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Two counter-examples for the Okada theorem in C^n (**)

1 - Statement of the problem

Let Ω be an open subset of C^n $(0 \in \Omega)$. Let

$$N(z) = (\sigma_{mk}(z)) \qquad N_1(z) = (\sigma_{mk}^{(1)}(z)) \qquad \dots N_n(z) = (\sigma_{mk}^{(n)}(z))$$

be $(n+1)$ infinite *triangular matrices* of complex-valued functions, defined in C^n, satisfying

$$\lim_{m \to +\infty} \sum_{k=0}^{+\infty} \sigma_{mk}(0) = 1 \quad \text{and} \quad \lim_{m_{\lambda_1} \to +\infty, \dots, m_{\lambda_\mu} \to +\infty} \sum_{k_1=0}^{+\infty} \sigma_{m_1 k_1}^{(1)}(0) \ \dots \sum_{k_n=0}^{+\infty} \sigma_{m_n k_n}^{(n)}(0) = 1 .$$

Here and in the sequel $z = (z_1, \dots, z_n)$ and the indices m, k, m_r, k_r, m_{λ_s} $(r = 1, \dots, n;\ s = 1, \dots, \mu;\ 1 \leq \mu \leq n)$ run over $0, 1, 2, \dots$.

The set of all functions holomorphic in Ω will be denoted by $H(\Omega)$. $H(\Omega)$ will always be considered as a topological space, with the topology of uniform convergence on compact subsets. If $f \in H(\Omega)$, suppose we know the power series expansion of f, around the origin

$$\sum_{\nu_1=0}^{+\infty} \dots \sum_{\nu_n=0}^{+\infty} \alpha_{\nu_1 \dots \nu_n}^{(f)} w_1^{\nu_1} \dots w_n^{\nu_n} . \tag{1}$$

A natural problem is *to construct a domain Λ, satisfying conditions*:

a. *the power series* (1) *is continued holomorphically into Λ by means of the summability transform $N(z)$ (or by means of the summability transform $(N_1(z), \dots, N_n(z))$)*

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b. *Λ is independent of f.*

This problem has been fairly sufficiently resolved for some special cases. The purpose of this paper is to discuss the problem in the general case.

2 - Preliminaries

Let us first introduce the notations and the terminology we need.

The *$N(z)$-transform of the sequence of the partial sums of f, around the origin,* is the sequence

$$\{\sum_{k=0}^{m} \sigma_{mk}(z) \sum_{\nu_1=0}^{k} \dots \sum_{\nu_n=0}^{k} \alpha^{(f)}_{\nu_1 \dots \nu_n} z_1^{\nu_1} \dots z_n^{\nu_n}\}.$$

The *$(N_1(z), \dots, N_n(z))$-transform of the sequence of the partial sums of f, around the origin,* is the sequence

$$\{\sum_{k_1=0}^{m_1} \sigma^{(1)}_{m_1 k_1}(z) \sum_{\nu_1=0}^{k_1} (\sum_{k_2=0}^{m_2} \sigma^{(2)}_{m_2 k_2}(z) \sum_{\nu_2=0}^{k_2} (\dots (\sum_{k_n=0}^{m_n} \sigma^{(n)}_{m_n k_n}(z) \sum_{\nu_n=0}^{k_n} \alpha^{(f)}_{\nu_1, \dots, \nu_n} z_1^{\nu_1} \dots z_n^{\nu_n}) \dots))\}.$$

Set

$\boldsymbol{P}^{\Omega}_{N(z)} = \{P' \subset \Omega \mid$ *for any $f \in \boldsymbol{H}(\Omega)$, the $N(z)$-transform of the sequence of the partial sums of f, around the origin, converges to $f(z)$, uniformly on every compact subset of P', if $m \to +\infty$* $\}$

$\boldsymbol{P}^{\Omega}_{(N_{\lambda_1}(z), \dots, N_{\lambda_\mu}(z))} = \{P'' \subset \Omega \mid$ *for any $f \in \boldsymbol{H}(\Omega)$, the $(N_1(z), \dots, N_n(z))$-transform of the sequence of the partial sums of f, around the origin, converges to $f(z)$, uniformly on every compact subset of P'', if $m_{\lambda_s} \to +\infty$* $\}$.

With this terminology, one way of stating our problem is the following

Given the summability transform $N(z)$ (or the summability transform $(N_1(z), \dots, N_n(z))$), construct a domain which is always contained in

$$\boldsymbol{E}^{n}_{N(z)}(\boldsymbol{H}(\Omega)) = \bigcup_{P' \in \boldsymbol{P}^{\Omega}_{N(z)}} P'$$

(respectively, in $\boldsymbol{E}^{n}_{N_{\lambda_1}(z), \dots, N_{\lambda_\mu}(z)}(\boldsymbol{H}(\Omega)) = \bigcup_{P'' \in \boldsymbol{P}^{\Omega}_{(N_{\lambda_1}(z), \dots, N_{\lambda_\mu}(z))}} P''$).

