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Capacities for Dirichlet forms (**)

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Introduction

The main object of this paper is to obtain various equivalent expressions for the variational capacity relative to Dirichlet forms and give an integral representation for the correspondent equilibrium potential.

Section 1 is devoted to expose the essential notions and properties of the variational capacity relative to Dirichlet forms. It summarizes some important results obtained on this subject by M. Fukushima [4].

Along classical lines (cfr. V. G. Maz'ja [7]), in Section 2 equivalent expressions for the variational capacity are showed.

Section 3 treats the integral representation for the potentials of measures of finite energy integrals, in terms of the Green's function. Additional hypotheses are required in this section to make use of some results proved on this subject by M. Biroli and U. Mosco [1].

Finally the concern of Section 4 is with another expression for the variational capacity in the particular case of square Hormander's operators.

1 - Preliminary hypotheses and properties

Let X be a locally compact, Hausdorff topological vector space and let m be a positive Radon measure with $\operatorname{supp} m = X$. Consider the Hilbert space

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$H = L^2(X, m)$ and let a be a *Dirichlet form* on H, that is $a(u, v)$ is a form defined on a dense subspace $D[a]$ of H, which satisfies the properties:

(1.1) *$a(u, v)$ is bilinear, symmetric and positive definite,*

(1.2) *$a(u, v)$ is closed, that is $D[a]$ is complete with respect to the intrinsic norm $a_1(u, u)^{\frac{1}{2}} = (a(u, u) + (u, u))^{\frac{1}{2}}$, where $(\cdot, \cdot)$ denotes the inner product of H,*

(1.3) *$a(u, v)$ is markovian, according to* [4].

The structure $(D[a], a)$ is said a *Dirichlet space.*

Let $C_0(X)$ be the space of all continuous functions u on X, with $\operatorname{supp} u \subset X$.

A *core* of a is by definition, a subset C_1 of $D[a] \cap C_0(X)$, dense in $D[a]$ with a_1 norm and dense in $C_0(X)$ with uniform norm. A further assumption on a is

(1.4) *$a(u, v)$ is «regular», i.e. a possesses a core C_1.*

In this case we say $(D[a], a)$ to be a *regular Dirichlet space.*

Now we expound some consequence of (1.1), (1.2), (1.3), (1.4), proved in [4].

To make exposition easier we put $\mathfrak{F} = D[a]$, so we deal with the Hilbert space $(\mathfrak{F}, a_1)$. Let Θ be the family of all open subsets of X. We define, for $A \in \Theta$ the set $\Lambda_A = \{u \in \mathfrak{F} \mid u \geq 1 \text{ m-a.e. on } A\}$ and

(1.5) $$\begin{aligned} \operatorname{cap}_1(A) &= \inf a_1(u, u) \qquad \text{for } u \in \Lambda_A \text{ and } \Lambda_A \neq \emptyset, \\ \operatorname{cap}_1(A) &= +\infty \qquad \text{if } \Lambda_A = \emptyset. \end{aligned}$$

For any set $Q \subset X$, we put

(1.6) $$\operatorname{cap}_1(Q) = \inf_{A \in \Theta,\, Q \subset A} \operatorname{cap}_1(A).$$

The capacity defined by (1.5), (1.6) is a *Choquet capacity* (see [4], Theorem 3.1.1).

The notions of property *quasi-everywhere* (q.e.) valid and of *quasi-continuous* (q.c.) function are standard. Given two functions u and v defined on X, we say that v is a *quasi-continuous modification* (q.c.m.) of u, if v is q.c. and $u = v$ m-a.e. Every $u \in \mathfrak{F}$ admits a q.c.m. on X (see [4], p. 65, Theorem 3.1.3) and, if two q.c. functions u, v coincide m-a.e. on X, then they coincide also q.e. ([4], Lemma 3.1.4).

