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Newlander-Nirenberg theorem on supermanifolds with boundary (**)

Introduction

Recall that in the classical context the Newlander-Nirenberg theorem states that on a sufficiently smooth integrable almost complex manifold, there exists local holomorphic coordinates. This is a statement about interior points on the manifold. For the case of a boundary point on an integrable almost complex manifold, such a result is not always true as was pointed out by one of the authors [5], [6], [7]. The up to the boundary analogue of the theorem is however true in the presence of pseudoconvexity. A simple proof in the strongly pseudoconvex case was given by N. Hanges and H. Jacobowitz [4] and a proof for the weakly pseudoconvex case was obtained by D. Catlin [2].

The super analogue for interior points was proved by A. McHugh [**13**]. Although our proof is based in part on his ideas for interior points, we have recast the argument in a more geometric language which is closer in spirit to the previous results. In fact, this article has two main goals: one is to prove the existence of supercoordinates up to the boundary, in the weakly pseudoconvex case, for a super integrable almost complex manifold. The other is to develop a geometric point of view, which clarifies the situation and makes it natural to consider the concept of a super CR-manifold. The authors hope to pursue generalizations along these lines in a future publication. It should be noted that in our proof we make essential use of analytical results of Catlin [2] and Kohn [9].

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1 - Preliminaries

First we recall the notion of a *real* C^∞ *supermanifold* [3], [10], [11], [12]. This consists of a triple $(X, \mathcal{A}, \alpha)$ where X is a C^∞ manifold, $\mathcal{A}$ is a sheaf over X of $\boldsymbol{Z}_2$ graded-commutative algebras over $\boldsymbol{R}$ and the augmentation map $\alpha: \mathcal{A} \to C^\infty$ is a sheaf homomorphism of algebras. The following axiom must be satisfied, which gives a local splitting: there exists a basis $\{V\}$ for the open sets of X such that for every V there is an isomorphism β_V which makes the diagram

$$\begin{array}{ccc} \mathcal{A}(V) & \xrightarrow{\beta_V} & C^\infty(V) \otimes \wedge^* \boldsymbol{R}^m \\ & \alpha \searrow \quad \swarrow \pi & \\ & C^\infty(V) & \end{array}$$

commutative. Here and in what follows we use the notation $\mathcal{S}(U)$ for the space of continuous sections over U of a sheaf $\mathcal{S}$ over X. In the diagram above, π is the natural projection. Such a supermanifold will be said to be of dimension (n, m) if the dimension of X is n.

A $\boldsymbol{Z}_2$ *grading of* $\mathcal{A}$ means that two subspaces $\mathcal{A}_0$ and $\mathcal{A}_1$ are fixed, the even and odd part respectively, such that

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 .$$

The elements of $\mathcal{A}_0$ commute with all elements in $\mathcal{A}$, while the elements in $\mathcal{A}_1$ anticommute with all elements in $\mathcal{A}_1$. Let $\mathcal{N}$ be the subsheaf of nilpotent elements of $\mathcal{A}$. It follows from the above diagram that for any open set U in X there is a map tilde induced by α

$$\begin{array}{c} \mathcal{A}(U) \to C^\infty(U) \xrightarrow{\cong} \mathcal{A}/\mathcal{N}(U) \\ f \to \tilde{f} \end{array} . \tag{1.1}$$

Sections $r_1, \ldots, r_n \in \mathcal{A}_0(U)$ are called an *even coordinate system* if the functions $\tilde{r}_1, \ldots, \tilde{r}_n \in C^\infty(U)$ form a coordinate system in U in the usual sense. Sections $s_1, \ldots, s_k$ of $\mathcal{A}_1(U)$ are algebraically independent if the product $s_1 \cdot s_2 \cdots s_k \neq 0$. The odd dimension m is defined as the smallest integer j such that $\mathcal{N}^{j+1} = 0$. Then m algebraically independent sections $s_1, \ldots, s_m \in \mathcal{A}_1(U)$ are said to form an *odd coordinate system*. Therefore a section f of $\mathcal{A}(U)$, called a C^∞ *superfunction*, can be written as $f = \sum_\mu f_\mu(r) s^\mu$ where $f_\mu(r) \in \mathcal{A}/\mathcal{N}(U)$. Here $\mu = (\mu_1, \ldots, \mu_m)$ is a multi-index with $\mu_j = 0$ or 1, and $s^\mu = s_1^{\mu_1} \cdot s_2^{\mu_2} \cdots s_m^{\mu_m}$. By (1.1) in a sufficiently small open set U, we can think of a superfunction as an expression of the form

$$f = \sum_{\mu} f_{\mu} s^{\mu}$$

with $f_{\mu} \in C^{\infty}(U)$. The grading is thus determined by $f \in \mathcal{A}(U)_0 \Leftrightarrow f = \sum_{\mu:\,|\mu|\ \text{even}} f_{\mu} s^{\mu}$ while $f \in \mathcal{A}(U)_1 \Leftrightarrow f = \sum_{\mu:\,|\mu|\ \text{odd}} f_{\mu} s^{\mu}$. The collection $\{r_i, s_i\}$ will be called a *supercoordinate system.*

