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Continuous dependence on modelling in penetrative convection with a nonlinear equation of state (**)

A Bianca Manfredi con amicizia e stima

1 - Introduction

Penetrative convection is a phenomenon whereby a convectively unstable layer of fluid is bounded by one (or more) stable layers. The motions in the unstable layer can penetrate deeply into the stable layer, and as such, the subject is of much importance in fields such as cloud physics, or studies of the structure of the interior of a star.

Various models of penetrative convection are reviewed in [18] and one of these consists of a layer of thermally conducting viscous fluid with a buoyancy law in which the density ρ is a nonlinear function of temperature Θ. Since there are a variety of choices for the $\rho(\Theta)$ behaviour, see [18], one wonders what effect the change in model has on the solution, since the same physical process is being described.

In this paper, therefore, we address the problem of examining the difference in solution behaviour between that for a model of convection employing a linear buoyancy law and that for a quadratic law. It will be clear how to treat the analogous problem of comparing quadratic against cubic, and various other models.

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The question of continuous dependence of the solution on changes in the model itself is one which has been receiving much recent attention. For example, such modelling questions are studied in a variety of contexts in Adelson [1], [2], Ames [3], [4], Bennett [5], Payne [8], [9], [10], Payne and Sather [12], Payne and Straughan [13], [14], [15], [16] and Song [17].

In this paper we show that a solution to the Navier-Stokes equations coupled with the convective heat equation, on a bounded domain, for the improperly posed backward in time problem, depends continuously on changes in the modelling of the buoyancy law, under the weak requirements that the perturbed velocity and temperature fields remain bounded, while the gradients of base velocity and temperature likewise remain bounded.

The proof we employ is based on a logarithmic convexity argument, cf. Payne [6], [7]; however, we here adapt a new variant developed by Payne [11] which enables us to obtain continuous dependence under the assumption of relatively weak bounds.

2 - The modelling problem

Without loss of generality, the equations for a heat conducting viscous fluid with a linear buoyancy law, backward in time, may be taken to be

$$(2.1)\qquad v_{i,t} = v_j v_{i,j} + p_{,i} - \Delta v_i - b_i \Theta \qquad v_{i,i} = 0 \qquad \Theta_{,t} = v_j \Theta_{,j} - \Delta\Theta$$

where v_i, p, Θ, b_i are velocity, pressure, temperature, and gravity. Equations (2.1) are defined on a bounded spatial domain $\Omega \subset \boldsymbol{R}^3$, and the time domain is $(0, T]$, for some $T < \infty$. On the boundary, Γ, we assume

$$(2.2)\qquad v_i(\boldsymbol{x}, t) = \hat{v}_i(\boldsymbol{x}, t) \qquad \Theta(\boldsymbol{x}, t) = \hat{\Theta}(\boldsymbol{x}, t)$$

where $\hat{v}_i$, $\hat{\Theta}$ are prescribed. The initial data are also given

$$(2.3)\qquad v_i(\boldsymbol{x}, 0) = h_i(\boldsymbol{x}) \qquad \Theta(\boldsymbol{x}, 0) = \Phi(\boldsymbol{x})$$

for prescribed functions h_i, Φ.

The equivalent problem for a heat conducting viscous fluid, but with a quadratic buoyancy law is

$$(2.4)\qquad v^*_{i,t} = v^*_j v^*_{i,j} + p^*_{,i} - \Delta v^*_i - b_i \Theta^* - \varepsilon b_i \Theta^{*2} \qquad v^*_{i,i} = 0 \qquad \Theta^*_{,t} = v^*_j \Theta^*_{,j} - \Delta\Theta^*$$

where ε is a measure of deviation of (2.4) from (2.1). Equations (2.4) are also defined on $\Omega \times (0, T]$.

Since we wish to study continuous dependence on changes in the model itself we suppose (v_i, Θ, p) and (v_i^*, Θ^*, p^*) satisfy the *same* boundary data (2.2) and the *same* initial data (2.3). Our aim is to show the difference in the solutions of (2.4) and (2.1) depends continuously on ε.

