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Decomposition of a K-midconvex (K-midconcave) function in a Banach space (**)

1 - Introduction

K. Nikodem, in 1984, during the International Conference on functional equations and inequalities, posed the following *conjecture* [6].

Let D be an open interval of $\boldsymbol{R}$ and $f, g\colon D \to \boldsymbol{R}$ be two functions respectively midconvex and midconcave such that

$$(\alpha) \qquad f(x) \leqslant g(x) \qquad \forall x \in D\,.$$

In these conditions, the author asks himself if there exist two functions $F, G\colon D \to \boldsymbol{R}$ respectively convex and concave and an additive function $A\colon \boldsymbol{R} \to \boldsymbol{R}$ with the properties

$$(1) \qquad f(x) = F(x) + A(x) \qquad \forall x \in D$$

$$(2) \qquad g(x) = G(x) + A(x) \qquad \forall x \in D\,.$$

In 1987 C. T. Ng [5] and K. Nikodem [8] proved that if D is an open and convex subset of $\boldsymbol{R}^n$, $f, g\colon D \to \boldsymbol{R}$ are two functions respectively midconvex and midconcave and such that there exists an open and convex subset N of D with the property (α) $\forall x \in N$, then there exist two functions $F, G\colon D \to \boldsymbol{R}$ respectively convex and concave and an additive function $A\colon \boldsymbol{R}^n \to \boldsymbol{R}$ satisfying (1), (2).

This theorem gives a first positive answer to the conjecture posed by K. Nikodem even in a more general context of that considered by K. Nikodem. In fact, for these Authors, D is not necessarily an open interval of $\boldsymbol{R}$ but an open and convex subset of $\boldsymbol{R}^n$ and moreover the inequality (α) is satisfied on an open and convex subset of D and not necessarily on D.

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Later, in 1989, Z. Kominek [3] obtained a proposition (Theorem 4) that is another positive answer to the conjecture of K. Nikodem. The Author proved that the proposition of C. T. Ng and K. Nikodem is still true in the more general case that the inequality (α) is satisfied on a subset T of D belonging to the class $\mathscr{A}$ of subsets of $\boldsymbol{R}^n$ defined by (6), which strictly contains the open sets (cf. Remark 2) (this class was introduced by R. Ger and M. Kuczma in 1968 [2]). Therefore Z. Kominek's theorem strictly contains the mentioned proposition of C. T. Ng and K. Nikodem.

In this paper we study the conjecture of K. Nikodem in a more general context. We first introduce the class $\mathscr{A}_K$ of subsets of $\boldsymbol{R}^n$ which contains, as we'll see in Sec. 2, the class introduced by R. Ger and M. Kuczma.

Let Y be a Banach space, K be a normal (Definition 1, Sec. 2) and closed cone in Y and $\mathscr{A}_K$ the class of the subsets T of $\boldsymbol{R}^n$ with the property that every K-midconvex function defined on a convex, open set D, $T \subset D$, taking its values in Y and K-upper bounded on T, is K-continuous on D (cf. (7)). We observe that, if in particular $Y = \boldsymbol{R}$ and $K = [0, +\infty[$, the class $\mathscr{A}_K$ is reduced to the class $\mathscr{A}$ of R. Ger and M. Kuczma (Remark 4).

Let now f and g be two functions defined on an open and convex subset D of $\boldsymbol{R}^n$, with values in a Banach space Y in which the order structure is endowed by a normal and closed cone K (cf. Remark 5), respectively K-midconvex and K-midconcave with the property

$$\exists T \in \mathscr{A}_K,\ T \subset D \mid f(x) \leqslant_K g(x) \qquad \forall x \in T$$

(cf. (8)).

In these conditions, we prove that there exist two functions F, G: $D \to Y$ respectively K-convex and K-concave and an additive function A: $\boldsymbol{R}^n \to Y$, satisfying (1), (2).

This result contains, as special case, Theorem 4 stated by Z. Kominek in [3]. In fact, if $Y = \boldsymbol{R}$ and $K = [0, +\infty[$, our proposition is reduced to the theorem of Z. Kominek (cf. Remarks 1 and 4). Therefore, even in this particular case, our theorem strictly contains the mentioned answer given to the conjecture of K. Nikodem by C. T. Ng and K. Nikodem himself.