The algebra of derivations Der $\mathcal{A}$ has a natural $\boldsymbol{Z}_2$ grading. Namely a derivation D has degree j iff $D(fg) = (Df)g + (-1)^{j \deg f} f(Dg)$ for all homogeneous f, $g \in \mathcal{A}$. Given $D \in (\operatorname{Der} \mathcal{A}(U))_0$ the nilpotent set $\mathcal{N}$ is stable under D and therefore, it generates a derivation $\tilde{D}$ of the quotient $\mathcal{A}/\mathcal{N}(U)$. Thus, $\tilde{D}$ can be thought as a vector field over U. The correspondence

$$(\operatorname{Der} \mathcal{A}(U))_0 \to \operatorname{Der}(C^{\infty}(U)) \qquad (1.2)$$
$$D \to \tilde{D}$$

is a Lie algebra epimorphism and we have $\widetilde{DF} = \tilde{D}\tilde{f}$ for all $f \in \mathcal{A}(U)$. In a supercoordinate system $\{r_i, s_j\}$ we have the partial derivations $\partial/\partial r_i$ and $\partial/\partial s_j$ defined by

$$\frac{\partial}{\partial r_i}(f_{\mu} s^{\mu}) = \frac{\partial f_{\mu}}{\partial r_i} s^{\mu} \qquad \frac{\partial}{\partial s_i}(f_{\mu} s^{\mu}) = \mu_j (-1)^p f_{\mu} s^{\mu'}$$

where $p = \mu_1 + \mu_2 + \ldots + \mu_{j-1}$ and $s^{\mu'} = s_1^{\mu_1} \cdot s_2^{\mu_2} \cdots s_j^{\mu_j - 1} \cdots s_m^{\mu_m}$. The even $\{\partial/\partial r_i\}$ commute and can be thought of as classical partial derivatives; whereas the odd $\{\partial/\partial s_j\}$ anticommute. Locally Der $\mathcal{A}$ is a free $\mathcal{A}(U)$-module with basis $\{\partial/\partial r_i, \partial/\partial s_j\}$, i.e., linear combinations of these basis elements with C^{∞} superfunction coefficients.

A C^{∞} hypersurface in the supermanifold $(X, \mathcal{A}, \alpha)$ is defined by an ideal $\mathfrak{J}$ in $\mathcal{A}$ which is locally generated by a superfunction r such that exterior derivative of $\tilde{r}$ is nonzero on the nonvoid zero locus of $\tilde{r}$.

The case of a *complex supermanifold* is entirely analogous. It is a triple $(X, \mathcal{B}, \alpha)$ where X is a complex manifold, $\mathcal{B}$ is a sheaf over X of $\boldsymbol{Z}_2$ graded-commutative algebras over $\boldsymbol{C}$, and the augmentation map $\alpha: \mathcal{B} \to \mathcal{O}$ is a sheaf homomorphism of algebras. Here $\mathcal{O}$ denotes the structure sheaf of holomorphic functions on X. The local splitting axiom now takes the form of an isomorphism β_V and a commutative diagram

$$\begin{array}{ccc} \mathcal{B}(V) & \xrightarrow{\beta_V} & \mathcal{O}(V) \otimes \wedge^* \boldsymbol{C}^m \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathcal{O}(V) & \end{array} .$$

Such a complex supermanifold will be said to be of *complex dimension* (n, m) if the complex dimension of X is n. A local section of $\mathcal{B}(U)$, called a *holomorphic superfunction*, can be thought of as

$$g = \sum_{\mu} g_{\mu} \eta^{\mu}$$

where $g_{\mu} = g_{\mu}(z_1, \dots, z_n) \in \mathcal{O}(U)$ and $\eta_1, \dots, \eta_m$ are algebraically independent sections of $\mathcal{B}_1(U)$. The *complex supercoordinates* $\{z_i, \eta_j\}$ are as before split in two groups, the even $\{z_i\}$ and the odd $\{\eta_j\}$ ones. Now the locally free $\mathcal{B}(U)$-module $\mathrm{Der}\,\mathcal{B}(U)$ has a basis $\{\partial/\partial z_i, \partial/\partial \eta_j\}$ with holomorphic superfunction coefficients. Note that in this context the superfunctions g satisfy the super Cauchy-Riemann equations $\dfrac{\partial g}{\partial \bar{z}_i} = 0$, $\dfrac{\partial g}{\partial \bar{\eta}_j} = 0$.

From now on we consider a C^{∞} supermanifold $(X, \mathcal{A}, \alpha)$ of real dimension $(2n, 2m)$. We proceed next to define the notion of a super integrable almost complex structure on $(X, \mathcal{A}, \alpha)$. First we use extension by real linearity to complexify both the superalgebra $\mathcal{A}$ and the algebra of derivations Der; let $\mathcal{A}_C$ and $\mathrm{Der}_C\,\mathcal{A}_C$ denote the respective complexifications. We also use real linearity to extend the supercommutator defined by

$$[X, Y] = XY - (-1)^{d(X)d(Y)}YX,$$

for homogeneous derivations X, Y, where $d(X)$ denotes the degree of X. If we have a real supercoordinate system

$$\{x^1, \dots, x^n, y^1, \dots, y^n, s^1, \dots, s^m, t^1, \dots, t^m\}$$

and set $z^k = x^k + \sqrt{-1}y^k$, $\eta_k = s^k + \sqrt{-1}\,t^k$, we may use the usual formulas from complex analysis:

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \sqrt{-1}\frac{\partial}{\partial y^k}\right) \qquad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial y^k}\right)$$

$$\frac{\partial}{\partial \eta^k} = \frac{1}{2}\left(\frac{\partial}{\partial s^k} - \sqrt{-1}\frac{\partial}{\partial t^k}\right) \qquad \frac{\partial}{\partial \bar{\eta}^k} = \frac{1}{2}\left(\frac{\partial}{\partial s^k} + \sqrt{-1}\frac{\partial}{\partial t^k}\right).$$

Then a super almost complex structure on the C^{∞}supermanifold $(X, \mathcal{A}, \alpha)$ consists in the prescription of a locally direct subsheaf $\mathcal{H}$ of the sheaf $\mathrm{Der}_C\,\mathcal{A}_C$ of $\mathcal{A}_C$-modules over X, of rank $n + m$, which satisfies

$$\mathcal{H} \cap \bar{\mathcal{H}} = 0. \tag{1.3}$$

It is called an integrable super almost complex structure if

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}. \tag{1.4}$$

Locally in U the prescription of $\mathcal{H}$ is equivalent to prescribing a basis $\{P_1, P_2, \ldots, P_{n+m}\}$, for $\mathcal{H}(U)$, of sections of $\mathrm{Der}_C\, \mathcal{A}_C(U)$; the requirements above are equivalent to