Define the difference variables u_i, π and θ by $u_i = v_i^* - v_i$, $\pi = p^* - p$, $\theta = \Theta^* - \Theta$. The solution (u_i, θ, π) may then be shown to satisfy the system of equations

$$\begin{aligned} u_{i,t} &= v_j^* u_{i,j} + u_j v_{i,j} + \pi_{,i} - \Delta u_i - b_i \theta - \varepsilon b_i \Theta^{*2} \qquad u_{i,i} = 0 \\ \theta_{,t} &= v_j^* \theta_{,j} + u_j \Theta_{,j} - \Delta\theta \end{aligned} \tag{2.5}$$

on the domain $(\boldsymbol{x}, t) \in \Omega \times (0, T]$, together with the boundary and initial conditions

$$\begin{aligned} u_i(\boldsymbol{x}, t) &= 0 \qquad \theta(\boldsymbol{x}, t) = 0 \qquad \boldsymbol{x} \in \Gamma, \; t \in [0, T] \\ u_i(\boldsymbol{x}, 0) &= 0 \qquad \theta(\boldsymbol{x}, 0) = 0 \qquad \boldsymbol{x} \in \Omega . \end{aligned} \tag{2.6}$$

We further assume the solutions v_i, Θ, v_i^*, Θ^* satisfy the bounds

$$|\nabla \boldsymbol{v}|, \quad |\nabla \Theta|, \quad |\boldsymbol{v}^*|, \quad |\Theta^*|^2 \leq M \tag{2.7}$$

for a known constant M. The restriction (2.7) defines the constraint set which is typically necessary in an improperly posed problem, see Payne [7]. We also suppose, without loss of generality, $|\boldsymbol{b}(\boldsymbol{x}, t)| \leq 1$.

3 - Continuous dependence on the buoyancy law

To apply the method of logarithmic convexity we define the functional $F(t)$ by

$$F(t) = \int_0^t (\|\boldsymbol{u}\|^2 + \|\theta\|^2)\, ds + \varepsilon^2 , \tag{3.1}$$

where $\|\cdot\|$ denotes the norm on $L^2(\Omega)$. We also denote integration over Ω by $\langle \cdot \rangle$.

Differentiate F to see that

$$F'(t) = \|\boldsymbol{u}\|^2 + \|\theta\|^2 = 2\int_0^t (\langle u_i u_{i,s}\rangle + \langle \theta\theta_{,s}\rangle)\, ds$$

$$(3.2) \qquad = 2\int_0^t \langle u_i u_j v_{i,j}\rangle\, ds - 2\int_0^t \langle u_i b_i \theta\rangle\, ds + 2\int_0^t \|\nabla \boldsymbol{u}\|^2\, ds$$

$$+2\int_0^t \langle \theta u_i \Theta_{,i}\rangle\, ds - 2\varepsilon\int_0^t \langle u_i b_i \Theta^{*2}\rangle\, ds + 2\int_0^t \|\nabla\theta\|^2\, ds$$

where in obtaining (3.2) we have used (2.5) and (2.6). Next, differentiate again, integrate by parts and use the boundary conditions $(2.6)_1$ to derive

$$F'' = 2\langle u_i u_j v_{i,j}\rangle - 2\langle u_i b_i \theta\rangle - 2\varepsilon\langle b_i \Theta^{*2} u_i\rangle + 2\langle \theta u_i \Theta_{,i}\rangle$$

$$-4\int_0^t \langle u_{i,s} \Delta u_i\rangle\, ds - 4\int_0^t \langle \theta_{,s} \Delta\theta\rangle\, ds$$

$$(3.3) \qquad = 2\langle u_i u_j v_{i,j}\rangle - 2\langle u_i b_i \theta\rangle - 2\varepsilon\langle b_i \Theta^{*2} u_i\rangle + 2\langle \theta u_i \Theta_{,i}\rangle$$

$$+4\int_0^t \langle u_{i,s}(u_{i,s} - v_j^* u_{i,j} - u_j v_{i,j} + b_i\theta + \varepsilon b_i \Theta^{*2})\rangle\, ds$$

$$+4\int_0^t \langle \theta_{,s}(\theta_{,s} - v_j^* \theta_{,j} - u_j \Theta_{,j}\rangle\, ds\,.$$

