t \in L(J)$; namely $tz \in R(L(I)) = I$. Hence it follows $t'tz = 0$ for all $t' \in L(I)$ and $z \in R(L(I)L(J))$. Since $I \subset J$ implies $L(J) \subseteq L(I)$, we have $L(J)^2 \subseteq L(I)L(J)$. So $R(L(I)L(J)) \subseteq R(L(J)^2) = R(L(J)) = J$ and $z \in J$. Finally $[z] \in J'$ and $R(L(J')) \subseteq J'$.

Moreover, if $J' = R(L(J'))$ and $J' \neq N'$ then $L(J')$ must be different from zero, because obviously the right annihilator of zero is N'.

We recall that a near-ring N is said *completely reducible* if it is a direct sum of simple ideal (see [8]) and N is called *reduced* if it has no nonzero nilpotent elements.

We have

Theorem 2. *Let N be an R-near-ring and let I be an ideal of N such that $L(I)$ is reduced. Then either N/I is integral and simple or N/I is completely reducible in integral near-rings.*

To make easier the discussion of the proof of Theorem 2, we first prove a lemma which is of interest in its own right.

Lemma 1. *Let N be an R-near-ring, and let I be an ideal of N such that $L(I)$ is reduced. Then N/I is a reduced R-near-ring.*

Since N is an R-near-ring, I and $R(L(I))$ coincide and I contains every nilpotent elements of N (Oss. 3 of [2]). Put $N' = N/I$. Let J' be an ideal of N' different from I and let J be the ideal of N such that $I \subset J$ and $J' = J/I$.

Now N' has no nonzero nilpotent elements because if $[y]^n = I$, for some $n \in \mathbf{N}$, then $y^n \in I$ and $xy^n = 0$, for all $x \in L(I)$. So it follows $(xy)^n = 0$, by Oss. 2 of [2] and $xy = 0$ (because $xy \in L(I)$ and $L(I)$ is reduced) and finally $y \in R(L(I)) = I$.

Moreover, since $I \subset J$, we have $L(J) \subsetneq L(I)$ and also $R(L(J)^2) = R(L(J)) = J$ (by Proposition 3).

Using techniques of the proof of Theorem 1, we can prove that $J' = R(L(J'))$ and so we have the result.

We may now complete the proof of Theorem 2.

Put $N' = N/I$; from Lemma 1, N' is a reduced R-near-ring. If N' is integral, then N' is simple (Proposition 5), otherwise N' has the IFP-property (see Lemma 1, [1]); hence N' contains a family $\mathscr{F}$ of completely prime ideals with trivial intersection (see Lemma 3 of [1]).

Let M be an element of $\mathscr{F}$. Thus N'/M is an R-near-ring; in fact N' is a reduced R-near-ring, and therefore every non-trivial ideal S of $N'^{\bullet}$ has no nonzero nilpotent elements and, from Proposition 3, it follows $R(S) = R(S^2)$, and finally, by Theorem 1, each homomorphic image of N' is an R-near-ring. Since N'/M is integral, it is simple and hence M is a maximal ideal of N'; moreover $M = R(L(M))$ and $L(M)$ is not void and different from zero.

Now we can consider M as in ideal of $N'^{\bullet}$ and we can note that it has no nilpotent elements. From Oss. 3 of [2], $R(M) = L(M) = A(M)$ holds and $A(M)$ is an ideal of N' such that $M \cap A(M) = 0$ (because N' is reduced). It follows that $N' = M \oplus A(M)$, where $A(M)$ is integral and simple because it is isomorphic to N'/M. To prove that N' is completely reducible, it is sufficient to show that every proper ideal of N' is a direct summand of N' (see [8], p. 55).

To this purpose, let J be a proper ideal of N'. If $J \in \mathscr{F}$, then J is a direct summand of N', by above observations.