(1.3)′ $$P_1, \ldots, P_{n+m}, \bar{P}_1, \ldots, \bar{P}_{n+m} \quad \textit{are linearly independent}$$

(1.4)′ $$[P_j, P_k] = \sum_{l=1}^{n+m} f_{j,k}^{l} P_l$$

where the $f_{j,k}^{l}$ are sections of $\mathcal{A}_C(U)$. We may futher choose our basis of the form $\{P_1, \ldots, P_{n+m}\} = \{L_1, \ldots, L_n, M_1, \ldots, M_m\}$, where $L_1, \ldots, L_n$ are of degree 0 and $M_1, \ldots, M_m$ are of degree 1, and further rewrite the conditions above as

(1.3)″ $$L_1, \ldots, L_n, \bar{L}_1, \ldots, \bar{L}_n, M_1, \ldots, M_m, \bar{M}_1, \ldots, \bar{M}_m$$

are linearly independent

(1.4)″ $$\begin{aligned} [L_i, L_j] &= \alpha_{ij}^{r} L_r + \beta_{ij}^{s} M_s \\ [M_p, M_q] &= \gamma_{pq}^{r} L_r + \delta_{pq}^{s} M_s \\ [L_i, M_q] &= \lambda_{iq}^{r} L_r + \mu_{iq}^{s} M_s \end{aligned}$$

using summation convention, for appropriate sections $\alpha_{ij}^{r}, \beta_{ij}^{s}, \gamma_{pq}^{r}, \delta_{pq}^{s}, \lambda_{iq}^{r}, \mu_{iq}^{s}$. Note that the first and last equations in (1.4)″ involve classical commutators; whereas the middle one involves the anticommutator. From the grading it follows that the $\alpha_{ij}^{r}, \gamma_{pq}^{r}, \mu_{iq}^{s}$ contain no odd terms and the $\beta_{ij}^{s}, \delta_{pq}^{s}, \lambda_{iq}^{r}$ contain no even terms; hence the later are nilpotent.

We can associate to the even derivations $L_1, \ldots, L_n \in \mathrm{Der}_C\, \mathcal{A}_C(U)$ the C^∞ complex vector fields $\mathcal{L}_j = \tilde{L}_j$ as well as their complex conjugates. By (1.3)″ $\mathcal{L}_1, \ldots, \mathcal{L}_n, \bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_n$ are linearly independent in U. Using (1.1) and the fact that the β_{ij}^{s} are nilpotent, we obtain that there are functions $a_{j,k}^{r} = \tilde{\alpha}_{j,k}^{r}$ such that

(1.5) $$[\mathcal{L}_j, \mathcal{L}_k] = \sum_{r=1}^{n} a_{j,k}^{r} \mathcal{L}_r \,.$$

We summarize this discussion with the following

Proposition 1. *A super integrable almost complex structure $\mathcal{H}$ on the supermanifold $(X, \mathcal{A}, \alpha)$ induces, via the augmentation map α, a classical integrable almost complex structure $\tilde{\mathcal{H}}$ on its reduced space X.*

Finally we consider the situation up to the boundary. Let $(X', \mathcal{A}, \alpha)$ be *a real* C^∞ $(2n, 2m)$ *dimensional supermanifold.* Consider an open domain $X \subset X'$ with a smooth boundary ∂X and closure $\bar{X}$. Assume we have an ideal $\mathfrak{J}$ which locally, in $U \subset X'$, is generated by some C^∞ real superfunction r such that $d\tilde{r} \neq 0$ in U and $\{\tilde{r} = 0\} \cap U = \partial X \cap U$. Here we think of X' as a neighborhood of $\bar{X}$, and $(\bar{X}, \mathcal{A}|_{\bar{X}}, \alpha, \mathfrak{J})$ as being a *supermanifold with a smooth boundary*. By an *integrable super almost complex structure* on it, which is smooth up to the boundary, we mean a super almost complex structure $\mathcal{H}$ on $(X', \mathcal{A}, \alpha)$ which is given to be integrable only on $(X, \mathcal{A}, \alpha)$. Suppose that U is a neighborhood of a point $p \in \partial X$ with $\bar{X} \cap U = \{\tilde{r} \leq 0\} \cap U$. Then the situation we are in is that (1.3)″ holds in U but that (1.4)″ is valid only for $\{\tilde{r} \leq 0\} \cap U$.

Note that it follows from Proposition 1 that the induced structure $\tilde{\mathcal{H}}$ gives a classical integrable almost complex structure on X that is smooth up to the boundary ∂X.

We may define the *Levi form* of the *super* CR *boundary structure* in terms of the classical Levi form for the induced CR structure on ∂X; we can assume without loss of generality that $\mathcal{L}_j \tilde{r} = 0$ for $j = 1, \ldots, n-1$ and that $\mathcal{L}_n \tilde{r} = 1$. If $N = \operatorname{Im} \mathcal{L}_n$ then $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}, \bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_{n-1}, N$ forms a basis for the complexified tangent space of the submanifold $\{\tilde{r} = 0\}$. We thus can define smooth functions b_{jk}, the Levi form of our supermanifold, by the usual expression

$$\frac{i}{2}[\mathcal{L}_j, \bar{\mathcal{L}}_k] = b_{jk} N \bmod \{\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}, \bar{\mathcal{L}}_1, \ldots, \bar{\mathcal{L}}_{n-1}\}. \tag{1.6}$$

The supermanifold with boundary is said to be *strictly* (*weakly*) *pseudoconvex* if the matrix (b_{jk}) is positive definite (positive semidefinite) on ∂X.

