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The necessity of the Wiener test for some nonlinear elliptic equations with quadratic growth in the gradient (**)

A Bianca Manfredi con amicizia e stima

1 - Introduction

Let Ω be a bounded open set in $\boldsymbol{R}^N$ and let $a_{ij}(x)$ be bounded measurable function on Ω such that $a_{ij} = a_{ji}$ and

$$\lambda|\xi|^2 \leqslant \sum_{ij=1}^{N} a_{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2 \tag{1.1}$$

for every $\xi \in R^n$ and a.e. in Ω. The bilinear form $a(u, v)$ on $H^1(\Omega)$ is defined by

$$a(u, v) = \sum_{ij=1}^{N} \int_\Omega a_{ij}\partial_{x_i}u\,\partial_{x_j}v\,\mathrm{d}x\,.$$

Moreover let $f(x, z, q)$ be a function on $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^N$ continuous in (z, q) for each fixed x and measurable in x for each fixed (z, q), such that $f(x, z, q) \leqslant a + b|q|^2$ for $q \in \boldsymbol{R}^n$, $z \in \boldsymbol{R}$ and for a.e. $x \in \Omega$.

We say that u is a bounded weak solution of the problem

$$-\sum_{ij}^{N} \partial_{x_j}(a_{ij}\partial_{x_i}u) = f(x, u, Du) \tag{1.2}$$

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(Du denotes the gadient of u) if $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and

$$a(u, \psi) = \int_\Omega f(x, u, Du)\, \psi\, dx \tag{1.3}$$

for every $\psi \in H_0^1(\Omega)$.

Let now Φ be a function in $H^1(\Omega) \cap L^\infty(\Omega)$ we say that u is a bounded weak solution of (1.3) with boundary value Φ if $(u - \Phi) \in H_0^1(\Omega)$.

Consider now a point $x_0 \in \partial\Omega$ we say that Φ is continuous at x_0 with respect to $\partial\Omega$ if $\mathrm{cap}(B(x_0, R) \cap \partial\Omega) > 0$ for some $R > 0$ ($\mathrm{cap}(E, O)$ denotes the usual capacity associated to the bilinear form a and to the open set O, we don't write O if $O = \boldsymbol{R}^N$) and q.e.-$\mathrm{osc}_{B(x_0, r) \cap \partial\Omega}\, \Phi$ converges to 0 as $r \to 0$. In this case we can take as value of Φ at x_0 the q.e.-limit of Φ when $x \in \partial\Omega$ converges to x_0.

We say that x_0 is a *regular* (*local*) *point* for $\partial\Omega$ with respect to (1.2) if for every open set $\Omega' \subseteq \Omega$ with $x_0 \in \partial\Omega \cap \partial\Omega'$ and every bounded weak solution u of (1.2) with boundary data $\Phi \in H^1(\Omega') \cap L^\infty(\Omega')$ on $\partial\Omega'$ continuous at x_0 we have

$$\lim_{x \to x_0,\, x \in \Omega'} u = \Phi(x_0).$$

We recall that there is a well known criterion, namely the *Wiener criterion*, [7] [5], for the regularity of x_0 in the case $f = 0$.

We recall here the above criterion

Theorem 1. *Let $f = 0$ and*

$$\delta(r) = \frac{\mathrm{cap}(\Omega^c \cap B(x_0, r), B(x_0, 2r))}{\mathrm{cap}(B(x_0, r), B(x_0, 2r))}. \tag{1.4}$$

Then a necessary and sufficient condition for the regularity of x_0 with respect to $\partial\Omega$ is that

$$\int_0^{R_0} \delta(r) \frac{dr}{r} = +\infty \tag{1.5}$$

where R_0 is positive and fixed.

The result in Theorem 1 was proved in the case of the Laplace operator by N. Wiener in 1924 [8], and after different extension the complete result of the above Theorem was proved in 1963 by Littman-Stampacchia-Weinberg [5]. For the nonlinear problem (1.2) it has been proved in [3] (in a more general framework) that (1.4) is again a sufficient for the regularity of x_0.