We next introduce the variables, cf. Payne [6]

$$(3.4) \qquad a_i = u_{i,t} - \frac{1}{2}\, v_j^* u_{i,j} \qquad \phi = \theta_{,t} - \frac{1}{2}\, v_j^* \theta_{,j}$$

so that equation (3.3) may be rewritten

$$F'' = 2\langle u_i u_j v_{i,j}\rangle - 2\langle u_i b_i \theta\rangle - 2\varepsilon\langle b_i \Theta^{*2} u_i\rangle + 2\langle u_i \Theta_{,i} \theta\rangle$$

$$-4\int_0^t \langle u_{i,s} u_j v_{i,j}\rangle\, ds - 4\int_0^t \langle \theta_{,s} u_i \Theta_{,i}\rangle\, ds$$

$$(3.5) \qquad +4\int_0^t \langle u_{i,s} b_i \theta\rangle\, ds + 4\varepsilon\int_0^t \langle u_{i,s} b_i \Theta^{*2}\rangle\, ds$$

$$+4\int_0^t (\|a_i\|^2 + \|\phi\|^2)\, ds - \int_0^t \langle v_j^* u_{i,j} v_k^* u_{i,k}\rangle\, ds - \int_0^t \langle (v_i^* \theta_i)^2\rangle\, ds\,.$$

To employ the method of logarithmic convexity it is necessary to form the expression $FF'' - (F')^2$, and so we derive

$$(3.6)\qquad FF'' - (F')^2 = 4S^2 + 4\varepsilon^2 \int_0^t (\|\boldsymbol{a}\|^2 + \|\phi\|^2)\, ds + \sum_{\alpha=1}^{6} I_\alpha + \sum_{\alpha=1}^{4} J_\alpha$$

in which S^2 has been defined by

$$(3.7)\qquad S^2 = \int_0^t (\|\boldsymbol{u}\|^2 + \|\theta\|^2)\, ds \int_0^t (\|\boldsymbol{a}\|^2 + \|\phi\|^2)\, ds - \Big(\int_0^t (\langle u_i a_i\rangle + \langle \theta\phi\rangle)\, ds\Big)^2$$

and is a non-negative quantity by virtue of the Cauchy-Schwarz inequality. The I_α and J_α terms are given below:

$$I_1 = 2F\langle u_i u_j v_{i,j}\rangle \qquad I_2 = -2F\langle u_i b_i \theta\rangle \qquad I_3 = -2\varepsilon F\langle b_i \Theta^{*2} u_i\rangle$$

$$I_4 = 2F\langle \theta u_i \Theta_{,i}\rangle \qquad I_5 = -F\int_0^t \langle v_j^* u_{i,j} v_m^* u_{i,m}\rangle\, ds \qquad I_6 = -F\int_0^t \|v_i^* \theta_{,i}\|^2\, ds$$

$$J_1 = -4F\int_0^t \langle u_{i,s} u_j v_{i,j}\rangle\, ds \qquad J_2 = -4F\int_0^t \langle \theta_{,s} u_i \Theta_{,i}\rangle\, ds$$

$$J_3 = 4F\int_0^t \langle b_i \theta u_{i,s}\rangle\, ds \qquad J_4 = 4\varepsilon F\int_0^t \langle b_i \Theta^{*2} u_{i,s}\rangle\, ds\,.$$

A useful inequality which we have recourse to employ later may be obtained from (3.2) and (3.7), namely

$$(3.8)\qquad \sqrt{\int_0^t (\|\boldsymbol{u}\|^2 + \|\theta\|^2)\, ds \int_0^t (\|\boldsymbol{a}\|^2 + \|\phi\|^2)\, ds} = \sqrt{S^2 + \frac{(F')^2}{4}} \leqslant S + \frac{F'}{2}\,.$$

The I_α terms are estimated with the aid of the bounds (2.7) as follows

$$I_1 \geqslant -2MF\|\boldsymbol{u}\|^2 \geqslant -2MFF' \qquad I_2 \geqslant -F(\|\boldsymbol{u}\|^2 + \|\theta\|^2) \geqslant -FF'$$

$$I_3 \geqslant -2\varepsilon MF\langle |\boldsymbol{u}|\rangle \geqslant -MF(\varepsilon^2 V + \|\boldsymbol{u}\|^2) \geqslant -MVF^2 - MFF'$$

where V is the volume of Ω

$$I_4 \geqslant -MFF' \qquad I_5 \geqslant -M^2 F\int_0^t \|\nabla \boldsymbol{u}\|^2\, ds \qquad I_6 \geqslant -M^2 F\int_0^t \|\nabla\theta\|^2\, ds\,.$$