Otherwise, there exists $M' \in \mathscr{F}$ such that J is not contained in M' (because $\mathscr{F}$ has trivial intersection). Suppose $K = J \cap M'$. If $K = 0$ then J is a direct summand of N' because M' is maximal in N'. If $K \neq 0$, then $L(J) = R(J)$ because J is an ideal of $N'^{\bullet}$ without nonzero nilpotent elements and $J \cap A(J) = 0$. So we have $L(J \oplus A(J)) = L(J) \cap L(R(J))$, by Proposition 1e. Since $J = R(A(J))$, it follows $A(J)J = 0$, and hence $JA(J) = 0$ (see [2], Oss. 2), and, indeed, $J = L(A(J))$. It follows that $L(J \oplus A(J)) = 0$ by Proposition 2 and therefore by Proposition 6, $J \oplus A(J) = N'$. This completes the proof.

Corollary 1. *A reduced near-ring N is a non-integral R-near-ring if and only if N is completely reducible in integral and zero-symmetric near-rings.*

In fact if N is a reduced and non-integral R-near-ring, N is completely reducible in integral near-rings, by Theorem 2. On the other hand, let N be reduced and completely reducible. If I is an ideal of N different from $\{0\}$, then I is a direct summand of N, namely $N = I \oplus J$. Now it is obvious that I and $R(J)$ coincide because N is reduced, so the result follows.

The following theorem gives another characterization of R-near-rings.

Theorem 3. *Let N be a near-ring in which $N^{\bullet}$ has no nonzero nilpotent ideals. Then N is an R-near-ring if and only if N is zero-symmetric and completely reducible.*

Let N be an R-near-ring and let I be a non-trivial ideal of N. Then $I = R(L(I))$, and also $(IL(I))^2 = 0$. Since $IL(I)$ is an ideal of $N^{\bullet}$, by hypothesis we have $IL(I) = 0$ and $L(I) \subseteq R(I)$; thus $R(I)$ is different from zero because $L(I) \neq 0$. On the other hand, $(R(I)I)^2 = 0$ and hence $R(I) \subseteq L(I)$, so it follows $R(I) = L(I) = A(I)$. Thus $A(I)$ is an ideal of N and, moreover, $I \cap A(I) = 0$ because $N^{\bullet}$ has no nonzero nilpotent elements.

Now we consider $L(I \oplus A(I)) = L(I) \cap L(A(I))$. It equals zero because $L(L(I)) \cap L(I)$ is a nilpotent ideal of $N^{\bullet}$. By Proposition 6, it follows $N = I \oplus A(I)$, and so I is a direct summand of N. Now we can conclude that N is completely reducible (see [8] Th. 2.48). Since $N^{\bullet}$ has no nonzero nilpotent element, the converse is obvious.

From now on, if $N = A \oplus B$, we shall denote by π_1, π_2, the first, the second projection map, respectively,

We come now to the main result of this section.

Theorem 4. *Let N be a near-ring. The following are equivalent:*

1) *N is an R-near-ring with an ideal I such that $L(I)$ is reduced*

2) *N equals $T \oplus S$, where T, S are near-rings such that:*

a) *T is either integral and simple, or completely reducible in integral near-rings*

b) *S is an R-near-ring with $L(S) = 0$*

c) *Each ideal I of N, I not contained in S or T, is the direct sum of $\pi_1(I)$ and $\pi_2(I)$.*

Let N be an R-near-ring. Let I be a proper ideal of N such that $L(I)$ is reduced. First of all, we prove that $N = I \oplus L(I)$. By Theorem 2, either N/I is simple or N/I is completely reducible.

In the first case the result is obvious.

In the other case, we observe that $L(L(I)) = R(L(I))$, because $L(I)$ is an ideal of $N^{\bullet}$ without nonzero nilpotent elements (see [2], Oss. 2), and so it follows that $L(L(I)) = I$ by hypothesis. Using Proposition 1e, we obtain that $L(I \oplus L(I)) = L(I) \cap L(L(I)) = L(I) \cap I = 0$ and consequently $N = I \oplus L(I)$ (Proposition 6). Since $L(I)$ is now isomorphic to N/I, $L(I)$ is zero-symmetric (Theorem 2) and either integral and simple or completely reducble in integral near-rings.