2 - Statements of results

Theorem 1. *Let $(\bar{X}, \mathcal{A}|_{\bar{X}}, \alpha, \mathfrak{J})$ be a C^∞ supermanifold of real dimension $(2n, 2m)$, with a smooth boundary. Let it be equipped with an integrable super almost complex structure that is smooth up to the boundary and weakly pseudoconvex. Then to each point $p \in \bar{X}$ there corresponds a neighborhood V and super coordinates $z_j = x_i + iy_j$. $\eta_k = s_k + it_k$ smooth on $\overline{V \cap \{\tilde{r} < 0\}}$ such that $\partial/\partial z_j$ and $\partial/\partial \eta_k$ form a basis for the structure $\mathcal{H}$ on $V \cap \{\tilde{r} \leq 0\}$.*

Theorem 2. *For $m \geq 0$ and $n \geq 2$ there are counterexamples to the above statement, in which the boundary has nondegenerate Levi form with exactly one degree of pseudo-concavity.*

Our proof of Theorem 1 can be read also thinking of p as an interior point of X; then it merely gives a rephrasing of A. McHugh's result [13]. But the goal of this paper is to concentrate on the local question at a boundary point $p \in \partial X$. It should be pointed out that our result means that for weakly pseudoconvex boundaries, the super Cauchy-Riemann equations can be written as

$$\frac{\partial f}{\partial \bar{z}^j} = 0, \qquad \frac{\partial f}{\partial \bar{\eta}^k} = 0,$$

up to the boundary, instead of merely as

$$\bar{L}_j f = 0, \qquad \bar{M}_k f = 0.$$

3 - Initial step in the proof

In order to prove Theorem 1, we shall need first of all to recall the classical result.

Theorem 3. *Assume that $n \geqslant 2$ and let $Z_1, \dots, Z_n$ define at $p \in \boldsymbol{R}^{2n}$ a C^∞ integrable almost complex structure on $\{r \leqslant 0\}$ which is C^∞ up to the boundary $\{r = 0\}$. If the structure is weakly pseudoconvex, then there exists a neighborhood U of p and functions $z^1, \dots, z^n \in C^\infty(U)$ such that $Z_j \bar{z}^k = 0$ in $\{r \leqslant 0\} \cap U$, for $j, k = 1, \dots, n$ and $\mathrm{d}z^1, \dots, \mathrm{d}z^n$ are linearly independent.*

A simple proof of this theorem in the strictly pseudoconvex case was given by N. Hanges and H. Jacobowitz [4]; a proof for the weakly pseudoconvex case was given by D. Catlin [2].

Applying this theorem to the almost complex structure defined by the vector fields $\mathcal{L}_i$ above, we can find complex coordinates $\tilde{z}^1, \dots, \tilde{z}^n$ in a neighborhood U of p, so that afther replacing the L_i by suitable linear combinations, we have

$$\mathcal{L}_i = \frac{\partial}{\partial \bar{\tilde{z}}^i} \qquad \text{for all} \quad i = 1, \dots, n,$$

on $U_- = U \cap \{\tilde{r} \leqslant 0\}$. Choose complex even coordinates $z^1, \dots, z^n$ such that z^i

corresponds to $\bar{z}^i$ via (1.1). It then follows that

$$L_i = \frac{\partial}{\partial z^i} + A,$$

where A is in the kernel of (1.2) on U_-. On the other hand, we can select an odd coordinate system $s^1, \ldots, s^{2m}$ such that, for some real even sections f_{jk} and g_{jk}, we have

$$M_j = \frac{\partial}{\partial s^j} + \sum_{k=1}^{m} (f_{jk} + ig_{jk}) \frac{\partial}{\partial s^{k+m}} + B \qquad j = 1, \ldots, m$$

for some derivation B which is in the span of $\mathcal{N} \cdot \mathrm{Der}_C\, \mathcal{A}_C(U)$. The condition (1.3)″ on the M_j's implies that the matrix (g_{jk}) is invertible on U_-. If we perform the linear change $s_j \to t_j$, $s_{j+m} \to f_{kj} t_k + t_{j+m}$, $1 \leqslant j \leqslant m$, we see that f_{jk} can be assumed to be identically zero. Let (h_{jk}) be the inverse transpose of the matrix (g_{jk}) and define new odd coordinates by taking the real and imaginary part of η, where $2\eta^j = s^j - i \sum_k h_{jk} s^{k+m}$. It follows readily that on U_-

$$M_j = \frac{\partial}{\partial \eta^j} + \tilde{B},$$

where $\tilde{B}$ is a derivation in the span of $\mathcal{N} \cdot \mathrm{Der}_C\, \mathcal{A}_C(U)$.

With this choice of supercoordinates made, we have

$$(3.7) \qquad \begin{aligned} L_i &= \frac{\partial}{\partial z^i} + A_i^{j,0} \frac{\partial}{\partial z^j} + B_i^{j,0} \frac{\partial}{\partial \bar{z}^j} + C_i^{q,0} \frac{\partial}{\partial \eta^q} + D_i^{q,0} \frac{\partial}{\partial \bar{\eta}^q} \\ M_j &= \frac{\partial}{\partial \eta^j} + A_j^{i,1} \frac{\partial}{\partial z^i} + B_j^{i,1} \frac{\partial}{\partial \bar{z}^i} + C_j^{q,1} \frac{\partial}{\partial \eta^q} + D_j^{q,1} \frac{\partial}{\partial \bar{\eta}^q} \end{aligned}$$

for some sections $A_j^{i,r}$, $B_j^{i,r}$, $C_j^{q,r}$ and $D_j^{q,r}$ in the complexified nilpotent ideal $\mathcal{N}_C$. In order not to burden the notation, we shall use $\mathcal{N}$ for $\mathcal{N}_C$. We now use induction and the filtration