In a recent paper Adams and Heard [1] have proved that (1.4) is necessary for the regularity of x_0 also in the nonlinear case under the additional assumption of Dini-continuity of the coefficients a_{ij} (see also [7] where the monotone case with irregular coefficient is studied).

The purpose of this paper is remove this last assumption and extend the necessary part of Theorem 1 in full generality.

Theorem 2. *Let $x_0 \in \partial\Omega$; (1.5) is a necessary and sufficient condition for the regularity of x_0.*

We recall that the sufficient part of Theorem 2 has been proved in [3]; then it is enough to prove the necessary part of Theorem 2. This extension is founded, roughly speaking, on a reduction to a linear case. This method work also in other cases and in particular in the case of nonlinear elliptic problems with a weight in the A_2 Muckenhoupt's class.

2 - Reduction to a linear problem

In this section we denote by O a bounded open set in $\boldsymbol{R}^N$ such that, denoted by $\lambda_1(O)$ the first eigenvalue of the Laplace operator in $H_0^1(O)$, we have $\dfrac{ab}{\lambda^2} < \lambda_1(O)$.

Proposition 1. *Let u be a bounded weak solution of (1.3) in O with boundary data Φ. Then the functions $\exp(\pm \frac{b}{\lambda} u)$ are subsolutions of the problem*

$$\sum_{ij=1}^{N} \int_O a_{ij}\,\partial_{x_i} V^{\pm}\,\partial_{x_j} v\,dx - \int_O \frac{ab}{\lambda} V^{\pm} v\,dx = 0 \qquad \forall v \in H_0^1(O) \tag{2.1}$$

with boundary value $\exp(\pm \frac{b}{\lambda} \Phi)$.

Proof. The proof follows by easy computations. Let v be in $H_0^1(O) \cap L^\infty(O)$ with $v \geq 0$, we have

$$\sum_{ij=1}^{N} \int_O a_{ij}\,\partial_{x_i}(\exp(\pm \frac{b}{\lambda} u))\,\partial_{x_j} v\,dx = \pm \frac{b}{\lambda} \sum_{ij=1}^{N} \int_O \exp(\pm \frac{b}{\lambda})\, a_{ij}\,\partial_{x_i} u\,\partial_{x_j} v\,dx$$

$$\pm \frac{b}{\lambda} \sum_{ij=1}^{N} \int_O a_{ij}\,\partial_{x_i} u\,\partial_{x_j}(\exp(\pm \frac{b}{\lambda} u)v)\,dx - (\frac{b}{\lambda})^2 \sum_{ij=1}^{N} \int_O \exp(\pm \frac{b}{\lambda} u)\, v a_{ij}\,\partial_{x_i} u\,\partial_{x_j} v\,dx$$

$$\leqslant \frac{ab}{\lambda} \int_O (\exp(\pm \frac{b}{\lambda} u) v \, dx + \int_O |Du|^2 (\exp(\pm \frac{b}{\lambda} u) v) \, dx$$

$$- (\frac{b}{\lambda})^2 \sum_{ij=1}^{N} \int_O \exp(\pm \frac{b}{\lambda} u) v a_{ij} \partial_{x_i} u \, \partial_{x_j} v \, dx \leqslant \frac{ab}{\lambda} \int_O \exp(\pm \frac{b}{\lambda} u) v \, dx .$$

Denote now by $V^{\pm}$ the solutions of (2.1) with boundary data $\exp(\pm \frac{b}{\lambda} \Phi)$ and define $u^{\pm} = \pm \frac{\lambda}{b} \log V^{\pm}$; then if u is a bounded weak solution of (1.3) with boundary data Φ we have

$$u^- \leqslant u \leqslant u^+ . \tag{2.2}$$

Proposition 2. *The problem* (1.3) *with boundary data* Φ *has at least one solution* u *and* $u^- \leqslant u \leqslant u^+$.

Proof. We only give the sketch of the different steps of the proof.

1. We regularize f by $f_\varepsilon = \dfrac{f}{1 + \varepsilon f}$ where $\varepsilon = \dfrac{1}{n}$ and we denote by P and P_ε the boundary problems relative to f and f_ε. It is easy to see that

$$f_\varepsilon(x, z, q) \leqslant f(x, z, q) .$$

The existence of a bounded weak solution on P_ε can be proved using a fixed point method in $H^1_{loc}(O)$ taking into account the local C^α estimate for the linear problem and the global L^∞ estimate for the global problem.

