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Curvatures on anti-Kaehler manifolds (**)

1 - Introduction and preliminaries

In this paper we introduce a special class of hermitian manifolds and call them *anti-Kaehler manifolds*. We characterize these manifolds in terms of their curvature properties and give examples of anti-Kaehler manifolds with flat hermitian curvatures. We obtain a local description of the anti-Kaehler manifolds with pointwise constant complex holomorphic sectional curvatures.

Let M be a *hermitian manifold* with complex structure J and hermitian metric h. The tangential space to M at a point $p \in M$ and its complexification are denoted by T_pM and T_p^CM, respectively. The algebras of real differentiable vector fields, complex differentiable vector fields and vector fields of type $(1, 0)$ on M are denoted by $\mathfrak{X}M$, $\mathfrak{X}^CM$ and $\mathfrak{X}^{1,0}M$, respectively.

If $\dim_C M = n$ and $z^1, \dots, z^n$ are holomorphic coordinate functions on M, then the complex vector fields $\partial_\alpha = \dfrac{\partial}{\partial z^\alpha}$ (resp. $\partial_{\bar\alpha} = \dfrac{\partial}{\partial \bar z^\alpha}$), form a basis for $T_p^{1,0}M$ (resp. $T_p^{0,1}M$).

Further, greek indices $\alpha, \beta, \gamma, \dots$ run from 1 to n, while latin letters $i, j, k, \dots$ run through $1, \dots, n, \bar 1, \dots, \bar n$.

The *fundamental 2-form* Φ on M is defined by $\Phi(X, Y) = h(JX, Y)$; $X, Y \in X^CM$.

In terms of local holomorphic coordinates we have

$$h_{\alpha\bar\beta} = h_{\bar\beta\alpha} \qquad \Phi_{\alpha\bar\beta} = -\Phi_{\bar\beta\alpha}\,.$$

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The exterior derivative $d\Phi$ has the essential (which may not be zero) components

$$d\Phi_{\alpha\beta\bar{\gamma}} = \partial_\alpha \Phi_{\beta\bar{\gamma}} - \partial_\beta \Phi_{\alpha\bar{\gamma}} .$$

On a hermitian manifold (M, J, h) we consider the following three connections:

a) The Levi-Civita connection ∇ of the metric h. The local components of ∇ are denoted by Γ^k_{ij} and its curvature tensor by R.

b) The hermitian connection D with torsion tensor T. This connection is characterized by the conditions: $Dh = DJ = 0$ and $T(JX, Y) = T(X, JY)$, $X, Y \in \mathfrak{X}^C M$. The curvature tensor of D is denoted by K. In local holomorphic coordinates we have:

$$D^\gamma_{\alpha\beta} = h^{\gamma\bar{\sigma}} \partial_\alpha h_{\beta\bar{\sigma}} \tag{1.1}$$

$$K^\lambda_{\alpha\bar{\beta}\gamma} = - \partial_{\bar{\beta}} D^\lambda_{\alpha\gamma} \qquad K_{\alpha\bar{\beta}\gamma\bar{\delta}} = K^\lambda_{\alpha\bar{\beta}\gamma} h_{\lambda\bar{\delta}} . \tag{1.2}$$

In this paper we use the following contractions of the curvature tensor K:

$$k_{\alpha\bar{\beta}} = h^{\lambda\bar{\mu}} K_{\alpha\bar{\beta}\lambda\bar{\mu}} \qquad \kappa = h^{\alpha\bar{\beta}} k_{\alpha\bar{\beta}} .$$

c) The associated connection $\tilde{D}$. This connection is defined by $\tilde{D} = D - \frac{1}{2} T$. The relation between ∇ and $\tilde{D}$ is given by

$$h(\tilde{D}_X Y, Z) = h(D_X Y, Z) + \frac{1}{4} (d\Phi(X, JY, Z) - d\Phi(JX, Y, Z)) \qquad X, Y, Z \in \mathfrak{X}^C M .$$

The essential components of $\tilde{D}$ are

$$\tilde{D}^\lambda_{\alpha\beta} = \frac{1}{2} h^{\lambda\bar{\sigma}} (\partial_\alpha h_{\beta\bar{\sigma}} + \partial_\beta h_{\alpha\bar{\sigma}}) = \Gamma^\lambda_{\alpha\beta} . \tag{1.3}$$

The associated connection $\tilde{D}$ has the following properties:

$\tilde{D}$ is torsion-free; $\tilde{D}J = 0$, i.e. $\tilde{D}$ is a complex connection;

$$\tilde{D}_\alpha h_{\beta\bar{\gamma}} = - \frac{i}{2} d\Phi_{\alpha\beta\bar{\gamma}} .$$