For any Borel set B

$$\text{(1.7)}\qquad \begin{aligned} &\mathrm{cap}_1^0(B)=\inf a_1(u,u), \text{ for } u\in\Lambda_B^0 \text{ if } \Lambda_B^0=\{u\in\mathfrak{F}\,|\,\tilde{u}\geq 1 \text{ q.e. on } B\}\neq\emptyset \\ &\mathrm{cap}_1^0(B)=+\infty \text{ if } \Lambda_B^0=\emptyset \end{aligned}$$

and the unique element $e_B^0\in\mathfrak{F}$ such that $a_1(e_B^0, e_B^0)=\mathrm{cap}_1^0(B)$ verifies also the properties (cfr. [4], p. 75)

$$\text{(1.8)}\qquad e_B^0=1 \text{ q.e. on } B \qquad a_1(e_B^0, v)\geq 0 \text{ for any } v\in\mathfrak{F},\ \tilde{v}\geq 0 \text{ q.e. on } B$$

where $\tilde{v}$ denotes a q.c.m. of v.

Moreover ([4], Theorem 3.3.1)

$$\text{(1.9)}\qquad \mathrm{cap}_1(B)=\mathrm{cap}_1^0(B) \quad \text{for any Borel set } B\,.$$

A positive Radon measure μ on X is said to be of *finite energy integral* if

$$\text{(1.10)}\qquad \int_X |v(x)|\,\mathrm{d}\mu(x)\leq c\sqrt{a_1(v,v)} \quad \text{for any } v\in\mathfrak{F}\cap C_0(X)$$

where c is a positive constant. Let S_1 be the set of all measures of finite energy integrals. Then (cfr. [4], Lemma 3.2.2 and Theorem 3.2.2)

(1.11) *for any $\mu\in S_1$ there exists a unique function $U_1^\mu\in\mathfrak{F}$ such that*

$$a_1(U_1^\mu, v)=\int_X \tilde{v}(x)\mu(\mathrm{d}x) \quad \text{for any } v\in\mathfrak{F}\,.$$

The function U_1^μ is called the (1-)*potential of the measure* μ. If $\mu\in S_1$, then

$$\text{(1.12)}\qquad a_1(\mu)=a_1(U_1^\mu, U_1^\mu)=\int_X \tilde{U}_1^\mu(x)\mu(\mathrm{d}x)$$

and we call $a_1(\mu)$ the (1-)*energy integral* of μ.

For any Borel set $B\subset X$, there exists a unique measure $\nu_B\in S_1$ supported on $\bar{B}$, such that e_B^0 is the (1-)*potential* of ν_B. ν_B is called the (1-)*equilibrium measure* of B and $e_B^0=U_1^{\nu_B}$ is called the (1-)*equilibrium potential* of B (cf. (1.8) and [4], Lemma 3.3.1). Therefore

$$\text{(1.13)}\qquad \mathrm{cap}_1(B)=a_1(e_B^0, e_B^0)=a_1(U_1^{\nu_B}, U_1^{\nu_B})=\int_X \tilde{U}_1^{\nu_B}(x)\nu_B(\mathrm{d}x)=\nu_B(\bar{B})\,.$$

A Dirichlet space $(\mathfrak{F}, a)$ relative to $L^2(X, m)$ is said to be *transient*, if there exists a bounded m-integrable function g, that is strictly positive m-a.e. on X and such that

$$\int_X |u| g\, m(\mathrm{d}x) \leq \sqrt{a(u, u)} \quad \text{for any } u \in \Im . \tag{1.14}$$

The function g is called a *reference function* of the transient Dirichlet space $(\Im, a)$.

It is worth remarking that $(\Im, a_1)$ is a Hilbert space, whereas usually $(\Im, a)$ is not even a pre-Hilbert space. However, if $(\Im, a)$ is transient with reference measure m, it is possible to extend $(\Im, a)$ to a Dirichlet space $(\Im_e, a)$ relative to $L^2(X, m)$, taking the completion of $(\Im, a)$ with respect to a. $(\Im_e, a)$ is a Hilbert space transient with reference m. A careful definition of extended (transient) Dirichlet space with reference measure m can be found in [4], p. 35.

If $(\Im, a)$ is a transient space we can define the capacity cap_1 with respect to $(\Im, a_1)$ and, moreover, a capacity cap_0 (*capacity of order zero*) with respect to $(\Im_e, a)$, simply denoted by cap. The properties and notions relative to cap_1, we mentioned before, hold for cap as well (cf. [4], p. 73).