Consider the following sequences, $\{p_m(x, z) = p_m(x_1, \ldots, x_n, z_1, \ldots, z_n)\}$ and $\{q_{(m_1, \ldots, m_n)}(x, z) = q_{(m_1, \ldots, m_n)}(x_1, \ldots, x_n, z_1, \ldots, z_n)\}$ of functions of the $2n$ complex variables $x_1, \ldots, x_n, z_1, \ldots, z_n$

$$p_m(x, z) = \sum_{k=0}^{m} \sigma_{mk}(z) \sum_{\nu_1=0}^{k} \cdots \sum_{\nu_n=0}^{k} x_1^{\nu_1} z_1^{\nu_1} \ldots x_n^{\nu_n} z_n^{\nu_n}$$

$$q_{(m_1, \ldots, m_n)}(x, z) = \sum_{k_1=0}^{m_1} \sigma_{m_1 k_1}^{(1)}(z) \sum_{\nu_1=0}^{k_1} (\ldots (\sum_{k_n=0}^{m_n} \sigma_{m_n k_n}^{(n)}(z) \sum_{\nu_n=0}^{k_n} x_1^{\nu_1} z_1^{\nu_1} \ldots x_n^{\nu_n} z_n^{\nu_n}) \ldots)).$$

Suppose that $\omega(N(\cdot))$ is the maximal open set in C^{2n}, in which the functions $p_m(x, z)$ are continuous and the sequence $\{p_m(x, z)\}$ converges, uniformly on every compact subset of $\omega(N(\cdot))$, to $(1 - x_1 z_1)^{-1} \ldots (1 - x_n z_n)^{-1}$. Further, assume that $\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot))$ is the maximal open set in C^{2n}, in which the functions $q_{(m_1, \ldots, m_n)}(x, z)$ are continuous and, the sequence $\{q_{(m_1, \ldots, m_n)}(x, z)\}$ converges, uniformly on any compact subset of $\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot))$, to $(1 - x_1 z_1)^{-1} \ldots (1 - x_n z_n)^{-1}$, if $m_{\lambda_1} \to +\infty, \ldots, m_{\lambda_\mu} \to +\infty$. Next, set

$$g(\omega; \Omega) = \{(z_1, \ldots, z_n) \in \Omega \,|\, (\frac{1}{\zeta_1}, \ldots, \frac{1}{\zeta_n}, z_1, \ldots, z_n) \in \omega, \text{ for any } \zeta_j \in \bar{C} - \mathrm{pr}_j(\Omega)\}$$

where $\omega = \omega(N(\cdot))$, $\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot))$ and where we have used the notation $\mathrm{pr}_j(\Omega)$ $(j = 1, \ldots, n)$ for the set

$$\{\zeta \in C \,|\, \text{there is a } (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in C^{n-1}, \text{ such that } (z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) \in \Omega\}.$$

In 1984, M. Eiermann proved that if $n = 1$ and if Ω is any domain of C, then $g(\omega(N(\cdot)); \Omega) \subset E^n_{N(z)}(H(\Omega))$ ([2]). It should be noted that Eiermann's result can be considered as an extension of Okada's theorem (in a generalized form, which is due to W. Gawronski and R. Trautner [4]). Using techniques, similar to those of Eiermann, we obtained an extension of Eiermann's theorem, in the case of an open polydisk in $\bar{C}^n$ ([1]).

The questions which may be asked are

i. Is the domain $g(\omega(N(\cdot)); \Omega)$ always contained in $E^n_{N(z)}(H(\Omega))$?

ii. Is the domain $g(\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot)); \Omega)$ always contained in $E^n_{N_{\lambda_1}(z), \ldots, N_{\lambda_\mu}(z)}(H(\Omega))$?

In Sections **4** and **5** we shall give two examples, which show that the answers are negative. In Section **3**, we recall some topics of complex analysis. Finally, in Section **6**, we modify the form of the sets $g(\omega; \Omega)$ and we construct two new domains

$$G(\omega(N(\cdot)); \Omega) \qquad G(\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot)); \Omega)$$

which are contained in

$$E^n_{N(z)}(H(\Omega)) \qquad E^n_{N_{\lambda_1}(z), \ldots, N_{\lambda_\mu}(z)}(H(\Omega))$$

respectively, under the assumption that Ω is a polydomain, i.e. a cartesian product of domains of C.