Now we show that I is an R-near-ring in any case. Let J be a proper ideal of I, then J is an ideal of N because I is a direct summand of N. Hence in N $J = R(L(J))$ and thus $L(I) \subset L(J)$, since $J \subset I$. Moreover, $L(J) \cap I \neq 0$: in fact, otherwise we should have $I \subseteq J$ which is excluded. Now, put $H = L(J) \cap I$. We have $L(J) = H \times L(I)$: in fact obviously $H \times L(I)$ is contained in $L(J)$, and moreover, if $(a, b) \in L(J)$ then $(a, b)(j, i) = (aj, 0) = (0, 0)$ for all $j \in J$, $b \in L(I)$, hence $aj = 0$; this implies $a \in L(J) \cap I = H$ and $b \in L(I)$.

Now we consider an element x in $I \cap R(H)$. Then, for all $h \in H$, $y \in L(I)$ we have $(h, y) = (x, 0) = (0, 0)$, and thus $(x, 0) \in R(L(J)) \cap I = J \cap I = J$: hence $A(H) \subseteq J$. Obviously $J \subseteq R(H)$, so $J = A(H)$ and $H \subseteq I$. Thus I is an R-near-ring with $L(I) = 0$. Let K be an ideal of $N = I \oplus L(I)$, $K' = \pi_1(K)$ and $K'' = \pi_2(K)$. Since $K \subseteq K' \oplus K''$, it follows $L(K' \oplus K'') \subseteq L(K)$. Moreover, $L(K) \subseteq L(K' \oplus K'')$. In fact if $(a, b) \in L(K)$, then $(a, b)(x, y) = (ax, by) = (0, 0)$ for all $(x, y) \in K$ and so $ax = 0$ and $by = 0$, for all $x \in \pi_1(K)$, $y \in \pi_2(K)$. From this it follows that (a, b) is in $L(K' \oplus K'')$. Hence $L(K) = L(K' \oplus K'')$ and, since K and $K' \oplus K''$ are ideals of N, which is an R-near-ring, $K = K' \oplus K''$ follows by Proposition 1c.

Conversely, let $N = T \oplus S$, where T is either an integral, simple and zero-symmetric near-ring or a completely reducible near-ring in integral near-rings and S is an R-near-ring with $L(S) = 0$. We have $S = R(T)$: in fact, obviously $S \subseteq R(T)$ and if $(r, 0)(a, b) = (0, 0)$, for all $r \in T$, then $ra = 0$, with $a \in T$ and this implies $a = 0$, because T has no nonzero nilpotent elements in any case.

In the same way, we can prove that in N $L(S) = T$, because S contains no nonzero left annihilators of S.

Let I be a nonzero ideal of N. If $I \subseteq S$ then in S, $I = R(L(I))$ with $L(I) \subseteq S$, because S is an R-near-ring. In this case we show that in N, $I = R(T \times L(I))$. In fact, obviously $I \subseteq R(T \times L(I))$, and moreover, if $(a, b) \in R(T \times L(I))$, then, for all $x \in T$, $y \in L(I)$ $(x, y)(a, b) = (xa, yb) = (0, 0)$. From this it follows $xa = 0$, $yb = 0$. Thus $a = 0$, because T has no nilpotent elements and $b \in S$, $b \in R(L(I))$, namely $b \in I$. Hence $(a, b) \in I$.

Now let I be not contained in S. Then $I \cap T \neq 0$, because otherwise it would be $IT = TI = 0$, and $I \subseteq R(T) = S$ which is excluded.

If $I \cap S = 0$ then $I \subseteq T$, and hence either $I = T$ or I is a direct summand of T. In the first case $I = R(H)$, in the second one $T = I \oplus H$, with H ideal of T and in T we have $I = R(H)$, by Corollary 1. Thus in N we have $I = R(H \times S)$ and so the result follows.

Finally, if $I \cap S \neq 0$, we put $I' = \pi_1(I)$ and $I'' = \pi_2(I)$. By hypothesis we have $I = I' \oplus I''$. Moreover, since T, S are R-near-rings, we have $I' = R(L(I'))$ in T and $I'' = R(L(I''))$ in S. Obviously $L(I') \times L(I'')$ is an ideal of $N^\bullet$, and so $I' \oplus I'' = R(L(I') \times L(I''))$. This is sufficient to prove the result.

3 - On strongly R-near-rings

Definition B. A near-ring N is said *strongly R-near-ring* if for each non-trivial ideal I of N there exists an ideal A of $N^\bullet$ such that $I = R(x)$, for all $x \in A \setminus \{0\}$.

Obviously a strongly R-near-ring is an R-near-ring.