With regard to cap, S_0 will be the set of all positive Radon measures of finite (0-)energy integrals, as well as U^μ will be the (0-)*potential* of the measure $\mu \in S_0$, and $a(\mu) = a(U^\mu, U^\mu) = \int_X \tilde{U}^\mu \mu(\mathrm{d}x)$ will be the (0-)*energy integral* of μ. Here $\tilde{v}$ denotes a (0-)q.c.m. of $v \in \Im_e$. Moreover, to any Borel set $B \subset X$ we associate the relative (0-)equilibrium potential and measure.

To complete our speech we want to mention Example 1.5.2 of [3], where cap is the variational capacity.

From now on we consider only *transient spaces* and their (*order zero*) *capacities*.

Let X_0 be an open subset of X such that $(D[a] \cap C_0(X_0), a)$ is a transient space. Then the extended transient space of $(D[a] \cap C_0(X_0), a)$ will be denoted by $(D_0[X_0], a)$ or simple by $D_0[X_0]$. Later on, cap will denote the (order zero) capacity relative to $D_0[X_0]$, and $S_0, U^\mu, \ldots$ will be referred to $D_0[X_0]$ as well.

Let $D_0[x_0]^*$ be the topological dual space of $D_0[X_0]$. The usual Hilbert techniques assure the existence, for any $T \in D_0[x_0]^*$, of a unique solution $u \in D_0[x_0]$ to the problem

$$a(u, v) = \langle T, v \rangle \quad u \in D_0[x_0] \qquad \text{for any } v \in D_0[x_0] \tag{1.15}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $D_0[x_0]^*$ and $D_0[X_0]$. If u is the solution of (1.15) we will say that u solves the *formal equation* $Lu = T$ and $G\colon D_0[x_0]^* \to D_0[x_0]$ will be the *Green operator*, which to any $T \in D_0[x_0]^*$ associates the relative solution $u \in D_0[x_0]$ of (1.15).

We shall say that a positive Radon measure on X_0 *belongs to* $D_0[x_0]^*$, if there exists a constant $c > 0$ such that

$$|\int \tilde{v}\mu(\mathrm{d}x)| \leqslant c\|v\|_{D_0[x_0]} \quad \text{for any } v \in D_0[x_0].$$

If $\mu \in S_0$, then $\mu \in D_0[x_0]^*$ and $G(\mu) = U^\mu$. In fact

$$|\int \tilde{v}\mu(\mathrm{d}x)| = |a(U^\mu, v)| \leqslant \|U^\mu\|_{D_0[x_0]}\,|v|_{D_0[x_0]} \quad \text{for any } v \in D_0[x_0].$$

2 - Equivalent capacities

Because of the capacitability of cap, we shall devote ourselves to find equivalent expressions for cap(K), where K is a compact subset of X_0. We shall deal mainly with classical techniques.

From now on μ_K and $u_K = U^{\mu_K}$ will be the equilibrium measure and, respectively, the equilibrium potential of K, whereas $S_0(K)$ will be the set of all measures $\mu \in S_0$ such that $\operatorname{supp}\mu \subseteq K$.

Proposition 1. *Define* $\underline{\mathrm{cap}}(K) = \sup\{\mu(K)\,|\,\mu \in S_0(K),\ \tilde{U}^\mu(x) \leqslant 1 \text{ q.e. on } K\}$. *Then* $\underline{\mathrm{cap}}(K) \cong \mathrm{cap}(K)$.

Proof. We have $\mu_K \in S_0(K)$ and $\tilde{U}^{\mu_K}(x) \leqslant 1$ q.e. on K. Therefore $\mathrm{cap}(K) = \mu_K(K) \leqslant \underline{\mathrm{cap}}(K)$. Vice-versa, if $\mu \in S_0(K)$, $\tilde{U}^\mu(x) \leqslant 1$ q.e. on K, if $\varphi \in D_0[x_0]$ and $\tilde{\varphi} \geqslant 1$ q.e. on K, then $\mu(K) \leqslant \int \tilde{\varphi}\mu(\mathrm{d}x) \leqslant \|\varphi\|_{D_0[X_0]}\|\mu\|_{D_0[X_0]^*}$. On the other hand

$$\|\mu\|^2_{D_0[X_0]^*} \leqslant a(G(\mu), G(\mu)) = \langle \mu, G(\mu)\rangle = \int \tilde{U}^\mu \mu(\mathrm{d}x) \leqslant \mu(K).$$