3 - Some definitions and results of complex analysis

If A and B are two subsets of C^n, then $A \subset\subset B$ means that A is contained in a compact subset of B. We denote by $\Delta^n(0, 1)$ the *unit polydisk* in C^n. For the *boundary* of a set Ω in C^n we use the notation $\partial\Omega$. If $\Omega = \Omega_1 \times \ldots \times \Omega_n$ is a polydomain of C^n, then $b\Omega$ denotes the *Shilov boundary* of Ω, that is its distinguished boundary $\partial\Omega_1 \times \ldots \times \partial\Omega_n$.

An open set in C^n is called a *domain of holomorphy* if we cannot find a connected open set D intersecting $\partial\Omega$ and a connected component D' of $D \cap \Omega$, such that for every $f \in H(\Omega)$ there exists $g \in H(D)$ with $f = g$ on D'. Recall that in the complex plane, all open sets are domains of holomorphy and that the situation is totally different in C^n, when $n > 1$ (see [5]). A well known result, that will be useful later, is

Proposition 1 ([3], p. 17). *If Ω is a domain of holomorphy of C^n and S is an hypersurface of C^n, then $\Omega - S$ is a domain of holomorphy.*

If Ω is an open subset of C^n and if K is a compact subset of Ω, we define the *$H(\Omega)$-hull* of K by

$$\widehat{K}_{H(\Omega)} = \{z \in \Omega \mid |f(z)| \leq \sup_K |f|, \text{ for any } f \in H(\Omega)\}.$$

If Ω and Ω' are two open sets in C^n, such that Ω' contains Ω and $H(\Omega')$ is a dense subset of $H(\Omega)$, then we say that (Ω, Ω') is a *Runge pair*. In particular, if (Ω, C^n) is a Runge pair, then we say that Ω is a Runge domain of C^n. For later use, we recall

Theorem 1 ([5], Theorem 4.3.3). *Let $\Omega \subset \Omega'$ be domains of holomorphy. Then the following conditions are equivalent*

(Ω, Ω') is a Runge pair

For every compact set $K \subset \Omega$, there holds $(\widehat{K}_{H(\Omega')} \cap \Omega) \subset\subset \Omega$.

4 - The first counter-example

Let Ω be an open subset of C^n, $0 \in \Omega$, such that:

(2) $$\mathrm{pr}_j(\Omega) = \Delta^1(0, 1) \qquad j = 1, 2, \ldots, n$$

(3) $$(\Omega, \Delta^n(0, 1)) \qquad \textit{is not a Runge pair.}$$

We shall show that there are $(n+1)$ infinite triangular matrices $N(z), N_1(z), \ldots, N_n(z)$ satisfying:

(4) $$\boldsymbol{g}(\omega(N(\cdot)); \Omega) \not\subset \boldsymbol{E}^n_{N(z)}(\boldsymbol{H}(\Omega))$$

(5) $$\boldsymbol{g}(\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot)); \Omega) \not\subset \boldsymbol{E}^n_{N_{\lambda_1}(z), \ldots, N_{\lambda_\mu}(z)}(\boldsymbol{H}(\Omega)).$$

Let $f \in \boldsymbol{H}(\Omega)$. Let (1) be the power series expansion of f, around the origin. Choosing $N(z) = N_1(z) = \ldots = N_n(z) = (\delta_{mk})$ (δ_{mk} Kronecker's symbol), it is obvious that

(6) $$\boldsymbol{g}(\omega(N(\cdot)); \Omega) = \boldsymbol{g}(\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot)); \Omega) = \Omega.$$

Assume now that

(7) $$\boldsymbol{g}(\omega(N(\cdot)); \Omega) \subset \boldsymbol{E}^n_{N(z)}(\boldsymbol{H}(\Omega)).$$

Combination of (6) and (7) shows that $\boldsymbol{E}^n_{N(z)}(\boldsymbol{H}(\Omega)) = \Omega$. Consequently

$$\lim_{m \to +\infty} \sum_{k=0}^{m} \delta_{mk} \cdot \sum_{\nu_1=0}^{k} \ldots \sum_{\nu_n=0}^{k} \alpha^{(f)}_{\nu_1 \ldots \nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n} = f(z_1, \ldots, z_n)$$

uniformly on every compact subset of Ω and therefore, f is the limit of a sequence of holomorphic functions in $\Delta^n(0, 1)$. Hence, $(\Omega, \Delta^n(0, 1))$ is a Runge pair, which is in direct contrast with the hypothesis (3). We conclude that the assumption (7) is false and thus we have proved