$$0 = \mathcal{N}^{2m+1}(U) \to \mathcal{N}^{2m}(U) \to \ldots \to \mathcal{N}(U) \to \mathcal{A}(U)$$

of ideals in $\mathcal{A}_C(U)$ to improve our guess of supercoordinates, and eliminate the coefficients $A_j^{i,r}$, $C_j^{q,r}$, above. Indeed, assume that (3.7) holds with these coefficients in $\mathcal{N}^p$. The sections $A_i^{j,0}$ and $C_j^{q,1}$ are even, while the sections $A_j^{i,1}$ and $C_i^{q,0}$ are odd. So by changing our derivations to

$$L_i - (A_i^{j,0} L_j + C_i^{q,0} M_q) \qquad M_j - (A_j^{i,1} L_i + C_j^{q,1} M_q)$$

we obtain a new family generating the same super integrable almost complex structure for which (3.7) will hold with the coefficients $A_j^{i,r}$ and $C_j^{q,r}$ of ∂_{z^i} and ∂_{η^q} in $\mathcal{N}^{p+1}$. Iterating the procedure $m+1$ times we obtain supercoodinates, the real and imaginary part of (z^j, η^q), such that on U_-

$$(3.8) \qquad L_i = \frac{\partial}{\partial z^i} + B_i^{j,0}\frac{\partial}{\partial \bar{z}^j} + D_i^{q,0}\frac{\partial}{\partial \bar{\eta}^q} \qquad M_j = \frac{\partial}{\partial \eta^j} + B_j^{i,1}\frac{\partial}{\partial \bar{z}^i} + D_j^{q,1}\frac{\partial}{\partial \bar{\eta}^q}$$

for some other choice of coefficients $B_i^{j,r}$, $D_i^{q,r}$ in $\mathcal{N}$. The question now is how to eliminate them. The analysis for this has to be finer because to get to this point, we have only used (1.5), which is not a sufficient condition to prove the theorem.

4 - The second step of the proof

Consider the coefficients in (3.8). Since they are nilpotent and of degree 0, the sections $B_i^{j,0}$ and $D_j^{q,1}$ must vanish modulo $\mathcal{N}^2$. Furthermore, modulo $\mathcal{N}^2$ we have

$$D_i^{q,0} = a_{ik}^q \eta^k + b_{ik}^q \bar{\eta}^k \qquad B_j^{i,1} = c_{jt}^i \eta^t + d_{jt}^i \bar{\eta}^t$$

for some functions a_{ik}^q, b_{ik}^q, c_{jt}^i and d_{jt}^i in U. Therefore, neglecting higher order nilpotent terms, we have

$$[L_i, M_j] = -[a_{ij}^q + (c_{jt}^p \eta^t + d_{jt}^p \bar{\eta}^t)\,\partial_{\bar{z}^p}(a_{ik}^q \eta^k + b_{ik}^q \bar{\eta}^k)]\frac{\partial}{\partial \bar{\eta}^q}$$

$$+[\partial_{z^i}(c_{jt}^p \eta^t + d_{jt}^p \bar{\eta}^t) + (a_{ik}^q \eta^k + b_{ik}^q \bar{\eta}^k)\, d_{jq}^p]\frac{\partial}{\partial \bar{z}^p}\,.$$

Since this bracket must be in the span of L_j, M_l, it follows that the sections a_{ij}^q must vanish identically, for all i, j, q. But then looking at the bracket $[L_i, L_j]$, we obtain

$$[L_i, L_j] = [\partial_{z^i} b_{jt}^r - \partial_{z^j} b_{it}^r + (\sum_q b_{it}^q b_{jq}^r - \sum_q b_{jt}^q b_{iq}^r)]\,\bar{\eta}^t \frac{\partial}{\partial \bar{\eta}_r} \qquad \mathrm{mod}\ \mathcal{N}^2\ \mathrm{Der}_C\ \mathcal{A}_C(U)\,.$$

Once again, since this bracket must be in the span of L_j, M_l, we conclude that

$$(4.9) \qquad \partial_{z^i} b_{jt}^r - \partial_{z^j} b_{it}^r + (\sum_q b_{it}^q b_{jq}^r - \sum_q b_{jt}^q b_{iq}^r) = 0\,.$$

It is now convenient to switch our viewpoint to superforms which behave covariantly under coordinate transformations. Recall that if r, s are supercoordinates, the super exterior derivative is computed by

$$\mathrm{d} = \mathrm{d}r^i \frac{\partial}{\partial r^i} + \mathrm{d}s^l \frac{\partial}{\partial s^l} .$$

In the coordinates found so far the forms θ^i, φ^k dual to L_i, M_k, can be expressed on U_- as

$$(4.10) \qquad \theta^i = \mathrm{d}z^i + \mathrm{d}\bar{\eta}^k C_k^i \qquad \varphi^k = \mathrm{d}\eta^k - \mathrm{d}\bar{z}^r \eta^t \bar{b}_{rt}^k \qquad \mathrm{mod}\ \mathcal{N}^2\Omega_C^1(U)$$

where $\bar{b}_{rt}^j$ is the conjugate of the coefficient above, and C_k^j is some nilpotent section over U, which when expanded as $\eta^l f_{kl}^i + \bar{\eta}^l g_{kl}^i$, implies that $f_{kl}^i = -\bar{d}_{kl}^i$ and $g_{kl}^i = -\bar{c}_{kl}^i$. Here $\Omega_C^1(U)$ denote the space of C^∞ complexified super one-forms in U.