Let $\widetilde{K}$ denote the curvature tensor of $\widetilde{D}$. In local holomorphic coordinates we have

$$\widetilde{K}^{\lambda}_{\alpha\bar{\beta}\gamma} = -\partial_{\bar{\beta}}\Gamma^{\lambda}_{\alpha\gamma} \tag{1.4}$$

$$\widetilde{K}^{\lambda}_{\alpha\beta\gamma} = \partial_{\alpha}\Gamma^{\lambda}_{\beta\gamma} - \partial_{\beta}\Gamma^{\lambda}_{\alpha\gamma} + \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\lambda}_{\alpha\sigma} - \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\lambda}_{\beta\sigma} = R^{\lambda}_{\alpha\beta\gamma}\,. \tag{1.5}$$

Some properties of $\widetilde{D}$ have been considered in [1].

2 - Anti-Kaehler manifolds

In this section we introduce the class of anti-Kaehler manifolds and find some geometric characterizations for them.

Definition 1. A hermitian manifold (M, J, h) is said to be *anti-Kaehler* if there exists an open covering of M with holomorphic coordinate neighbourhoods (U, z^{α}), so that

$$\partial_{\alpha} h_{\beta\bar{\gamma}} = -\partial_{\beta} h_{\alpha\bar{\gamma}}$$

with respect to the local holomorphic coordinates $z^1, \ldots, z^n$ in U.

We recall that on a Kaehler manifold $\partial_{\alpha} h_{\beta\bar{\gamma}} = \partial_{\beta} h_{\alpha\bar{\gamma}}$ with respect to any local system of holomorphic coordinates.

The next theorem gives a tensor characterization for anti-Kaehler manifolds.

Proposition 1. *For a hermitian manifold* (M, J, h) *the following conditions are equivalent:*

i) *The manifold* (M, J, h) *is anti-Kaehler*
ii) *The associated connection* $\widetilde{D}$ *is flat.*

Proof. Let (M, J, h) be anti-Kaehler and (U, z^{α}) be a holomorphic coordinate system, satisfying the conditions of Definition 1. Taking into account (1.3), (1.4) and (1.5) we find $\widetilde{K} = 0$.

Conversely, let $\widetilde{K} = 0$. Since $\widetilde{D}$ is a torsion-free and complex connection, then there exist locally holomorphic coordinate systems (U, z^{α}) such that $\widetilde{D}^{\lambda}_{\alpha\beta} = 0$ with respect to the coordinates $z^1, \ldots, z^n$. From (1.3) it follows that $\partial_{\alpha} h_{\beta\bar{\gamma}} = -\partial_{\beta} h_{\alpha\bar{\gamma}}$ in U, which proves the statement.

Using (1.1), (1.2), (1.3) and (1.4) we find that on a hermitian manifold

$$2\tilde{K}_{\alpha\bar{\beta}\gamma\bar{\delta}} = K_{\alpha\bar{\beta}\gamma\bar{\delta}} + K_{\gamma\bar{\beta}\alpha\bar{\delta}} . \tag{2.1}$$

The proofs of the following lemmas are standard.

Lemma 1. *Let K be the hermitian curvature tensor on a hermitian manifold. The following conditions are equivalent:*

1) $K_{\alpha\bar{\beta}\gamma\bar{\delta}} + K_{\gamma\bar{\beta}\alpha\bar{\delta}} = 0$

2) $K(Jx, y)z + K(Jy, z)x + K(Jz, x)y = 0 \qquad x, y, z \in T_pM, \; p \in M.$

Lemma 2. *Let R be the riemannian curvature tensor on a hermitian manifold. The following conditions are equivalent:*

1) $R^{\lambda}_{\alpha\beta\gamma} = 0$

2) $R(x, y, z, u) = R(Jx, Jy, Jz, Ju) \qquad x, y, z, u \in T_pM, \; p \in M.$

Applying consequently Proposition 1, Lemma 1, Lemma 2, (1.5) and (2.1), we obtain the following characterization of anti-Kaehler manifolds in terms of their hermitian and riemannian curvatures.

Proposition 2. *A hermitian manifold with hermitian curvature tensor K and riemannian curvature R is anti-Kaehler iff*

$$K(Jx, y)z + K(Jy, z)x + K(Jz, x)y = 0$$

$$R(x, y, z, u) = R(Jx, Jy, Jz, Ju)$$

for arbitrary vectors x, y, z, u in T_pM, $p \in M$.