Proposition 2. *Let Ω be an open subset of C^n $(0 \in \Omega)$, such that $\mathrm{pr}_j(\Omega) = \Delta^1(0, 1)$ $(j = 1, \ldots, n)$ and $(\Omega, \Delta^n(0, 1))$ is not a Runge pair. Then there holds $\boldsymbol{g}(\omega(N(\cdot)); \Omega) \not\subset \boldsymbol{E}^n_{N(z)}(\boldsymbol{H}(\Omega))$.*

Repetition of the proof of Proposition 2, making only formal changes in substituting $\omega(N(\cdot))$ by $\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot))$ and $\boldsymbol{E}^n_{N(z)}(\boldsymbol{H}(\Omega))$ by $\boldsymbol{E}^n_{N_{\lambda_1}(z), \ldots, N_{\lambda_\mu}(z)}(\boldsymbol{H}(\Omega))$, gives

Proposition 3. *Let Ω be an open subset of $\boldsymbol{C}^n$ $(0 \in \Omega)$, such that* $\mathrm{pr}_j(\Omega) = \Delta^1(0, 1)$ $(j = 1, 2, \ldots, n)$ *and* $(\Omega, \Delta^n(0, 1))$ *is not a Runge pair. Then there holds*

$$\boldsymbol{g}(\omega(\boldsymbol{N}_{\lambda_1}(\cdot), \ldots, \boldsymbol{N}_{\lambda_\mu}(\cdot)); \Omega) \notin \boldsymbol{E}^n_{N_{\lambda_1}(z), \ldots, N_{\lambda_\mu}(z)}(\boldsymbol{H}(\Omega)).$$

Next, we shall construct an example of an open subset of $\boldsymbol{C}^2$, satisfying (2) and (3). Let Ω be the domain of $\boldsymbol{C}^2$ defined by

$$\Omega = \Delta^2(0, 1) - \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, z_1 + z_2 = 1\}. \tag{8}$$

Obviously, $0 \in \Omega$ and $\mathrm{pr}_1(\Omega) = \mathrm{pr}_2(\Omega) = \Delta^1(0, 1)$. Further, the open set Ω is a domain of holomorphy. In fact, it is sufficient to see that $\Delta^2(0, 1)$ is a domain of holomorphy of $\boldsymbol{C}^2$ and that $\{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, z_1 + z_2 = 1\}$ is an hypersurface in $\boldsymbol{C}^2$. In order to show that $(\Omega, \Delta^2(0, 1))$ is not a Runge pair, it suffices to apply Theorem 1, that is to find a compact subset K of Ω satisfying

$$\widehat{K}_{H(\Delta^2(0, 1))} \cap \Omega \not\subset\subset \Omega. \tag{9}$$

If we choose $K = \{(\frac{1}{2}, \frac{3}{4} e^{i\theta}) \,|\, 0 \leqslant \theta \leqslant 2\pi\}$, then $(\frac{1}{2}, \frac{1}{2}) \in \widehat{K}_{H(\Delta^2(0, 1))}$. Since $(\frac{1}{2}, \frac{1}{2}) \in \partial\Omega$, we obtain (9).

5 - The second counter-example

Let Ω be an open subset of $\boldsymbol{C}^2$ defined by

$$\Omega = \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, |z_1| < 1,\ |z_2| < 1,\ |z_1 + z_2| < 1\}. \tag{10}$$

It is clear that $0 \in \Omega$ and that $\mathrm{pr}_1(\Omega) = \mathrm{pr}_2(\Omega) = \Delta^1(0; 1)$. Moreover, it is easily verified that $(\Omega, \Delta^2(0, 1))$ is a Runge pair. As in Section 4, we shall show that there are $(n + 1)$ infinite triangular matrices $\boldsymbol{N}(z), \boldsymbol{N}_1(z), \boldsymbol{N}_2(z), \ldots, \boldsymbol{N}_n(z)$ satisfying (4) and (5).

Choose

$$f\colon \Omega \to \boldsymbol{C};\ (z_1, z_2) \to f(z_1, z_2) = \frac{1}{1 - (z_1 + z_2)} \in \boldsymbol{H}(\Omega).$$

Clearly, f can be expressed, in a neighborhood of 0, as

$$f(w_1, w_2) = \sum_{\nu=0}^{+\infty} (w_1 + w_2)^\nu = \sum_{\nu=0}^{+\infty} \Big(\sum_{p=0}^{\nu} C_\nu^p w_1^{\nu-p} w_2^p \Big) = \sum_{m,\,p=0}^{+\infty} C_{m+p}^p w_1^m w_2^p$$

where $m = \nu - p$ and $C_\nu^p = \binom{\nu}{p}$.