From (2.2) we obtain $\qquad u^- \leqslant u_\varepsilon \leqslant u^+$

a.e. in O; then the sequence u_ε is uniformly bounded with respect to ε.

2. From [4] we obtain easily that the sequence u_ε is bounded in C^α_{loc} uniformly in ε.

3. At least after extraction of subsequences we that $u_\varepsilon - \Phi$ is weakly convergent in $H^1_0(O)$ and strongly convergent in $L^p(O)$, $1 < p < \infty$ to u. Moreover from the C^α_{loc} estimate we can assume that u_ε converges to u in $L^\infty_{loc}(O)$; then u_ε converges to u strongly in $H^1_{loc}(O)$. The convergences in 3 prove that u is a solution of (1.3) in O with boundary value Φ.

Now we recall the well known results on the estimates for the Green function, see for example [5] in the uniformly elliptic case ($ab = 0$), which can be easily adapted to our framework using the methods in [2].

Proposition 3. *For every $x \in O$ there exists a Green function for* (2.1) *with singularity at x denoted by G^x_O. Moreover choosing $O = B(x, R)$ we have*

$$G^x_{B(x,R)} \approx r^{2-N} \qquad on\ \partial B(x, r)$$

for $r \leq \frac{R}{2}$. Moreover if we denote by $G^x_{\rho, B(x,R)}$ the regularized Green function (the definition is analogous to the one in [6] *for the case $ab = 0$) we have also*

$$G^x_{\rho, B(x,R)} \approx r^{2-N} \qquad on\ \partial B(x, r)\ for\ 2\rho < r < R$$

and

$$\lim_{\rho \to 0} G^x_{\rho, B(x,R)} = G^x_{B(x,r)}$$

in $C^\alpha_{\text{loc}}(B(x, R) - \{x\}) \cap L^\infty(B(x, R) - \{x\})$.

Taking into account Propositions 1 and 2 the result will be proved if, in the case of convergence of the Wiener integral, we construct a solution V^+ of (2.2) relative to a boundary data $\Psi \in H^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, $\Psi > \varepsilon > 0$ and to the set $\Omega_r = \Omega \cap B(x_0, r)$ with

$$\lim_{x \to x_0,\, x \in \Omega_r} \Psi = 1 = \Psi(x_0)$$

such that

$$\liminf_{x \to x_0,\, x \in \Omega_r} V^+ < 1.$$

In fact if u is the solution of (1.3) in Ω_r with boundary data

$$\Phi = \frac{\lambda}{b} \log \Psi \in H^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$$

then

$$\liminf_{x \to x_0,\, x \in \Omega_r} u \leq \liminf_{x \to x_0,\, x \in \Omega_r} u^+ < 0 = \Phi(x_0)$$

where $u^+ = \frac{\lambda}{b} \log V^+$.

3 - Proof of Theorem 2

In this section all the potentials and Green functions are taken with respect to the form in (2.1) and we can assume without loss of generality $x_0 = 0$. We recall also that we assume again $\frac{ab}{\lambda^2} < \lambda_1(O)$.

Proposition 4. *Let μ be a bounded positive measure in $H^{-1}(B(2R)$, $O = B(2R)$, with support in $B(R)$. Let v_R be the potential of μ in $B(2R)$. Assume*

$$\int_0^{2R} \mu(B(\rho)) \frac{\rho^2}{m(B(\rho))} \frac{d\rho}{\rho} < +\infty . \tag{3.1}$$

Denote by G the Green function with singularity at 0 with respect to $B(2R)$. Then $G(x, 0)$ is integrable with respect to the measure μ and the value

$$\hat{v}_R(0) = \int_{B(2R)} G(x, 0)\mu(dx)$$

is well defined.