In term of the superforms θ^i and φ^k, the integrability condition (1.4)″ is equivalent to the fact that their super exterior derivative is in the ideal generated by them. Using this condition on $\theta^i = \mathrm{d}z^i + \mathrm{d}\bar{\eta}^k \eta^l f_{kl}^i + \mathrm{d}\bar{\eta}^k \bar{\eta}^l g_{kl}^i$, we conclude that $g_{kl}^i + g_{lk}^i = 0$. Consequently, if we take as our new even set of coordinates the real and imaginary parts of $w^j = z^j + \bar{\eta}^k \eta^l f_{kl}^j + \bar{\eta}^k \bar{\eta}^l g_{kl}^j$, we obtain that on U_-

$$\theta^i = \mathrm{d}w^i - \bar{\eta}^k \varphi^l f_{kl}^i \qquad \mathrm{mod}\ \mathcal{N}^2\Omega_C^1(U).$$

If we change the form θ^i, to $\theta^i + \bar{\eta}^k \varphi^l f_{kl}^i$, we do not change the super almost complex structure and we get

$$\theta^i = \mathrm{d}w^i \qquad \mathrm{mod}\ \mathcal{N}^2\Omega_C^1(U).$$

We rename the w^i's as z^i's. Thus we have the desired result for the forms θ^i, namely on U_-

$$\theta^i = \mathrm{d}z^i \qquad \mathrm{mod}\ \mathcal{N}^2\Omega_C^1(U).$$

In order to make an analogous simplification in the φ^j, we consider $U \times \boldsymbol{C}^m$ with the coordinates $z^1, \ldots, z^n$ we have found in U, and the standard coordinates $w^1, \ldots, w^m$ in $\boldsymbol{C}^m$. On $U \times \boldsymbol{C}^m$ we introduce the complex vector fields

$$(4.11) \qquad W^j = \frac{\partial}{\partial w^j} \qquad Z^j = \frac{\partial}{\partial z^j} + b_{jt}^r \bar{w}^t \frac{\partial}{\partial \bar{w}^r} .$$

The brackets of the W^j's are clearly zero, as well as the brackets of the W^j's with the Z^{l}'s. That the brachets of the Z^{l}'s are all zero in $U_- \times C^m$ follows from (4.9). Thus in $U_- \times C^m$ we have a classical integrable almost complex structure.

Proposition 2. *There exists a neighborhood G of $(p, 0)$ in $U \times C^m$, such that the holomorphic coordinates $z^1, \dots, z^n$ can be completed to a holomorphic coordinate system $z^1, \dots, z^n, v^1, \dots, v^m$ in $G_- = G \cap (U_- \times C^m)$ with respect to the structure defined by* (4.11).

This proposition follows from Theorem 3 by observing that the Levi form of $\partial U_- \times C^m$ is positive semidefinite in a neighborhood of $(p, 0)$, because of the positive semidefiniteness of the Levi form of ∂U_-.

The functions v^l are annihilated by $\bar{W}^l$ and thus v^l admits an expansion of the form

$$v^l = h^l + h^l_j w^j + o(|w|).$$

Since this function must be annihilated by $\bar{Z}^k$, it follows that

$$\partial_{\bar{z}^j} h^k_t = -\bar{b}^r_{jt} h^k_r .$$

The linear independence of the forms dz^i, dv^j implies that the matrix (h^k_j) has maximal rank. By a suitable shrinking of U_-, the statements about h^l_j hold on U_-. This provides us with an up to the boundary version of a lemma of McHugh [13], which will suffice to continue the proof.

Indeed, on U_- mod $\mathcal{N}^2\Omega^1_C(U_-)$, we have

$$h^k_j \varphi^j = h^k_j d\eta^j - h^k_j d\bar{z}^t \bar{b}^j_{tl} \eta^l = h^k_j d\eta^j + d\bar{z}^t \partial_{\bar{z}^t} h^k_l \eta^l = d(h^k_j \eta^j) - \theta^l \partial_{z^l} h^k_s \eta^s .$$

Hence by changing the forms φ^k to $h^k_j \varphi^j + \theta^l \partial_{z^l} h^k_s \eta^s$ and leaving the θ^i's the same, we do not change the given structure on U_-. Setting $h^k_j \eta^j$ as our new supercoordinate η^k, we obtain that on U_-

$$\theta^i = dz^i \qquad \varphi^j = d\eta^j \qquad \mod \mathcal{N}^2\Omega^1_C(U_-).$$

We remark that this suffices to show that $\mathcal{N}/\mathcal{N}^2$ is a sheaf of sections of a holomorphic vector bundle over U_-. Indeed, if we have a change of odd coordinates that preserves the super almost complex structure, which mod $\mathcal{N}^2$ we can write as

$$\zeta^l = p^l_k \eta^k + q^l_k \bar{\eta}^k ,$$

then $\zeta^l = p^l_k \eta^k$ with p^l_k holomorphic. This follows by using the exterior derivative and the fact that the conjugates of L_i, M_j would have to be in the annihilator of $d\zeta^l$, which implies

$$d\bar{z}^r \partial_{\bar{z}^r} p^l_k \eta^k + d\bar{\eta}^l q^k_l = 0 \qquad \mathrm{mod}\ \mathcal{N}^2 \Omega^1_C(U).$$

Here $q^k_l = 0$ and $\partial_{\bar{z}^r} p^l_k = 0$.

5 - Last step in the proof

In order to finish the proof we shall need the theorem below, which uses the work of Kohn [9] and which is essentially in Amar [1]. We prefer to give our own proof.

Consider a C^∞ *weakly pseudoconvex hypersurface* Σ in $\boldsymbol{C}^n$, and a point $p \in \Sigma$. Near this point we represent Σ as $\{\rho = 0\}$ with $d\rho \neq 0$ and the surface weakly pseudoconvex from the side $\{\rho < 0\}$. If U is a neighborhood of p in $\boldsymbol{C}^n$, we indicate by U_- the set $U \cap \{\rho \leqslant 0\}$.