For any $(w_1, w_2) \in \Omega$, set

$$S_k(w_1, w_2) = \sum_{m,\,p=0}^{k} C_{m+p}^p w_1^m w_2^p \qquad (k = 0, 1, 2, \ldots).$$

Let us study the difference $S_{k+1}(w_1, w_2) - S_k(w_1, w_2)$ $((w_1, w_2) \in \Omega)$. Suppose (z_1, z_2) is a point of Ω. We have

$$S_{k+1}(z_1, z_2) - S_k(z_1, z_2) = \sum_{p=0}^{k} C_{k+1+p}^p z_1^{k+1} z_2^p + \sum_{m=0}^{k} C_{m+k+1}^{k+1} z_1^m z_2^{k+1} + C_{2k+2}^{k+1} z_1^{k+1} z_2^{k+1}.$$

In particular, when $z_2 = -z_1$ and $k + 1 = 2k'$ (i.e. even), the above difference becomes

$$S_{k+1}(z_1, z_2) - S_k(z_1, z_2) = S_{2k'}(z_1, -z_1) - S_{2k'-1}(z_1, -z_1)$$

$$= \sum_{p=0}^{2k'-1} C_{2k'+p}^p (-1)^p z_1^{2k'+p} + \sum_{m=0}^{2k'-1} C_{2k'+m}^{2k'} z_1^{2k'+m} + C_{4k'}^{2k'} z_1^{4k'}.$$

If we restrain our attention to the case where $z_1 > 0$, then it is easily seen that

$$\sum_{p=0}^{2k'-1} C_{2k'+p}^p (-1)^p z_1^{2k'+p} + \sum_{m=0}^{2k'-1} C_{2k'+m}^{2k'} z_1^{2k'+m} > 0 \qquad C_{4k'}^{2k'} z_1^{4k'} > 0.$$

Assuming that

(11) $$\lim_{k \to +\infty} S_k(w_1, w_2) = \frac{1}{1 - (w_1 + w_2)} \qquad \text{for any } (w_1, w_2) \in \Omega$$

we obtain $\lim_{k \to +\infty} (S_{k+1}(w_1, w_2) - S_k(w_1, w_2)) = 0$ for any $(w_1, w_2) \in \Omega$ and consequently

(12) $$\lim_{k' \to +\infty} C_{4k'}^{2k'} z_1^{4k'} = 0.$$

But $$C_{4k'}^{2k'} = \frac{(4k')!}{((2k')!)^2}$$

and it follows from Stirling's formula that

$$(13)\qquad \lim_{k'\to+\infty} C_{4k'}^{2k'} z_1^{4k'} = \lim_{k'\to+\infty} \frac{(\frac{4k'}{e})^{4k'}\sqrt{8\pi k'}}{(\frac{2k'}{e})^{4k'}(\sqrt{4\pi k'})^2} z_1^{4k'} = \lim_{k'\to+\infty} \frac{2^{4k'} z_1^{4k'}}{\sqrt{2\pi k'}} \neq 0$$

for a $z_1 > 0$, suitably choosen and near to 1. Comparison of (12) and (13) shows that the assumption (11) is false. Therefore, the point $(z_1, -z_1)$ ($z_1 > 0$, z_1 near to 1) satisfies

$$(14)\qquad f(z_1, -z_1) \neq \sum_{\nu_1=0}^{+\infty} \sum_{\nu_2=0}^{+\infty} \alpha_{\nu_1\nu_2}^{(f)} z_1^{\nu_1}(-z_1)^{\nu_2}.$$

Choosing $N(z) = N_1(z) = \dots = N_n(z) = (\delta_{mk})$ (δ_{mk} Kronecker's symbol), it is easy to see that

$$\lim_{m\to+\infty} \sum_{k=0}^{m} \delta_{mk} \sum_{\nu_1,\nu_2=0}^{k} \alpha_{\nu_1\nu_2}^{(f)} z_1^{\nu_1}(-z_1)^{\nu_2} = \sum_{\nu_1,\nu_2=0}^{+\infty} \alpha_{\nu_1\nu_2}^{(f)} z_1^{\nu_1}(-z_1)^{\nu_2}$$

$$\lim_{m_{\lambda_1}\to+\infty,\dots,m_{\lambda_\mu}\to+\infty} \sum_{k_1=0}^{m_1} \delta_{m_1k_1} \sum_{\nu_1=0}^{k_1} \Big(\sum_{k_2=0}^{m_2} \delta_{m_2k_2} \sum_{\nu_2=0}^{k_2} \alpha_{\nu_1\nu_2}^{(f)} z_1^{\nu_1}(-z_1)^{\nu_2}\Big)$$

$$= \sum_{\nu_1,\nu_2=0}^{+\infty} \alpha_{\nu_1\nu_2}^{(f)} z_1^{\nu_1}(-z_1)^{\nu_2}$$

and hence it follows from (14):

$$(15)\qquad (z_1, -z_1) \notin \boldsymbol{E}_{N(z)}^{n}(\boldsymbol{H}(\Omega))$$

$$(16)\qquad (z_1, -z_1) \notin \boldsymbol{E}_{N_{\lambda_1}(z),\dots,N_{\lambda_\mu}(z)}^{n}(\boldsymbol{H}(\Omega)).$$