2 - Definitions and remarks

Let X and Y be two real topological vector spaces (satisfying the T_0 separation axiom).

Given two real numbers α, β and two sets $S, T \subset Y$, we put

$$\alpha S + \beta T = \{ y \in Y \mid y = \alpha s + \beta t, \ \ s \in S, \ \ t \in T \}.$$

A set $K \subset Y$ is said to be a *cone* ([9], p. 9) if it satisfies the following conditions:

$$K + K \subset K \qquad \alpha K \subset K \qquad \forall \alpha \in [0, +\infty[.$$

Definition 1. A cone $K \subset Y$ is said to be a *normal cone* ([9], p. 9) if there exists a base $\mathcal{V}(0)$ of neighbourhoods of zero in Y such that

$$V = (V + K) \cap (V - K), \qquad \forall V \in \mathcal{V}(0).$$

Definition 2. Given an open, convex and nonempty subset D of X, a function f: $D \to Y$ is said to be *K-lower bounded, K-upper bounded* on a set $A \subset Y$ ([9], p. 48, 34) if there exists a bounded set $B \subset Y$ such that

$$\bigcup_{x \in A} f(x) \subset B + K \qquad \bigcup_{x \in A} f(x) \subset B - K \qquad \text{respectively.}$$

The function f is said to be *K-bounded* on A if it is *K-lower bounded* and *K-upper bounded* on A.

Let $\mathcal{U}(0)$ and $\mathcal{W}(0)$ be two bases of neighbourhoods of zero, respectively in X and in Y. The function f is said to be *K-lower semicontinuous* (K-l.s.c.) in a point $x_0 \in D$ ([7], p. 394) if $\forall W \in \mathcal{W}(0)$ there exists a neighbourhood $U \in \mathcal{U}(0)$, $x_0 + U \subset D$, such that $f(x_0) \in f(x) + W + K$, $\forall x \in x_0 + U$.

The function f is said to be *K-upper semicontinuous* (K-u.s.c.) in $x_0 \in D$ ([7], p. 394) if $\forall W \in \mathcal{W}(0)$ there exists a neighbourhood $U \in \mathcal{U}(0)$, $x_0 + U \subset D$, such that $f(x) \in f(x_0) + W + K$, $\forall x \in x_0 + U$.

The function f is said to be *K-continuous* in the point $x_0 \in D$, if it is K-lower semicontinuous and K-upper semicontinuous in this point.

Now we recall, for the function f, the definitions of K-convexity, K-midconvexity, K-concavity and K-midconcavity.

The function f is *K-convex* ([9], (2.1)) if

$$tf(x) + (1-t)f(y) \in f(tx + (1-t)y) + K \tag{3}$$

for all $x, y \in D$ and $t \in [0, 1]$, while f is called *K-midconvex* ([9], (3.1)) if (3) holds for $t = \dfrac{1}{2}$.

Moreover, the function f is said to be *K-concave* (cf. [9], (2.2)) if

$$f(tx + (1-t)y) \in tf(x) + (1-t)f(y) + K \tag{4}$$

for all $x, y \in D$ and $t \in [0, 1]$, while f is called *K-midconcave* ([9], (4.1)) if (4) holds for $t = \frac{1}{2}$.

Remark 1. We observe that, in the particular case $Y = \boldsymbol{R}$ and $K = [0, +\infty[$, the definitions of K-convexity, K-concavity, K-midconvexity and K-midconcavity are reduced respectively to the definitions of convexity, concavity, midconvexity and midconcavity.

In the case that Y is a normed space, let ε be a non negative number. A function $f\colon D \to Y$ is *ε-Jensen* on D ([3], p. 499) if

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \varepsilon \tag{5}$$

for all $x, y \in D$. Every function $f\colon D \to Y$ satisfying (5) with $\varepsilon = 0$, is called a *Jensen* function ([3], p. 499).

Then we recall that R. Ger and M. Kuczma [2], in 1968, introduced the following class of subsets of $\boldsymbol{R}^n$:

(6) $\mathscr{A} = \{T \subset \boldsymbol{R}^n$: every midconvex function defined on a convex, open set D, $T \subset D$, taking its values in $\boldsymbol{R}$ and bounded above on T, is continuous on $D\}$.