For later use we employ (2.7) in (3.2) to derive

$$-\int_0^t (\|\nabla \boldsymbol{u}\|^2 + \|\nabla\theta\|^2)\,\mathrm{d}s \geqslant -\frac{1}{2}F' - h_1 F \tag{3.9}$$

where $h_1 = \max\{2M + \frac{1}{2}, \frac{1}{2}MVT\}$.

The lower bounds for I_α are now used in (3.6) and (3.9) is further employed to find

$$\begin{aligned} FF'' - (F')^2 &\geqslant 4S^2 + 4\varepsilon\int_0^t (\|\boldsymbol{a}\|^2 + \|\phi\|^2)\,\mathrm{d}s \\ &-M(V + h_1 M)F^2 - (\frac{1}{2}M^2 + 4M + 1)FF' + \sum_{\alpha=1}^{4} J_\alpha . \end{aligned} \tag{3.10}$$

We now estimate the J_α terms. This involves use of the Cauchy-Schwarz and arithmetic-geometric mean inequalities, together with the constraint set (2.7)

$$J_1 \geqslant -4MF\sqrt{\int_0^t \|\boldsymbol{u}\|^2 \int_0^t \|\boldsymbol{a}\|^2\,\mathrm{d}s} - M^2F(\int_0^t \|\nabla\boldsymbol{u}\|^2\,\mathrm{d}s + \int_0^t \|\boldsymbol{u}\|^2\,\mathrm{d}s) \tag{3.11}$$

$$\begin{aligned} J_2 &= -4F\int_0^t \langle \phi u_i \Theta_{,i}\rangle\,\mathrm{d}s - 2F\int_0^t \langle v_k^* \theta_{,k} u_i \Theta_{,i}\rangle\,\mathrm{d}s \\ &\geqslant -4MF\sqrt{\int_0^t \|\theta\|^2\,\mathrm{d}s \int_0^t \|\boldsymbol{u}\|^2\,\mathrm{d}s} - M^2F(\int_0^t \|\nabla\theta\|^2\,\mathrm{d}s + \int_0^t \|\boldsymbol{u}\|^2\,\mathrm{d}s) \end{aligned} \tag{3.12}$$

$$\begin{aligned} J_3 &= 4F\int_0^t \langle b_i \theta a_i\rangle\,\mathrm{d}s + 2F\int_0^t \langle b_i \theta v_j^* u_{i,j}\rangle\,\mathrm{d}s \\ &\geqslant -4F\sqrt{\int_0^t \|\theta\|^2\,\mathrm{d}s \int_0^t \|\boldsymbol{a}\|^2\,\mathrm{d}s} - MF(\int_0^t \|\theta\|^2\,\mathrm{d}s + \int_0^t \|\nabla\boldsymbol{u}\|^2\,\mathrm{d}s) \end{aligned} \tag{3.13}$$

$$\begin{aligned} J_4 &= 4\varepsilon F\int_0^t \langle b_i \Theta^{*2} a_i\rangle\,\mathrm{d}s + 2\varepsilon F\int_0^t \langle b_i \Theta^{*2} v_j^* u_{i,j}\rangle\,\mathrm{d}s \\ &\geqslant -4MF\sqrt{V\varepsilon^2 T\int_0^t \|\boldsymbol{a}\|^2\,\mathrm{d}s} - M^2VTF\varepsilon^2 - M^2F\int_0^t \|\nabla\boldsymbol{u}\|^2\,\mathrm{d}s . \end{aligned} \tag{3.14}$$

The estimates (3.11)-(3.14) are now employed simultaneously and with further use of the arithmetic-geometric mean inequality and (3.9) we may deduce

$$\sum_{\alpha=1}^{4} J_\alpha \geqslant -c_1 F^2 - c_2 FF' - 4\varepsilon^2 \int_0^t \|\boldsymbol{a}\|^2 ds \tag{3.15}$$
$$- 4\sqrt{2}(M+1) F \sqrt{\int_0^t (\|\boldsymbol{u}\|^2 + \|\theta\|^2) ds \int_0^t (\|\boldsymbol{a}\|^2 + \|\phi\|^2) ds}$$

where $c_1 = 2M^2(VT + h_1 + 1) + M(1 + h_1) \qquad c_2 = M(M + \frac{1}{2})$.