But on the other hand there holds:

$$(17)\qquad (z_1, -z_1) \in \boldsymbol{g}(\omega(\boldsymbol{N}(\cdot)); \Omega) = \Omega$$

$$(18)\qquad (z_1, -z_1) \in \boldsymbol{g}(\omega(\boldsymbol{N}_{\lambda_1}(\cdot), \dots, \boldsymbol{N}_{\lambda_\mu}(\cdot)); \Omega) = \Omega.$$

Combination of (15) and (17) proves (4), while combination of (16) and (18) proves (5).

6 - A modified form of Okada's theorem in C^n

Consider the family of sets

$(\mathcal{C})_n^\infty = \{\Omega$ *open set in* $C^n \mid 0 \in \Omega$ *and for any* $z \in \Omega$, *there is a simply connected polydomain* $D_z = D_z^{(1)} \times \ldots \times D_z^{(n)}$ *such that* $\{0, z\} \subset D_z \subset\subset \Omega$ *and* $\partial D_z^{(j)}$ *is smooth* (C^∞), $j = 1, 2, \ldots, n\}$.

It is clear that if Ω is a polydomain of C^n, then $\Omega \in (\mathcal{C})_n^\infty$. For $\omega = \omega(N(\cdot))$, $\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot))$ and for $\Omega \in (\mathcal{C})_n^\infty$, set

$$\omega_z = \{(x_1, \ldots, x_n) \in C^n \mid (x_1, \ldots, x_n, z_1, \ldots, z_n) \in \omega\}$$

$$G(\omega; \Omega) = \{z \in \Omega \mid \textit{there is a } D_z \textit{ such that } [bD_z]^{-1} \subset \omega_z\}$$

where we have used the notation $[bD_z]^{-1} = \{(\frac{1}{t_1}, \ldots, \frac{1}{t_n}) \mid (t_1, \ldots, t_n) \in bD_z\}$.

The following theorem can be regarded as *a modified form of Okada's theorem.* In fact, according to Okada's classical theorem of the case $n = 1$, if Ω is a domain of C $(0 \in \Omega)$, then the $N(z)$-transform of the sequence of the partial sums of f, around 0, converges to f, compactly on

$$g(\omega; \Omega) = \{z \in \Omega \mid (\frac{1}{\zeta}, z) \in \omega, \textit{ for any } \zeta \in \bar{C} - \Omega\} \qquad \text{for any } f \in H(\Omega).$$

Our next theorem shows that we can obtain the compact convergence of the same sequences into the new domain

$$G(\omega; \Omega) = \{z \in \Omega \mid (\frac{1}{\zeta}, z) \in \omega, \textit{ for any } \zeta \in \partial D_z \textit{ and a } D_z\}.$$

Theorem 2. *If* $\Omega \in (\mathcal{C})_n^\infty$, *then there holds*

i. $$G(\omega(N(\cdot)); \Omega) \subset E^n_{N(z)}(H(\Omega))$$

ii. $$G(\omega(N_{\lambda_1}(\cdot), \ldots, N_{\lambda_\mu}(\cdot)); \Omega) \subset E^n_{N_{\lambda_1}(z), \ldots, N_{\lambda_\mu}(z)}(H(\Omega)).$$

In order to prove this theorem we use a lemma, which is a direct consequence of Cauchy's integral formula.

Lemma 1. *Let $\Omega \in (\mathcal{C})_n^\infty$ and let $w \in \Omega$.*

a. *Suppose that there is a D_w, such that the functions $p_m(x, z)$ $(m = 0, 1, 2, \ldots)$ are continuous in $[bD_w]^{-1} \times \{w\}$. If $f \in \boldsymbol{H}(\Omega)$, then there holds*

$$|f(w) - \sum_{k=0}^{m} \sigma_{mk}(w) \sum_{\nu_1=0}^{k} \ldots \sum_{\nu_n=0}^{k} \alpha_{\nu_1 \ldots \nu_n}^{(f)} w_1^{\nu_1} \ldots w_n^{\nu_n}|$$

$$\leqslant L(f, D_w) \sup_{t \in [bD_w]^{-1}} |(1 - t_1 w_1)^{-1} \ldots (1 - t_n w_n)^{-1} - p_m(t, z)|$$

where $L(f, D_w)$ is a constant which depends on f and D_w, but is independent of m.