Theorem 4. *Given a fundamental sequence $\{U\}$ of neighborhoods of p, there exists a corresponding fundamental sequence $\{V\}$ of neighborhoods of p in U_-; i.e., each $V \subset U_-$ with the following properties*:

1. *The $\{V \cap \Sigma\}$ form a fundamental sequence of neighborhoods of p in Σ.*

2. *Each V is a compact domain having a C^∞ weakly pseudoconvex boundary.*

3. *Given a C^∞ $\bar{\partial}$-closed $(0, 1)$-form f in U_-, there exists a corresponding C^∞ function u on V such that $\bar{\partial} u = f$ on V*

Note that this theorem gives a solution u which is C^∞ up to the boundary near p. In order not to interrupt the flow of ideas, we proceed with the proof of Theorem 1, and postpone the proof of Theorem 4 until after that.

We use an induction argument. Indeed we now have coordinates such that for some sections B^i_j, C^i_q, D^k_j and E^k_q in $\mathcal{N}^l$, on U_- the following relation holds

$$(5.12) \qquad \begin{aligned} \theta^i &= dz^i + d\bar{z}^j B^i_j + d\bar{\eta}^q C^i_q \\ \varphi^k &= d\eta^k + d\bar{z}^j D^k_j + d\bar{\eta}^q E^k_q \end{aligned} \qquad \mathrm{mod}\ \mathcal{N}^{l+1} \Omega^1_C(U)$$

with $l = 2$. Given that (5.12) is true on U_- we want to prove that such result holds for l replaced $l + 1$. We assume that l is even, as the case where l is odd follows by the same argument.

If we expand the superforms θ^j, φ^k modulo nilpotent terms of degree $l+1$, we get

$$\theta^i = \mathrm{d}z^i + \mathrm{d}\bar{z}^j \eta^I \bar{\eta}^J b^i_{j,I,J} + \mathrm{d}\bar{\eta}^q \eta^I \bar{\eta}^J c^i_{q,I,J}$$
$$\varphi^k = \mathrm{d}\eta^k + \mathrm{d}\bar{z}^j \eta^I \bar{\eta}^J d^k_{j,I,J} + \mathrm{d}\bar{\eta}^q \eta^I \bar{\eta}^J e^k_{q,I,J}$$

where in each sum, I, J are multi indices such that $|I| + |J| = l$. The integrability condition readily inplies that the only nontrivial contributions in the expression above correspond to the cases where the multi index J is 0. So we actually have:

$$\begin{aligned} \theta^i &= \mathrm{d}z^i + \mathrm{d}\bar{z}^j \eta^I b^i_{j,I,0} + \mathrm{d}\bar{\eta}^q \eta^I c^i_{q,I,0} \\ \varphi^k &= \mathrm{d}\eta^k + \mathrm{d}\bar{z}^j \eta^I d^k_{j,I,0} + \mathrm{d}\bar{\eta}^q \eta^I e^k_{q,I,0} \end{aligned} \qquad \bmod \mathcal{N}^{l+1}\Omega^1_C(U).$$

Since $\mathrm{d}\bar{\eta}^q \eta^I$ is odd while $\mathrm{d}\bar{z}^j \eta^I$ is even, we also conclude that $c^i_{q,I,0}$ and $d^k_{j,I,0}$ are both zero, and we have

$$\theta^i = \mathrm{d}z^i + \mathrm{d}\bar{z}^j \eta^I b^i_{j,I,0} \qquad \varphi^k = \mathrm{d}\eta^k + \mathrm{d}\bar{\eta}^t \eta^I e^k_{t,I,0} \qquad \bmod \mathcal{N}^{l+1}\Omega^1_C(U).$$

If we now compute $\mathrm{d}\theta^i$ from this expression, we conclude from the integrability condition that the superform $\mathrm{d}\bar{z}^j \wedge \mathrm{d}\bar{z}^k \partial b^i_{j,I,0}/\partial \bar{z}^k$ must be zero on U_-. Thus the (0, 1)-form $\mathrm{d}\bar{z}^j b^i_{j,I,0}$ is $\bar{\partial}$ closed on U_-. Using Theorem 4 we obtain functions h^i_I on some neighborhood of p such that $\bar{\partial} h^i_I = \mathrm{d}\bar{z}^l b^i_{l,I}$ on U_-. We switch our super coordinates to $z^i + h^i_I \eta^I$ and $\eta^k + \bar{\eta}^t \eta^I e^k_{t,I,0}$. Then we have that

$$\begin{aligned} \mathrm{d}z^i &= \theta^i + \theta^j \frac{\partial h^i_I}{\partial z^j} \eta^I + (-1)^{\delta_{I,p}} \varphi^p \eta^{I'} h^i_I \\ \mathrm{d}\eta^t &= \varphi^t - (-1)^{\delta_{I,q}} \varphi^q \bar{\eta}^p \eta^{I'} e_{q,I,0} \end{aligned} \qquad \bmod \mathcal{N}^{l+1}\Omega^1_C(U)$$

for some new neighborhood U_- possibly smaller than the one in (5.12). Here, $\delta_{I,p} = \pm 1$ and $\eta^p \eta^{I'} = \eta^I$. If we now replace the superforms θ^i, φ^t by the linear combinations on the right in the above expression, we do not modify the given integrable super almost complex structure, and (5.12) holds with l replaced by $l+1$. This completes the induction and the proof of Theorem 1.

Proof of Theorem 4. We can assume that the full complex Hessian of ρ is positive semidefinite on Σ, by replacing ρ by $e^{\lambda \rho} - 1$ for some $\lambda > 0$ sufficiently large. We can choose p to be the origin and assume that $\rho = h(x_1, z_2, \ldots, z_n) - y_1$, where h vanishes to second order at 0. Now we have that $\partial\bar{\partial}\rho \geqslant 0$ in a full neighborhood of p.

Given the set U, we choose a sufficiently small ball B centered about the origin which we write as $\{\varphi \leqslant 0\}$ for $\varphi = \|z\|^2 - \varepsilon^2$.