Proposition 2. *Define* $\underline{\underline{\mathrm{cap}}}(K) = \sup\{\mu(K)\,|\,\mu \in S_0(K),\ a(\mu)^{\frac{1}{2}} \leqslant 1\}$. *Then* $\underline{\underline{\mathrm{cap}}}(K) \cong \mathrm{cap}(K)^{\frac{1}{2}}$.

Proof. If $\mu \in S_0(K)$, $a(\mu)^{\frac{1}{2}} \leqslant 1$, $\varphi \in D_0[X_0]$ and $\tilde{\varphi} \geqslant 1$ q.e. on K, then the same method of the second part of the proof of Proposition 1 gives $\mu(K) \leqslant \|\varphi\|_{D_0[X_0]}$ hence $\mathrm{cap}(K)^{\frac{1}{2}} \geqslant \underline{\underline{\mathrm{cap}}}(K)$. Vice-versa $\mathrm{cap}(K)^{\frac{1}{2}} = \mu_K(K)^{\frac{1}{2}} = \nu_K(K)$, where $\nu_K = \mu_K/\mu_K(K)$ and $a(\nu_K)^{\frac{1}{2}} = a(\mu_K)^{\frac{1}{2}}/\mu_K(K)^{\frac{1}{2}} \leqslant 1$. Then $\mathrm{cap}(K)^{\frac{1}{2}} \leqslant \underline{\underline{\mathrm{cap}}}(K)$.

Proposition 3. *Define* $\operatorname{cap}^*(K) = (\inf\{a(\mu) \mid \mu \in S_0(K),\ \mu(K) = 1\})^{-1}$ $= \sup\{a(\mu)^{-1} \mid \mu \in S_0(K),\ \mu(K) = 1\}$. *Then* $\operatorname{cap}^*(K) \cong \operatorname{cap}(K)$.

Proof. If $\mu \in S_0(K)$, $\mu(K) = 1$, $\varphi \in D_0[X_0]$ and $\tilde{\varphi} \geqslant 1$ q.e. on K, then

$$1 \leqslant \int \tilde{\varphi}\mu(\mathrm{d}x) \leqslant \|\varphi\|_{D_0[X_0]} \qquad \|\mu\|_{D_0[X_0]^*} \leqslant \|\varphi\|_{D_0[X_0]}\, a(\mu)^{\frac{1}{2}}$$

(see the second part of Proposition 1), hence $\operatorname{cap}(K) \geqslant \operatorname{cap}^*(K)$. Vice-versa, if $\operatorname{cap}(K) > 0$, then $\nu_K = \operatorname{cap}(K)^{-1}\mu_K \in S_0(K)$ and $\nu_K(K) = 1$. Therefore $\operatorname{cap}^*(K)^{-1} \leqslant a(\mu_K)(\operatorname{cap}(K))^{-1} \leqslant (\operatorname{cap}(K))^{-1}$. If $\operatorname{cap}(K) = 0$, then the first part of this proof gives $a(\mu) = +\infty$ for any $\mu \in S_0(K)$, $\mu(K) = 1$.

Proposition 4. *Define* $\overline{\operatorname{cap}}(K) = \inf\{\mu(K) \mid \mu \in S_0(K),\ \tilde{U}^{\mu} \geqslant 1$ q.e. on $K\}$. *Then* $\overline{\operatorname{cap}}(K) \cong \operatorname{cap}(K)$.

Proof. We have $\overline{\operatorname{cap}}(K) \leqslant \mu_K(K) = \operatorname{cap}(K)$. Vice-versa, given $\lambda \in S_0(K)$ such that $\tilde{U}^{\lambda} \geqslant 1$ q.e. on K, then

$$\operatorname{cap}(K) \leqslant \int \tilde{U}^{\lambda}\mu_K(\mathrm{d}x) = a(U^{\mu_K}, U^{\lambda}) = a(U^{\lambda}, U^{\mu_K}) = \int \tilde{U}^{\mu_K}\lambda(\mathrm{d}x) \leqslant \lambda(K)$$

hence $\operatorname{cap}(K) \leqslant \overline{\operatorname{cap}}(K)$.

