Riv. Mat. Univ. Parma (5) 2 (1993), 279-282

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On commutativity of near-rings (**)

1 - Introduction

Long ago H. E. Bell [7] proved that a periodic distributively generated (d.g.) near ring such that to each $x \in R$ and $u \in N$ there exist integers $n = n(u, x)$ and $m = m(u, x)$ each greater than 1, satisfying $[u, x]^n = [u, x]$ and $[x, u]^m = [u, x]$, is commutative.

We consider now the *identities*

$$[x, y] = [x^n, y^m] \tag{1.1}$$

where $n = n(x, y) > 1$ and $m = m(x, y) > 1$,

$$[x, y] = x^m [x, y^n] x^q \tag{1.2}$$

where m and q are fixed positive integers and $n = n(y) > 1$.

In Sec. 2 we prove that a periodic d.g. near ring satisfying (1.1) or (1.2) is commutative (Theorems 1, 2).

Throughout this paper R denotes the left near ring with multiplicative centre $Z(R)$ and $N(R)$ stands for the set of nilpotent elements of R. For the definitions of distributive near rings, distributively generated (d.g.) near rings, strongly distributively generated (s.d.g.) near rings, periodic near rings, ideal, zero symmetric and zero commutative property see [12].

We mention below some well known lemmas due to A. Fröhlich and to H. E. Bell.

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(**) Received December 15, 1992. AMS classification 16 A 30.

Lemma 1 ([8]). *A d.g. near ring R is distributive if and only if R^2 is additively commutative.*

Lemma 2 ([8]). *A d.g. near ring R with unity 1 is a ring if R is distributive or if R^+ is commutative.*

Lemma 3 ([7]). *A periodic ring is commutative if its nilpotent elements are central.*

Lemma 4 ([5]). *If R is a distributively generated (d.g.) near ring with its nilpotent elements lying in the centre, then the set N of nilpotent elements of R forms an ideal, and if R/N is periodic, then R must be commutative.*

2 - Main results

Lemma 5. *Let R be a d.g. near ring satisfying* (1.1) *for all x, y in R, $n = n(x, y) > 1$ and $m = m(x, y) > 1$, then $N(R) \subseteq Z(R)$.*

Proof. Let R be a d.g. near ring satisfying (1.1) and the corresponding assumptions about n and m. If $a \in N(R)$ and $x \in R$ then there exist integers $n_1 = n(x, a) > 1$ and $m_1 = m(x, a) > 1$ such that $[x, a] = [x^{n_1}, a^{m_1}]$.

Now choose $n_2 = n(x^{n_1}, a^{m_1}) > 1$ and $m_2 = m(x^{n_1}, a^{m_1}) > 1$ such that $[x^{n_1}, a^{m_1}] = [x^{n_1 n_2}, a^{m_1 m_2}]$. Hence we have $[x, a] = [x^{n_1 n_2}, a^{m_1 m_2}]$.

Continuing in this way for an arbitrary integer t, we obtain integers $n_1, n_2, \ldots, n_t > 1$ and $m_1, m_2, \ldots, m_t > 1$ such that

$$[x, a] = [x^{n_1 n_2 \ldots n_t}, a^{m_1 m_2 \ldots m_t}].$$

Since $a \in N(R)$ we get $a^{m_1 m_2 \ldots m_t} = 0$ for sufficiently large t; thus $[x, a] = 0$ for a in $N(R)$ and x in R, i.e. a is central.

Theorem 1. *Let R be a periodic d.g. near ring satisfying* (1.1) *for each x, y in R, $n = n(x, y) > 1$ and $m = m(x, y) > 1$. Then R is commutative.*

Proof. By Bell's [5] Lemma 1 and our Lemma 5 we derive that $N(R)$ is a two sided ideal. Now applying the main theorem of Bell [5] (cf. Lemma 4), we obtain that R is commutative.

Lemma 6. *Let R be a d.g. near ring satisfying* (1.2) *for all x, y in R, m and q fixed positive integers and $n = n(y) > 1$, then $N(R) \subseteq Z(R)$.*

Proof. If $a \in N(R)$ and $x \in R$ then there exists $n_1 = n(a) > 1$ such that $[x, a] = x^m[x, a^{n_1}]x^q$. Now choose $n_2 = n(a^{n_1}) > 1$ such that $[x, a^{n_1}] = x^m[x, a^{n_1 n_2}]x^q$. Hence $[x, a] = x^{2m}[x, a^{n_1 n_2}]x^{2q}$. It is now obvious that for any positive integer t, we have

$$[x, a] = x^{tm}[x, a^{n_1 n_2 \cdots n_t}]x^{tq}.$$

Thus $[x, a] = 0$ for sufficiently large t, hence a is central.

Theorem 2. *Let R be a periodic d.g. near ring satisfying* (1.2) *for each x, y in R, fixed positive integers m, q and $n = n(y) > 1$. Then R is commutative.*

Proof. By using Lemma 6 and same arguments as we have used in the proof of Theorem 1, we get the result.

Theorem 3. *Let R be a periodic d.g. near ring satisfying* (1.1) *for each x, y in R, $n = n(x, y) > 1$ and $m = m(x, y) > 1$. If $R^2 = R$ then R is a commutative ring.*

Proof. In view of Theorem 1 a periodic d.g. near ring satisfying (1.1) is commutative. Thus for any x, y, z in R, we have

$$(y + z)x = x(y + z) = xy + xz = yx + zx.$$

This implies that R is distributive and hence by Lemma 1, R^2 is additively commutative. Now $R^2 = R$ implies that R is also additively commutative. Hence R is a commutative ring.

Remark 1. If the condition $R^2 = R$ of Theorem 3 is replaced by the condition that R has unity, then the result follows trivially by Lemma 2 and Theorem 3. Similarly if R is an s.d.g. periodic near ring satisfying identity (1.1), then by Theorem 1 and Lemma 1, R^2 is additively commutative. Hence the additive group R^+ of an s.d.g. near ring is also commutative. Thus R is a commutative ring.

Remark 2. A result, analogous to Theorem 1 can be proved by replacing identity (1.1) by identity (1.2) for fixed positive integers m, q and $n = n(y) > 1$. As in Remark 1 corresponding modifications for the result can also be obtained.

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Sommario

Si dimostra che un quasi anello periodico distributivamente generato, soddisfacente l'identità (1.1) *ovvero l'identità* (1.2), *è commutativo. Con opportune ipotesi aggiuntive il quasi anello risulta essere un anello commutativo.*

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