For some δ suitably fixed with respect to ε, we shall use a nonnegative even cutoff function $\chi(t) \in C_c^\infty(\boldsymbol{R})$, with support in $[-2\delta, 2\delta]$ such that:

1. $\chi(t) \equiv 1$ on $[-\delta, \delta]$.

2. $\dot{\chi}(t) \geqslant 0$ $[-2\delta, 0]$.

3. $\chi(t)$ has only one inflection point at $(-\frac{3\delta}{2}, \frac{1}{2})$ in the interval $[-2\delta, 0]$ and on $[-\frac{3\delta}{2}, 0]$ we have $-M \leqslant \ddot{\chi}(t) \leqslant 0$ for some $M = M(\delta)$.

Our V will be constructed as a domain of the form $V = \{\psi \leqslant 1\}$ where $\psi = (\chi(\rho)e^{n\rho} + \chi(\varphi)e^{n\varphi})^{\frac{1}{n}}$, for some n sufficiently large which will be chosen later.

It is clear that $\psi = \rho$ when $\varphi \leqslant -2\delta$ and $-\delta \leqslant \rho \leqslant 0$, and $\psi = \varphi$ when $\rho \leqslant -2\delta$ and $-\delta \leqslant \varphi \leqslant 0$. Choosing n large enough we can make $d\psi \neq 0$ when $\psi = 1$, and also $\psi < 1$ when $\rho \leqslant -\delta$ and $\varphi \leqslant -\delta$. Observe that what this accomplishes is to round the corners near the intersection of Σ with ∂B.

We proceed to study the Levi form of the boundary of V. In order to show that the boundary of V is weakly pseudoconvex, it will suffice to show that the Levi form of $\tilde{\psi} = \log \psi$ is positive semidefinite on the boundary of V. If $A = \chi(\rho)e^{n\rho} + \chi(\varphi)e^{n\varphi}$, then

$$\partial\bar{\partial}\tilde{\psi} = -\frac{1}{nA^2}\partial A \wedge \bar{\partial}A + \frac{1}{nA}\partial\bar{\partial}A .$$

The first term in the right side of the expression above does not contribute to the Levi form, because on $\psi = 1$ the $(1, 0)$-vectors annihilated by ∂A are precisely the $(1, 0)$-vectors tangent to the bondary. Thus it suffices to show that $\partial\bar{\partial}A \geqslant 0$ at $\psi = 1$.

Obviously we need only consider the corners where $\psi = 1$ concides neither with Σ nor ∂B. In the region where $-\delta \leqslant \rho \leqslant 0$ and $-\delta \leqslant \varphi \leqslant 0$ we have that $\partial\bar{\partial}A = \partial\bar{\partial}(e^{n\rho}) + \partial\bar{\partial}(e^{n\varphi})$, which is positive definite in all directions. In the region where $-\delta \leqslant \rho \leqslant 0$ and $\varphi \leqslant -\delta$, we obtain that

$$\partial\bar{\partial}A = \partial\bar{\partial}(e^{n\rho}) + (n\chi + \dot{\chi})e^{n\varphi}\partial\bar{\partial}\varphi + (n^2\chi + 2n\dot{\chi} + \ddot{\chi})e^{n\varphi}\partial\varphi \wedge \bar{\partial}\varphi$$

where $\chi = \chi(\varphi)$. We can make the above positive semidefinite in all directions by choosing $n > \sqrt{2M}$. In the remaining region $-\delta \leqslant \varphi \leqslant 0$ and $\rho \leqslant -\delta$ the same argument with the roles of ρ and φ interchanged shows that $\partial\bar{\partial}A$ is positive definite in all directions.

We now have produced a compact domain $V \subset U_-$ having a C^∞ weakly pseudoconvex bondary, which coincides with Σ in a neighborhood of p. Given a $\bar{\partial}$-closed (0, 1)-form f that is C^∞ on U_-, we may use the result of Kohn [9] to write its restriction to V as $\bar{\partial}u$ for some function u which is C^∞ in V. This completes the proof of Theorem 4.

6 - Proof of Theorem 2

Suppose we have a real $2n$ dimensional manifold X with smooth boundary, and that $L_1, \ldots, L_n$ are complex vector fields on X which define a classical integrable almost complex structure smooth up to the boundary. Given m consider the supermanifold $(X, \mathcal{A}, \alpha)$ where $\mathcal{A} = C^\infty(X) \otimes \wedge^* \boldsymbol{R}^{2m}$ and α is the natural projection onto $C^\infty(X)$. Choose real coordinates (s, t) for $\boldsymbol{R}^{2m}$ and let $M_k = \frac{1}{2}(\frac{\partial}{\partial s^k} - i\frac{\partial}{\partial t^k})$ be the odd derivations of the complexified exterior algebra. The family $\{L_1, \ldots, L_n, M_1, \ldots, M_m\}$ defines an integrable super almost complex structure on our supermanifold which is smooth up to the boundary.

Suppose that in some neighborhood of a boundary point there exist supercoordinates (z, η) so that $\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \eta^q}$ generate the super almost complex structure. Then coordinates $\tilde{z}^1, \ldots, \tilde{z}^n$ corresponding to $z^1, \ldots, z^n$ via the augmentation map give classical holomorphic coordinates up to the boundary point in X. Thus we have a counterexample to Theorem 1 whenever we can produce a counterexample to the existence to these $\tilde{z}$'s.

For Levi signature of the form $-0+\ +\ldots+$ such counterexamples were found in [5], [6], [7]. By a refinement of these results, which involves a bending in the fiber direction, similar counterexamples can be constructed having Levi signature $-++\ldots+$. The reader is referred to the paper of Hill and Nacinovich [8].

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Sommario

Si ottiene una versione del teorema di Newlander e Nirenberg per supervarietà complesse a contorno debolmente pseudoconvesso. Si mostra inoltre che esistono controesempi nei quali si ha esattamente un grado di pseudoconcavità.

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