3 - Representation of potentials

Assume the form a has the *strong local property* (i.e. a is *of diffusion type*), that is, according to our notations in **1**

(3.1) $\quad a(u, v) = 0$ *for any* $u, v \in \Im$ *with* v *constant on* $\operatorname{supp} u$.

If the norms $a_1(u, u)^{\frac{1}{2}}$ and $a(u, u)^{\frac{1}{2}}$ are equivalent (and $\Im$ is transient) then (3.1) assure that the support of the equilibrium measure (of order zero) of a Borel set B is contained in ∂B.

In fact let ν_B and e_B be the equilibrium respectively measure and potential of B. If the sequence u_n, belonging to the core C_1, approximates u in the a-norm and $u_n \equiv 1$ on B, then, for any $v \in \Im$ with $\operatorname{supp} v \subseteq B$, $a(u_n, v) = 0$. So $a(u, v) = 0$, or $\int \tilde{v}\nu_B(\mathrm{d}x) = 0$.

If b is a regular Dirichlet form, then $b(u, v) = \int_X \mu_b(u, v)(\mathrm{d}x)$, for any

$u, v \in D[a]$ where μ_b is a Radon-measure-valued, positive semidefinite, bilinear form on $D[b]$, uniquely associated to b, called the *energy measure of b* ([4], p. 152).

A regular Dirichlet form b has the *separation property* if (C_1 being the core of $D[b]$):

$$\forall x, y \in X,\ x \neq y,\ \exists \varphi \in C_1 \text{ such that } \mu_b(\varphi, \varphi) \leqslant m \text{ on } X \text{ and } \varphi(x) \neq \varphi(y).$$

Assume

(3.2) *a has the separation property.*

This enable us to define an *(intrinsic) distance* $d = d_a \colon X \times X \to [0, +\infty]$ as

$$d(x, y) = \sup \{\varphi(x) - \varphi(y) \mid \varphi \in C_1,\ \mu_a(\varphi, \varphi) \leqslant m \text{ on } X\}$$

and the related intrinsic balls $B(x, r) = \{y \in X \mid d(x, y) < r\}$.

We assume also:

(3.3) *the topology generated on X by the metric d is equivalent to the initial topology*

(3.4) *the measure m has the doubling property with respect to the intrinsic balls.*

(3.3) and (3.4) give to X the structure of *homogeneous space* according to [2].

Finally we assume *Poincaré's and Poincaré-Sobolev's inequalities* for all functions locally belonging to $D[a]$.

Let X_0 be an open, connected, relatively compact subset of X. There exist some constants $c', c'' > 0$, $s > 2$ and an integer $k \geqslant 1$ such that, for any $x \in X_0$ and $r > 0$ with $B(x, r) \subset X_0$ the following inequalities hold:

$$(3.5) \qquad \int_{B(x, \frac{r}{k})} |u - \bar{u}|^2 m(dx) \leqslant c' r \int_{B(x, r)} \mu_a(u, u)\, dx$$

for any $u \in D_{\text{loc}}[X_0]$, where $\bar{u} = \dfrac{1}{m(B(x, \frac{r}{k}))} \displaystyle\int_{B(x, \frac{r}{k})} u m(dx)$

$$(3.6) \qquad \left(\frac{1}{m(B(x, r))} \int_{B(x, r)} |u|^s m(dx)\right)^{\frac{1}{s}} \leqslant c'' r \left(\frac{1}{m(B(x, r))} \int_{B(x, r)} \mu_a(u, u)(dx)\right)^{\frac{1}{2}}$$

for any $u \in D_{\text{loc}}[X_0]$, where $\operatorname{supp} u \subset B(x, r)$.

A function u belongs to $D_{\text{loc}}[X_0]$ if it is measurable on X_0 and there exists $w \in D_{\text{loc}}[a]$ such that $\varphi u = \varphi w$ m-a.e. for every $\varphi \in C_1$, $\operatorname{supp} \varphi \subset X_0$.