Remark 2. We want to remind that every subset of $\boldsymbol{R}^n$ having positive inner Lebesgue measure belongs to the class $\mathscr{A}$ ([2], p. 158). On the other hand, there exist ([4], p. 210) subsets of $\boldsymbol{R}^n$ with Lebesgue measure equal to zero which belong to the class $\mathscr{A}$.

Now we introduce, for every fixed cone K in Y, the following class of subsets of X:

(7) $\mathscr{A}_K = \{T \subset X$: every K-midconvex function defined on a convex, open set D, $T \subset D$, taking its values in Y and K-upper bounded on T, is K-continuous on $D\}$.

Remark 3. For every cone K in Y, the class $\mathscr{A}_K$ is nonempty. In fact, every subset T of X, with nonempty interior, belongs to this class ([7], Theorem 1).

Remark 4. Moreover, in the particular case that $X = \boldsymbol{R}^n$, $Y = \boldsymbol{R}$ and $K = [0, +\infty[$, the class $\mathcal{A}_K$, is reduced to the class $\mathcal{A}$ of R. Ger and M. Kuczma (cf. Remark 1).

Remark 5. Finally, the cone K of the space Y endows the space Y with an order structure, namely

$$\forall x, y \in Y \qquad x \preccurlyeq_K y \Leftrightarrow y - x \in K. \tag{8}$$

It is obvious that, with this order structure, the definitions of convexity, concavity, midconvexity and mindconcavity for functions f: $D \to Y$ are the same definitions of K-convexity, K-concavity, K-midconvexity and K-midconcavity (cf. (3) and (4)), respectively.

3 - A sufficient condition

In this section we present a sufficient condition for a K-midconvex (K-midconcave) function to be the sum of a K-convex (K-concave) function and an additive function.

Theorem. *Let Y be a Banach space, K be a normal and closed cone in Y, D be an open and convex subset of $\boldsymbol{R}^n$ and f, g: $D \to Y$ be two functions such that:*

i) *f is K-midconvex on D;*
ii) *g is K-midconcave on D;*
iii) $\exists T \in \mathcal{A}_K \mid f(x) \preccurlyeq_K g(x)$, $\forall x \in T$.

Then there exist two functions F, G: $D \to Y$ respectively K-convex and K-concave and an additive function A: $\boldsymbol{R}^n \to Y$ satisfying (1), (2).

We start with introducing a function H: $D \to Y$, defined by

$$H(x) = f(x) - g(x) \qquad \forall x \in D. \tag{9}$$

By i), ii) if follows that H is K-midconvex on D. Now from iii) we get $H(x) \in \{0\} - K$, $\forall x \in T$, that is H is K-upper bounded on T. Since $T \in \mathcal{A}_K$ we can say that H is K-continuous on D.

Therefore, fixed $x_0 \in D$ and a positive number M, there exists an open, convex and bounded neighbourhood $U(0)$, $x_0 + U(0) \subset D$, such that

$$H(x_0) \in H(x) + M\boldsymbol{B} + K \qquad \forall x \in x_0 + U(0) \tag*{(10)$_1$}$$

$$H(x) \in H(x_0) + M\boldsymbol{B} + K \qquad \forall x \in x_0 + U(0) \tag*{(10)$_2$}$$

where $\boldsymbol{B}$ is the unit closed ball in Y. Now, if we choose $\delta > 0$ with the property $H(x_0) + M\boldsymbol{B} \subset \delta\boldsymbol{B}$, from $(10)_1$, $(10)_2$ we obtain

$$(11)_1 \qquad H(x) \in \delta\boldsymbol{B} - K \qquad \forall x \in x_0 + U(0)$$

$$(11)_2 \qquad H(x) \in \delta\boldsymbol{B} + K \qquad \forall x \in x_0 + U(0)$$

that is H is K-bounded on $x_0 + U(0)$. Taking (9), i), $(11)_1$ and $(11)_2$ into account, we get

$$(12) \qquad 2g(\frac{x+y}{2}) - g(x) - g(y) \in 4\,\delta\boldsymbol{B} - K \qquad \forall x,\, y \in x_0 + U(0)\,.$$

On the other hand, by ii), it follows

$$(13) \qquad 2g(\frac{x+y}{2}) - g(x) - g(y) \in 4\,\delta\boldsymbol{B} + K \qquad \forall x,\, y \in D\,.$$