Moreover the limit $\quad v_R(0) = \lim_{\rho \to 0} \frac{1}{m(B(\rho))} \int_{B(\rho)} v_R(x)\, dx$

exists finite and

$$v_R(0) = \hat{v}_R(0) \leq C \int_0^{2R} \mu(B(\rho)) \frac{\rho^2}{m(B(\rho))} \frac{d\rho}{\rho} . \tag{3.2}$$

The proof is analogous to the proof of the same result in the usual uniformly elliptic case given in [**6**].

Proposition 5. *Let E_ρ, $\rho > 0$, be subsets of R^N such that*

$$E_r \cap B(\rho) \subseteq E_\rho \subseteq B(\rho) \subseteq B(r) \subseteq O$$

for every $0 < \rho < r$. Let μ_ρ be the capacitary measure of E_ρ in $B(2\rho)$; then for every $r > 0$ and $0 < \rho < r$ we have

$$\mu_r(B(\rho)) \leq \mu_\rho(\overline{B(\rho)}) .$$

Proof. Let w_ρ be the potential of E_ρ in $B(2\rho)$. We have

$$\sum_{ij}^{N} \int_O a_{ij}\, \partial_{x_i} w_\rho\, \partial_{x_j} w_\rho\, dx \geq \sum_{ij}^{N} \int_O a_{ij}\, \partial_{x_i} w_\rho\, \partial_{x_j} w_r\, dx$$

where $0 < \rho < r$.

We can rewrite the above relation using the capacitary measure and we obtain $\int_O w_\rho \mu_\rho(\mathrm{d}x) = \mu_\rho(\overline{B(\rho)}) \geqslant \int_O w_\rho \mu_r(\mathrm{d}x) \geqslant \mu_r(B(\rho))$.

The result is so proved.

Consider now the set $\Omega_{2r} = \Omega \cap B(2r)$. We observe that for 0, $r < R$, R suitable, we have $\frac{ab}{\lambda^2} < \lambda_1(\Omega_{2r})$ so we can use all the above results with $O = \Omega_{2r}$.

Proof of Theorem 2. Let us suppose

$$\int_0^R \delta(\rho)\,\frac{\mathrm{d}\rho}{\rho} < +\infty . \tag{3.3}$$

To prove Theorem 2 it is enough to prove that for a suitable r with $0 < r < R$ there exists w_r solution of (2.1) in Ω_{2r} with boundary data $\Psi \in H^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, $\Psi < \varepsilon > 0$, such that

$$\liminf_{x \to 0} w_r(x) < 1 . \tag{3.4}$$

If we prove that, denoted by v_r the potential of $\Omega^c \cap B(r)$ in $B(2R)$, we have

$$\liminf_{x \to 0} v_r(x) < 1 \tag{3.5}$$

the maximum principles gives that (3.4) also holds for $w_r = v_r + \varepsilon$ with ε small enough.

To prove (3.5) it is enough to prove that for r suitable we have

$$\lim_{\rho \to 0} \frac{1}{m(B(\rho))} \int_{B(\rho)} v_r\,\mathrm{d}x = v_r(0) < 1 . \tag{3.6}$$

Let μ_r be the capacitary measure of Ω_r with respect to $B(2r)$.

For every $r > 0$ we have $\operatorname{supp}(\mu_r) \subseteq \overline{B(r)}$ then form (3.3) and from Proposition 5 we obtain

$$\int_0^{2r} \mu_r(B(\rho))\,\frac{\rho^2}{m(B(\rho))}\,\frac{\mathrm{d}\rho}{\rho} < +\infty .$$

By Proposition 4 with $\mu = \mu_r$ we obtain

$$v_r(0) \leq C \int_0^{2r} \mu_r(B(\rho)) \frac{\rho^2}{m(B(\rho))} \frac{d\rho}{\rho} .$$

Then from Proposition 5 we have $v_r(0) \leq C \int_0^{2r} \delta(\rho) \frac{d\rho}{\rho}$.

By letting $r \to 0$, we obtain from (3.3)

$$\lim_{r \to 0} v_r(0) = 0 . \tag{3.7}$$

From (3.7) the relation (3.6) follows for suitable fixed r.

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Sommario

Si dimostra che la condizione di Wiener per la regolarità di un punto del contorno è necessaria anche nel caso di problemi quasi ellittici con crescita quadratica nel gradiente.

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