$D_{\text{loc}}[a]$ denotes the space of all measurable functions w on X such that for every open, relatively compact subset A of X there exists a function $v \in D_{\text{loc}}[a]$ such that $w = v$ m-a.e. on A. Moreover the measure $\mu(w, w)$ is defined as $1_\lambda \mu(w, w) = 1_\lambda \mu(v, v)$.

From now on Ω will be an open set and $X_0 = B(x_0, R_0)$ will be an intrinsic ball of center x_0 and radius R_0 such that $\bar{\Omega} \subset B(x_0, \frac{R_0}{4})$, $B(x_0, R_0) \subset X$.

From (3.6) for $s = 2$ we have $a_1(u, u)^{\frac{1}{2}} \cong a(u, u)^{\frac{1}{2}}$ for any $u \in D_0[X_0]$. Here $(D_0[X_0], a)$ or simply $D_0[X_0]$, is a transient space. As in **2**, $D_0[X_0]^*$ denotes the topological dual space of $D_0[X_0]$ and we are interesting again to the solutions of the problem

$$(3.7) \qquad a(u, v) = \langle T, v \rangle \quad u \in D_0[x_0] \qquad \text{for any } v \in D_0[x_0]$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $D_0[X_0]^*$ and $D_0[X_0]$.

On account of (1.1)-(1.4) and (3.1)-(3.6) M. Biroli and U. Mosco [**1**] proved that the solution u to the problem

$$(3.8) \qquad a(u, v) = \int_{X_0} fv m(\mathrm{d}x) \quad u \in D_0[x_0] \qquad \text{for any } v \in D_0[x_0]$$

$f \in L^p(X_0, m)$, $p > p_0$, p_0 suitable structural constant, is locally Hölder-continuous in X_0 with respect to the intrinsic distance, and u has a representation formula

$$(3.9) \qquad u(x) = \int_{X_0} G^x(y) f(y) m(\mathrm{d}y) \quad \text{m-a.e..}$$

The function $G^x(y) \in L^q(X_0, m)$, $\frac{1}{q} + \frac{1}{p} = 1$, is the *Green function* relative to X_0 with singularity at x.

Our purpose is now to extend the representation formula (3.9) to the solution $u \in D_0[X_0]$ to the problem

$$(3.10) \quad a(u, v) = \int_{X_0} \tilde{v} \mu(\mathrm{d}x) \;\; u \in D_0[X_0] \quad \text{for any } v \in D_0[X_0],\ \mu \in S_0,\ \operatorname{supp} \mu \subset \Omega .$$

To this end we shall follow the classic method of [**6**]. It lies in proving that u is the *weak solution vanishing on* ∂X_0 *of* $Lu = \mu$ according to the definition below, and proving afterwards the representation formula for every such solution.

We continue to denote by G: $D_0[X_0]^* \to D_0[X_0]$ the Green operator, relative to (3.7).

Definition 1. For $\mu \in S_0$ we shall say that $u \in L^1(X_0, m)$ is a *weak solution vanishing on ∂X_0 of $Lu = \mu$*, if and only if

$$\int_{X_0} u\psi m(\mathrm{d}x) = \int_{X_0} \tilde{G}(\psi)\mu(\mathrm{d}x) \tag{3.11}$$

for every $\psi \in C(\bar{X}_0)$. (There is at most one solution to this problem).

Remark 1. If u solves (3.10) for a certain $\mu \in C_0$, then u is the weak solution vanishing on ∂X_0 of $Lu = \mu$, according to Definition 1. In fact, if $\psi \in C(\bar{X}_0)$ and $\varphi = G(\psi)$, then $a(\varphi, v) = \int \psi v m(\mathrm{d}x)$, for every $v \in D_0[X_0]$. If $v = u$, then $a(\varphi, u) = \int \psi u m(\mathrm{d}x)$. Moreover $a(u, w) = \int \tilde{w}\mu(\mathrm{d}x)$, for energy $w \in D_0[X_0]$. If $w = \varphi$ then $a(u, \varphi) = \int \tilde{\varphi}\mu(\mathrm{d}x)$. Therefore $\int_{X_0} u\psi m(\mathrm{d}x) = \int_{X_0} \tilde{G}(\psi)\mu(\mathrm{d}x)$.