The last term in (3.15) is now bounded with the aid of (3.8) to derive

$$\sum_{\alpha=1}^{4} J_\alpha \geqslant -c_1 F^2 - c_3 FF' - c_4 FS - 4\varepsilon^2 \int_0^t \|\boldsymbol{a}\|^2 ds \tag{3.16}$$

where $c_3 = c_2 + \sqrt{2}(M+1) \qquad c_4 = 4\sqrt{2}(M+1)$.

Inequality (3.16) is next employed in (3.10), where we observe that we have deliberately arranged that the $\|\boldsymbol{a}\|^2$ term in (3.16) is balanced by that in (3.10), and we obtain

$$FF'' - (F')^2 \geqslant 4S^2 - c_5 F^2 - c_6 FF' - c_4 FS \tag{3.17}$$

where $c_5 = c_1 + M(V + h_1 M) \qquad c_6 = c_3 + \frac{1}{2} M^2 + 4M + 1$.

We complete the square in (3.17) to finally arrive at

$$FF'' - (F')^2 \geqslant -k_1 FF' - k_2 F^2 \tag{3.18}$$

where k_1, k_2 are given by $k_1 = c_6 \qquad k_2 = c_5 + \frac{c_4^2}{16}$.

Inequality (3.18) is integrated, cf. Payne [7], by putting $\sigma = e^{-k_1 t}$ to find

$$\frac{d^2}{d\sigma^2}(\log F(\sigma)) + \frac{k_2}{k_1^2 \sigma^2} \geqslant 0 \tag{3.19}$$

or, alternatively

$$\frac{d^2}{d\sigma^2}\{\log[F(\sigma)\,\sigma^{k_2 k_1^{-2}}]\} \geqslant 0. \tag{3.20}$$

We now identify the time interval $[0, T]$ with $[\sigma_2, \sigma_1]$, so that $\sigma_1 = 1$, $\sigma_2 = e^{-k_1 T}$, and the basic convexity inequality (3.20) allows us to deduce

$$F(t) \leqslant [F(0)]^{(\sigma - \sigma_2)(1-\sigma_2)^{-1}} [F(T)\, e^{\mu T}]^{(1-\sigma)(1-\sigma_2)^{-1}}\, e^{-\mu t} \tag{3.21}$$

where $\mu = \dfrac{k_2}{k_1}$.

To use (3.21) to establish continuous dependence on the modelling, i.e. on ε, we suppose a bound is known for $F(T)$, for example, we assume $F(T)\, e^{\mu T} \leqslant K$. Then, from (3.21) we see that

$$\int_0^t (\|\boldsymbol{u}\|^2 + \|\theta\|^2)\, ds \leqslant e^{-\mu t} K^{(1-\sigma)(1-\sigma_2)^{-1}}\, \varepsilon^{2(\sigma - \sigma_2)(1-\sigma_2)^{-1}}. \tag{3.22}$$

Inequality (3.22) clearly establishes Hölder continuous dependence of the solution on compact subintervals of $[0, T)$, and continuous dependence on the buoyancy is achieved.

Remark. We have here only considered the question of changes in the buoyancy from a linear to a quadratic law. In [18] various buoyancy laws which have been used in the literature are reviewed, such as cubic, fifth and sixth order ones, and others. We could, for example, consider the equivalent modelling problem of comparing the quadratic and cubic models. The details are easily derived following the lines presented here.

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Sommario

Viene analizzato l'andamento della soluzione di un problema di convenzione termica in un fluido viscoso conduttore del calore, al variare della legge di forza.

Per il problema (mal posto) «all'indietro» si mostra che la soluzione dipende con continuità (nel senso di Hölder) dai cambiamenti del modello.

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