b. *Suppose that there is a D_w, such that the functions $q_{(m_1, \ldots, m_n)}(x, z)$ $(m_j = 0, 1, 2, \ldots)$ are continuous in $[bD_w]^{-1} \times \{w\}$. If $f \in \boldsymbol{H}(\Omega)$, then there holds*

$$|f(w) - \sum_{k_1=0}^{m_1} \sigma_{m_1 k_1}^{(1)}(w) \sum_{\nu_1=0}^{k_1} (\ldots (\sum_{k_n=0}^{m_n} \sigma_{m_n k_n}^{(n)}(w) \sum_{\nu_n=0}^{k_n} \alpha_{\nu_1 \ldots \nu_n}^{(f)} w_1^{\nu_1} \ldots w_n^{\nu_n}) \ldots)|$$

$$\leqslant M(f, D_w) \sup_{t \in [bD_w]^{-1}} |(1 - t_1 w_1)^{-1} \ldots (1 - t_n w_n)^{-1} - q_{(m_1, \ldots, m_n)}(t, z)|$$

where $M(f, D_w)$ is a constant which depends on f and D_w, but is independent of $(m_1, \ldots, m_n)$.

Proof. It suffices to note that

$$f(w) = \frac{1}{(2\pi i)^n} \int_{\zeta \in bD_w} \frac{f(\zeta)}{(\zeta_1 - w_1) \ldots (\zeta_n - w_n)} \, d\zeta_1 \ldots d\zeta_n$$

$$= \frac{1}{(2\pi i)^n} \int_{\zeta \in [bD_w]^{-1}} \frac{f(\zeta^{-1})}{(1 - w_1 \zeta_1) \ldots (1 - w_n \zeta_n)} \zeta_1 \ldots \zeta_n \, d\zeta_1 \ldots d\zeta_n,$$

$$\sum_{k=0}^{m} \sigma_{mk}(w) \sum_{\nu_1=0}^{k} \ldots \sum_{\nu_n=0}^{k} \alpha_{\nu_1 \ldots \nu_n}^{(f)} w_1^{\nu_1} \ldots w_n^{\nu_n}$$

$$= \sum_{k=0}^{m} \sigma_{mk}(w) \sum_{\nu_1=0}^{k} \ldots \sum_{\nu_n=0}^{k} \Big(\frac{1}{(2\pi i)^n} \int_{\zeta \in bD_w} \frac{f(\zeta)}{\zeta_1^{\nu_1+1} \ldots \zeta_n^{\nu_n+1}} \, d\zeta_1 \ldots d\zeta_n \Big) w_1^{\nu_1} \ldots w_n^{\nu_n}$$

$$= \frac{1}{(2\pi i)^n} \int_{\zeta \in bD_w} f(\zeta) \Big(\sum_{k=0}^{m} \sigma_{mk}(w) \sum_{\nu_1=0}^{k} \ldots \sum_{\nu_n=0}^{k} \Big(\frac{w_1}{\zeta_1}\Big)^{\nu_1} \ldots \Big(\frac{w_n}{\zeta_n}\Big)^{\nu_n} \Big) \zeta_1^{-1} \ldots \zeta_n^{-1} \, d\zeta_1 \ldots d\zeta_n$$

$$= \frac{1}{(2\pi i)^n} \int_{\zeta \in [bD_w]^{-1}} f(\zeta^{-1}) p_m(\zeta, w) \zeta_1 \ldots \zeta_n \, d\zeta_1 \ldots d\zeta_n$$

$$\sum_{k_1=0}^{m_1} \sigma^{(1)}_{m_1 k_1}(w) \sum_{\nu_1=0}^{k_1} (\dots(\sum_{k_n=0}^{m_n} \sigma^{(n)}_{m_n k_n}(w) \sum_{\nu_n=0}^{k_n} \alpha^{(f)}_{\nu_1\dots\nu_n} w_1^{\nu_1}\dots w_n^{\nu_n})\dots)$$
$$= \frac{1}{(2\pi i)^n} \int_{\zeta\in[bD_w]^{-1}} f(\zeta^{-1})\, q_{(m_1,\dots,m_n)}(\zeta, w)\, \zeta_1\dots\zeta_n \, d\zeta_1\dots d\zeta_n .$$