Theorem 1. *If $\mu \in S_0$, $\operatorname{supp}\mu \subset \Omega$, then*

$$u(x) \equiv \int_{y \in \operatorname{supp}\mu} G^x(y)\mu(\mathrm{d}y) \tag{3.12}$$

is finite m-a.e. *and it is the weak solution vanishing on ∂X_0 of $Lu = \mu$.*

Proof (see [6], Theorem 6.1). Take $\varphi = G(\psi)$, $\psi \in C(\bar{X}_0)$. From (3.8) and the symmetry of the Green function

$$\varphi(y) = \int G^x(y)\psi(x)m(\mathrm{d}x). \tag{3.13}$$

Because of the local Hölder-continuity of φ proved in [1], we have $\tilde{\varphi} = \varphi$ q.e. on $\operatorname{supp}\mu$. Then

$$\int \tilde{\varphi}\mu(\mathrm{d}x) = \int_{\operatorname{supp}\mu} \varphi\mu(\mathrm{d}x) + \int_{X - \operatorname{supp}\mu} \tilde{\varphi}\mu(\mathrm{d}x) = \int_{\operatorname{supp}\mu} \varphi\mu(\mathrm{d}x).$$

Hence, from Fubini's theorem, $\int_{y \in \operatorname{supp}\mu} G^x(y)\mu(\mathrm{d}y)$ exists m-a.e., and

$$\int \tilde{\varphi}\mu(\mathrm{d}x) = \int_{\substack{x \in X_0 \\ y \in \operatorname{supp}\mu}} G^x(y)\psi(x)m(\mathrm{d}x)\mu(\mathrm{d}y) = \int \psi(x)u(x)m(\mathrm{d}x).$$

4 - The case of Hörmander's vector fields

In this case we assume m being the *Lebesgue's measure*, and $D_0[X_0] = H_0^1(X_0)$ will be the closure of $C_0^\infty(X_0)$ with respect to the norm coming from the form

$$a(u, v) = \sum_{i=1}^{q} \int_{X_0} X_i(u)X_i(v)$$

X_i, $i = 1, \ldots, q$ are vector fields satisfying Hörmander's condition [4] and $X_0 = B(x_0, R_0)$ is an intrinsic ball contained in an open subset X of $\boldsymbol{R}^N$.

The topological dual space of $H_0^1(X_0)$ will be denoted by $H^{-1}(X_0)$. Formally $a(u, v) = \langle Lu, v \rangle$, where the operator $L\colon H_0^1(X_0) \to H^{-1}(X_0)$ is defined as

$$L = \sum_{i=1}^{q} X_i^*(X_i)$$

and X_i^* denotes the formal adjoint of X_i, for any $i = 1, \ldots, q$.

Let's now suppose

(4.1) *the Green's function of X_0 has integrable «first order partial derivatives» $X_i G^x$, $i = 1, \ldots, q$ in X_0.*

Then any function $\varphi \in C_0^\infty(X_0)$ admits a representation formula in terms of the gradient of G^x and the gradient of φ too. In fact, from the representation formula (3.9) we have

$$\varphi(x) = \int G^x(y) L\varphi(y)\, dy = \sum_{i=1}^{q} \int X_i G^x(y) X_i \varphi(y)\, dy. \tag{4.2}$$

In this way we associate to any function $\varphi \in C_0^\infty(X_0)$ a unique vector

$$(w_i)_{i=1}^q \equiv (X_i \varphi)_{i=1}^q \in (C_0^\infty(X_0))_{i=1}^q \quad \text{such that} \quad \varphi(x) = \sum_{i=1}^{q} \int X_i G^x w_i.$$

This application can be inverted. Given $w \equiv (w_i)_{i=1}^q \in (C_0^\infty(X_0))_{i=1}^q$, put

$$G'(w)(x) = \sum_{i=1}^{q} \int X_i G^x w_i. \tag{4.3}$$

We prove $G'(w) \in C_0^\infty(X_0)$. Let $T \in H^{-1}(X_0)$ be the *divergence* of w, that is $T = \sum_{i=1}^{q} X_i^*(w_i)$ and let u be the unique element $u \in H_0^1(X_0)$ such that $Lu = T$. This means that

$$\sum_{i=1}^{q} \int X_i(u) X_i(v) = \langle T, v \rangle = \sum_{i=1}^{q} \int w_i X_i(v) \quad \text{for any } v \in H_0^1(X_0).$$

Then $w_i = X_i(u) + \psi_i$, $i = 1, \ldots, q$, where $\sum_{i=1}^{q} X_i^*(\psi_i) = 0$.