Let (M, J, h) be an anti-Kaehler manifold. From Lemma 1 it follows that for all X, Y, Z, U in $T_p^{1,0}M$

$$K(X, \bar{Y}, Z, \bar{U}) = -K(Z, \bar{Y}, X, \bar{U}) = K(Z, \bar{U}, X, \bar{Y}) . \tag{2.2}$$

Let $E = \text{span}\{X, Y\}$ be a complex holomorphic section, i.e. a complex 2-plane in the holomorphic part $T_p^{1,0}M$ of the complex tangent space T_p^CM at $p \in M$. Taking into account the symmetries (2.2) of the hermitian curvature tensor K, we define the *complex holomorphic sectional curvature* of E with respect to K by the equality

$$K(E; p) = \frac{K(X, \bar{X}, Y, \bar{Y})}{h(X, \bar{X})h(Y, \bar{Y}) - |h(X, \bar{Y}|^2}. \tag{2.3}$$

It is clear that $K(E; p)$ is a real number depending in general on $p \in M$ and E in $T_p^{1,0}M$.

Remark. A more general definition for complex holomorphic sectional curvature is given in [3].

Definition 2. An anti-Kaehler manifold is said to be *of pointwise constant complex holomorphic sectional curvature* ν if the complex holomorphic sectional curvatures do not depend on the complex holomorphic section, i.e. $K(E; p) = \nu(p)$.

Taking into account (2.3) and (2.2) it is easy to prove

Lemma 3. *Let* (M, J, h) *be an anti-Kaehler manifold with* $\dim_C M \geq 3$. *Then* M *is of pointwise constant complex holomorphic sectional curvatures* ν *iff*

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nu(h_{\alpha\bar{\beta}}h_{\gamma\bar{\delta}} - h_{\gamma\bar{\beta}}h_{\alpha\bar{\delta}}) \tag{2.4}$$

where $n(n-1)\nu = \kappa$.

Remark. If $\dim_C M = 2$, the equality (2.4) is satisfied for an arbitrary anti-Kaehler manifold.

3 - Anti-Kaehler manifolds of pointwise constant complex holomorphic sectional curvatures

In this section we study anti-Kaehler manifolds with hermitian curvature tensor K, satisfying (2.4).

First we consider the case $\nu = \kappa = 0$.

Proposition 3. *Let* $M = \boldsymbol{G}$ *be a complex Lie group. The standard hermitian structure on* $M = \boldsymbol{G}$ *is anti-Kaehler, iff the group* $\boldsymbol{G}$ *is two-step nilpotent.*

Proof. Let $\{Z_\alpha\}$, $\alpha = 1, \ldots, n$ be a basis of the Lie algebra $\mathfrak{g}$ of $\boldsymbol{G}$. The standard hermitian metric h on $\boldsymbol{G}$ is defined by the equalities $h(Z_\alpha, Z_\beta) = 0$, $h(Z_\alpha, Z_{\bar\beta}) = \delta_{\alpha\bar\beta}$, where $\delta_{\alpha\bar\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\bar\beta} = 0$ if $\alpha \neq \beta$. The canonical flat connection D defined by $DZ_\alpha = DZ_{\bar\beta} = 0$ is the hermitian connection on $(\boldsymbol{G}, h)$ with torsion tensor T satisfying the equalities $T(Z_\alpha, Z_\beta) = -[Z_\alpha, Z_\beta]$. If ∇ is the Levi-Civita connection of the metric h, then $\nabla_{Z_\alpha} Z_\beta = \frac{1}{2}[Z_\alpha, Z_\beta]$ and the riemannian curvature tensor R satisfies the condition $R(Z_\alpha, Z_\beta)Z_\gamma = -\frac{1}{4}[[Z_\alpha, Z_\beta], Z_\gamma]$. This equality and Proposition 2 imply the assertion.

From Proposition 3 we obtain

Examples 1. Let $M = \boldsymbol{G}$ be a two-step nilpotent complex Lie group. Then the standard hermitian structure on $\boldsymbol{G}$ is anti-Kaehler with flat hermitian connection D and parallel torsion T ($DT = 0$). Every complex Heisenberg group is in the above set of examples.

Examples 2. Let $\boldsymbol{H}$ be a discrete subgroup of a two-step nilpotent complex Lie group $\boldsymbol{G}$. Then the quotient space $\boldsymbol{G}/\boldsymbol{H}$ with the standard hermitian structure is a compact anti-Kaehler manifold with flat hermitian connection. The Iwasawa manifold (see [2]) is in the above examples.

Further we consider anti-Kaehler manifolds satisfying the condition (2.4) with $\kappa \neq 0$ ($\nu \neq 0$) on M.