We recall that a near-ring ideal I is called *completely prime* (see [**1**]) if $xy \in I$ implies $x \in I$ or $y \in I$. Now we can show

Proposition 8. *Let N be a strongly R-near-ring. The non-trivial ideals of N are completely prime.*

Let I be a proper ideal of N, $I \neq 0$, and $xy \in I$. Let A be the ideal of $N^\bullet$ such that $I = R(z)$, for each $z \in A \setminus \{0\}$. Since $Axy = 0$, if $Ax \neq 0$ we have $y \in R(Ax) = I$ (because $Ax \subseteq A$), hence $y \in I$. Otherwise $Ax = 0$ and $x \in R(A) = I$. This proves the Proposition 8.

Proposition 9. *Let N be a strongly R-near-ring. If N is subdirectly reducible, then N is reduced.*

In fact, let x be a nonzero nilpotent element of N. Then there exists $k \geq 2$ such that $x^k = 0$ and $x^{k-1} \neq 0$. Let $\mathcal{G}$ denote the family of nonzero ideals of N. Since in $\mathcal{G}$ every ideal I is completely prime, it follows that $x^{k-1} \in I$, and thus x^{k-1} is an element of the intersection of $\mathcal{G}$, but such intersection equals $\{0\}$ by hypothesis, so $x^{k-1} = 0$ and this implies $x = 0$, and hence the result.

Now we obtain a characterization concerning strongly R-near-rings.

Theorem 5. *A near-ring N is subdirectly reducible strongly R-near-ring if and only if N is direct sum of two integral, simple and zero-symmetric near-rings.*

Let N be a subdirectly reducible strongly R-near-ring. Then N is reduced (Proposition 9) and hence N is completely reducible in integral, zero-symmetric near-rings (Corollary 1).

Moreover, the non-trivial ideals of N are completely prime and N satisfies condition 2 of Th. 1 of [3], therefore N is completely reducible in two integral zero-symmetric ideals. The converse is obvious.

Proposition 10. *Let N be a non-simple subdirectly irreducible strongly R-near-ring. Then the set $Q(N)$ of nilpotent elements of N is not trivial and it is contained in the minimal ideal I of N. Moreover, if N/I is not simple, it is not an R-near-ring.*

Since N is a non-simple subdirectly irreducible near-ring, N contains a minimum ideal I, $I \neq 0$.

Moreover, N has nonzero nilpotent elements. In fact, if N is reduced, N is a direct sum of integral near-rings (cfr. [1]), and hence N is integral and simple (Proposition 5), which is excluded by hypothesis. Yet N/I is integral, because I is completely prime (Proposition 8), and thus $Q(N) \subseteq I$.

Suppose that N/I is not simple. Let J' be a proper ideal of N/I and J the ideal of N such that $J' = J/I$. Since $L(J') = \{[x] \in N/I \colon [x]J' = I\}$, it follows that $xj \in I$, for all $j \in J$, $x \in L(J')$, and hence $x \in I$ because $J \setminus I$ contains (at least) one element j. Thus $L(J') = I$ and J' equals the right annihilator of zero, but this result is excluded in an R-near-ring if $J' \neq N/I$.

Recall that the set $\mathscr{S}$ of all ideals of N is *linearly ordered* if, for any $A, B \in \mathscr{S}$, $A \subseteq B$ or $B \subseteq A$ and N is said *irreducible* if $nN = 0$ or $nN = N$, for all $n \in N$.

Proposition 11. *Let N be an irreducible strongly R-near-ring. Then all proper ideals of N are modular.*

Let I be a proper ideal of N. Then there exists $A \subseteq N^{*}$ such that $I = R(x)$, for any $x \in A \setminus \{0\}$. It follows $xI = 0$ and $xN = N$ (because $xN = 0$ implies $N \subseteq R(x)$, which is a contradiction). Thus I is modular by Th. 3.23, p. 84 of [8].

We conclude with a result, which follows easily from Propositions 8, 10 and Th. 1 of [3].

Theorem 6. *Let N be a non-simple subdirectly irreducible strongly R-near-ring. Then all ideals of N are linearly ordered, and N is a non-integral near-ring, all whose proper factors are integral.*

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Summary

See Introduction.

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