In particular, because of $T \in C_0^\infty(X_0)$ and L is hyphoelliptic, we have

$u \in C_0^\infty(X_0)$, then $\psi_i \in C_0^\infty(X_0)$, $i = 1, \ldots, q$ and

$$G'(w)(x) = \sum_{i=1}^{q} X_i(G^x) X_i(u) + \sum_{i=1}^{q} \int X_i(G^x) \psi_i$$

where

$$\sum_{i=1}^{q} X_i(G^x) X_i(u) \in C_0^\infty(X_0) \quad \text{and} \quad \sum_{i=1}^{q} \int X_i(G^x)\psi_i = \int G^x \sum_{i=1}^{q} X_i^*(\psi_i) = 0\,.$$

Hence $G'(w) \in C_0^\infty(X_0)$ and $G'(w) = u$. Moreover

$$\|u\|_{H_0^1(X_0)} \cong \|T\|_{H^{-1}(X_0)} \leq c\|w\|_{(L^2(X_0))^q}.$$

Therefore, for K compact subset of $\Omega \subset B(x_0, \frac{R_0}{40})$ all the solutions of the following extremal problems are equivalent (to the variational capacity cap(K)):

$$\inf\{\|\varphi\|^2_{H_0^1(X_0)} \;|\varphi \in C_0^\infty(X_0) \text{ and } \varphi \geq 1 \text{ on } K\}$$

$$\inf\{\|w\|^2_{(L^2(X_0))^q} \;|w \in (C_0^\infty(X_0))^q \text{ and } G'(w) \geq 1 \text{ on } K\}$$

$$\inf\{\|u\|^2_{H_0^1(X_0)} \;|u \in H_0^1(X_0) \text{ and } \tilde{u} \geq 1 \text{ q.e. on } K\}$$

$$\inf\{\|F\|^2_{(L^2(X_0))^q} \;|F \in (L^2(X_0))^q \text{ and } G'(F) \geq 1 \text{ weakly on } K\}$$

where $G'(F) \geq 1$ *weakly on* K means that there exists a sequence $w^\nu \in (C_0^\infty(X_0))^q$ such that $G'(w^\nu)(x) = \sum_{i=1}^{q} \int X_i(G^x) w_i^\nu \geq 1$ on K and $w^\nu \to F$ in $(L^2(X_0))^q$.

References

[1] M. Biroli and U. Mosco, *A Saint-Venant principle for Dirichlet forms on discontinuous media*, Quaderno 56/P, Dip. Mat. Politecnico, Milano 1992.

[2] R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certaines éspaces homogènes*, Lecture Notes in Math. 242, Springer, Berlin 1971.

[3] E. Fabes, D. Jerison and C. Kenig, *The Wiener test for degenerate elliptic équations*, Ann. Inst. Fourier 3 (1982), 151-183.

[4] M. Fukushima, *Dirichlet forms and Markov processes*, North Holland and Kodansha, Amsterdam 1980.

[5] L. HÖRMANDER, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 141-171.

[6] W. LITTMAN, G. STAMPACCHIA and H. WEINBERGER, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 43-77.

[7] V. G. MAZ'JA and V. P. HAVIN, *Nonlinear potential theory*, Uspekhi Mat. Nauk **27** (1972), 67-138; Russian Math. Surveys **27** (1972), 71-148.

[8] A. NAGEL, E. STEIN and S. WEINGER, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), 103-147.

Sommario

Si dimostra l'equivalenza tra diverse definizioni della capacità variazionale associata a forme di Dirichlet. Si stabilisce una rappresentazione integrale per i potenziali di misura ad integrale dell'energia finito.

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