Theorem 1. *Let (M, J, h) be an anti-Kaehler manifold of pointwise constant complex holomorphic sectional curvatures $\nu = \dfrac{\kappa}{n(n-1)} \neq 0$. Then the metric*

$$g = \frac{2|\kappa|}{n(n-1)} h \tag{3.1}$$

is a Kaehler metric of constant holomorphic sectional curvature $-\varepsilon$ ($\varepsilon = \operatorname{sgn}\kappa$) and the scalar curvature κ satisfies the equalities

$$\partial_\alpha \partial_{\bar\beta} \ln|\kappa| = \varepsilon g_{\alpha\bar\beta}$$

$$\nabla^0_\alpha \partial_\beta \ln|\kappa| + \frac{1}{2} \partial_\alpha \ln|\kappa| \, \partial_\beta \ln|\kappa| = 0$$

where ∇^0 is the Levi-Civita connection of the metric g.

Proof. Under the assumptions of the theorem from Lemma 3 it follows that

$$k_{\alpha\bar{\beta}} = \frac{\kappa}{n} h_{\alpha\bar{\beta}} . \tag{3.2}$$

The second Bianchi identity for the tensor K and (3.2) imply the metric g is Kaehler. Let R^0 be the curvature tensor of the Levi-Civita connection ∇^0 of the metric g. Taking into account (3.1) we calculate

$$R^{0\,\lambda}_{\,\alpha\bar{\beta}\gamma} = K^{\lambda}_{\alpha\bar{\beta}\gamma} - \partial_\alpha \partial_{\bar{\beta}} \ln|\kappa| \delta^{\lambda}_{\gamma} . \tag{3.3}$$

From (3.1), (3.2) and (3.3) we find $\partial_\alpha \partial_{\bar{\beta}} \ln|\kappa| = \varepsilon g_{\alpha\bar{\beta}}$ and

$$R^{0\,\lambda}_{\,\alpha\bar{\beta}\gamma} = -\frac{\varepsilon}{2}(g_{\alpha\bar{\beta}}\delta^{\lambda}_{\gamma} + g_{\gamma\bar{\beta}}\delta^{\lambda}_{\alpha}) .$$

Hence, (M, J, g) is a Kaehler manifold of constant holomorphic sectional curvature $-\varepsilon$.

Under the assumptions of the theorem, Proposition 2, Lemma 2 and (3.1) imply the second equality of Theorem 1.

Following the proof of Theorem 1 in reverse order, we obtain

Theorem 2. *Let (M, J, g) be a Kaehler manifold of constant holomorphic sectional curvature $\varepsilon = \pm 1$, and u be a real C^∞ function on M satisfying the equations*

$$\partial_\alpha \partial_{\bar{\beta}} u = \varepsilon g_{\alpha\bar{\beta}}$$

$$\nabla^0_\alpha \partial_\beta u + \frac{1}{2} \partial_\alpha u \, \partial_\beta u = 0$$

where ∇^0 is the Levi-Civita connection of the metric g. Then the manifold (M, J, h), where $h = \frac{1}{2} n(n-1) e^{-u} g$, is an anti-Kaehler manifold of pointwise constant complex holomorphic sectional curvature $\nu = -\frac{\varepsilon e^u}{n(n-1)}$.

Applying Theorem 2 by direct computations we obtain

Examples 3. Let $M = \boldsymbol{D}^n$, where $\boldsymbol{D}^n$ is the unit ball: $\delta_{\alpha\bar{\mu}} z^\alpha z^{\bar{\mu}} = r^2 < 1$ in $\boldsymbol{C}^n$ and ν_0 be a positive constant. We put $\omega_\alpha = \delta_{\alpha\bar{\mu}} z^{\bar{\mu}}$ and construct the metric

$$h_{\alpha\bar{\beta}} = \frac{1}{\nu_0}((1 - r^2)\delta_{\alpha\bar{\beta}} + \omega_\alpha \omega_{\bar{\beta}}) .$$

The manifold $(\boldsymbol{D}^n, h)$ is an anti-Kaehler manifold of pointwise constant complex holomorphic sectional curvature $\nu = \nu_0 \, (1 - r^2)^{-2} > 0$.

Examples 4. Let $M = \boldsymbol{C}^n$ and

$$h_{\alpha\bar{\beta}} = -\frac{1}{\nu_0}\left((1 + r^2)\,\delta_{\alpha\bar{\beta}} - \omega_\alpha\,\omega_{\bar{\beta}}\right)$$

where ν_0 is a negative constant. Then the manifold $(\boldsymbol{C}^n, h)$ is an anti-Kaehler manifold of pointwise constant complex holomorphic sectional curvature $\nu = \nu_0 \, (1 + r^2)^{-2} < 0$.

References

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Sommario

Si introduce una nuova classe di varietà hermitiane, quella delle varietà anti-kaehleriane. Si indicano vari esempi di varietà anti-kaehleriane a curvatura hermitiana piatta. Si ottiene una descrizione locale delle varietà anti-kaehleriane a curvatura sezionale olomorfa complessa puntualmente costante.

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