Proof of Theorem 2. In a first step we shall prove **i**. Let $f \in \boldsymbol{H}(\Omega)$ and let $z^0 = (z_1^0, \dots, z_n^0) \in \boldsymbol{G}(\omega(N(\cdot)); \Omega)$. By the definition of the set $\boldsymbol{G}(\omega(N(\cdot)); \Omega)$, it follows that there is D_{z^0} with $[bD_{z^0}]^{-1} \subset (\omega(N(\cdot)))_{z^0}$. Consequently, the compact set $\{(x_1, \dots, x_n, z_1^0, \dots, z_n^0) \in \boldsymbol{C}^{2n} \mid (x_1, \dots, x_n) \in [bD_{z^0}]^{-1}\}$ is contained in the open set $\omega(N(\cdot))$. Applying Lemma 1, for $w = z^0$, we obtain

$$|f(z^0) - \sum_{k=0}^{m} \sigma_{mk}(z^0) \sum_{\nu_1=0}^{k} \dots \sum_{\nu_n=0}^{k} \alpha^{(f)}_{\nu_1\dots\nu_n} (z_1^0)^{\nu_1}\dots(z_n^0)^{\nu_n}|$$
$$\leq L(f, D_{z^0}) \sup_{t\in[bD_{z^0}]^{-1}} |(1-t_1 z_1^0)^{-1}\dots(1-t_n z_n^0)^{-1} - p_m(t, z^0)|$$

for $m = 0, 1, 2, \dots$. Since $\boldsymbol{G}(\omega(N(\cdot)); \Omega)$ is open, there is a compact neighborhood of z^0, $\bar{U}_{z^0} \subset \boldsymbol{G}(\omega(N(\cdot)); \Omega) \cap D_{z^0}$. Clearly, for any $z \in \bar{U}_{z^0}$, one can choose D_z equal to D_{z^0}. Repetition of the proof shows that

$$|f(z) - \sum_{k=0}^{m} \sigma_{mk}(z) \sum_{\nu_1=0}^{k} \dots \sum_{\nu_n=0}^{k} \alpha^{(f)}_{\nu_1\dots\nu_n} z_1^{\nu_1}\dots z_n^{\nu_n}|$$
$$\leq L(f, D_{z^0}) \sup_{t\in[bD_{z^0}]^{-1}} |(1-t_1 z_1)^{-1}\dots(1-t_n z_n)^{-1} - p_m(t, z)|$$

for $m=0, 1, 2, \dots$ and for any $z=(z_1, \dots, z_n)\in \bar{U}_{z^0}$. Hence, for $m=0, 1, 2, \dots$

$$\sup_{z\in\bar{U}_{z^0}} |f(z) - \sum_{k=0}^{m} \sigma_{mk}(z) \sum_{\nu_1=0}^{k} \dots \sum_{\nu_n=0}^{k} \alpha^{(f)}_{\nu_1\dots\nu_n} z_1^{\nu_1}\dots z_n^{\nu_n}|$$
$$\leq \sup_{z\in\bar{U}_{z^0}} (L(f, D_{z^0})) \sup_{z\in\bar{U}_{z^0}} (\sup_{t\in[bD_{z^0}]^{-1}} |(1-t_1 z_1)^{-1}\dots(1-t_n z_n)^{-1} - p_m(t, z)|).$$

By passing to the limit, when $m \to +\infty$, we see that

$$\lim_{m\to+\infty} \sum_{k=0}^{m} \sigma_{mk}(z) \sum_{\nu_1=0}^{k} \dots \sum_{\nu_n=0}^{k} \alpha^{(f)}_{\nu_1\dots\nu_n} z_1^{\nu_1}\dots z_n^{\nu_n} = f(z)$$

uniformly on $\bar{U}_{z^0}$, and the proof of part **i** follows. Similarly, we can prove part **ii**.

In particular, we have

Corollary 1. *If* $\Omega = \Omega_1 \times \dots \Omega_n$ *is a polydomain of* C^n $(0 \in \Omega)$. *Then there holds*

$$\text{i.} \qquad G(\omega(N(\cdot)); \Omega) \subset E^n_{N(z)}(H(\Omega))$$

$$\text{ii.} \qquad G(\omega(N_{\lambda_1}(\cdot), \dots, N_{\lambda_\mu}(\cdot)); \Omega) \subset E^n_{N_{\lambda_1}(z), \dots, N_{\lambda_\mu}(z)}(H(\Omega)).$$

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Sommario

Per un dato metodo di sommabilità, il teorema di Okada indica un dominio nel quale una arbitraria serie di potenze può essere olomorficamente prolungata. Il primo obiettivo di questo lavoro è mostrare che il corrispondente risultato è falso se la dimensione è maggiore di uno. Il secondo obiettivo è di ottenere una forma modificata del teorema di Okada, indipendente dalla dimensione.

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