Let $\mathcal{V}(0)$ be as in Definition 1; it is trivial to prove that

$$(14) \qquad \alpha V = (\alpha V + K) \cap (\alpha V - K) \qquad \forall V \in \mathcal{V}(0) \text{ and } \forall \alpha > 0\,.$$

Fixed a bounded neighbourhood $V \in \mathcal{V}(0)$, it is possible to find two positive numbers p, q such that

$$(15) \qquad p\boldsymbol{B} \subset V \subset q\boldsymbol{B}\,.$$

Using (12), (13), (15) and (14), we obtain

$$2g(\frac{x+y}{2}) - g(x) - g(y) \in \frac{4\,\delta q}{p}\,\boldsymbol{B} \qquad \forall x,\, y \in x_0 + U(0)$$

that is (cf. (5)) g is $\frac{4\,\delta q}{p}$-Jensen on $x_0 + U(0)$. By Theorem 3 of [3] it is possible to find a Jensen function $G_1\colon \boldsymbol{R}^n \to Y$ and a positive number σ such that

$$(16) \qquad \|G_1(x) - g(x)\| \leq \sigma \qquad \forall x \in x_0 + U(0)\,.$$

Now, we define the functions $A\colon \boldsymbol{R}^n \to Y$, $G\colon D \to Y$ and $F\colon D \to Y$ putting

$$(17)_1 \qquad A(x) = G_1(x) - G_1(0) \qquad \forall x \in \boldsymbol{R}^n$$

$$(17)_2 \qquad G(x) = g(x) - A(x) \qquad \forall x \in D$$

$$(17)_3 \qquad F(x) = H(x) + G(x) \qquad \forall x \in D\,.$$

First we prove that A is an *additive function* of $\boldsymbol{R}^n$. In fact, fixed $x,\, y \in \boldsymbol{R}^n$, since G_1 is a Jensen function by $(17)_1$ we have

$$A(x+y) = \frac{1}{2}\, G_1(2x) + \frac{1}{2}\, G_1(2y) - G_1(0) = A(x) + A(y)\,.$$

Moreover, by the additivity of the function A and by the ipothesis ii) it follows that G is a *K-midconcave function* on D. On the other hand, by $(17)_2$, $(17)_1$ and (16), we get

$$G(x) \in \sigma\boldsymbol{B} + G_1(0) + K \qquad\qquad \forall x \in x_0 + U(0)$$

that is G is K-lower bounded on $x_0 + U(0)$, and it satisfies, therefore, the assumptions of the Theorem 5.3 of [1]: there exists a point $z \in D$ in which G is K-upper semicontinuous. Using the Corollary 1 and the Theorem 5.4 of [1], we have that G is *K-concave* on D; therefore by $(17)_2$ it follows the equality (2) that we wanted to prove.

To obtain the equality (1), taking $(17)_3$, $(17)_2$ and (9) into account, it is sufficient to prove that F is K-convex on D. In fact, by $(17)_3$, (9), $(17)_2$ and by the additivity of A, we have that F is a *K-midconvex function* on D; F is also K-continuous and so by Theorem 4.2 of [1] it follows that F is *K-convex* on D.

Remark 6. Our proposition contains, as a particular case, Theorem 4 stated by Z. Kominek in [3]. In fact, if $Y = \boldsymbol{R}$ and $K = [0, +\infty[$, our theorem is reduced to the theorem of Z. Kominek (Remarks 1 and 4). Moreover, even in this particular case, our theorem strictly contains the answer given to the question posed by K. Nikodem in [6], by C. T. Ng ([5], p. 540) and by K. Nikodem himself in [8]. In fact Kominek's theorem generalizes the result obtained by C. T. Ng and K. Nikodem (cf. [3], p. 507 and Remark 2).

References

[1] A. AVERNA e T. CARDINALI, *Sui concetti di K-convessità (K-concavità) e di K-convessità* (K-concavità*)*, Riv. Mat. Univ. Parma **16** (1990), 311-330.

[2] R. GER and M. KUCZMA, *On the boundedness and continuity of convex functions and additive functions*, Aequationes Math. 4 (1970), 157-162.

[3] Z. KOMINEK, *On a local stability of the Jensen functional equation*, Demonstratio Math. 22 (1989), 499-507.

[4] M. KUCZMA, *An introduction to the theory of functional equations and inequalities*, PWN, Uniwersytet Slaski, Warszawa-Kraków-Katowice 1985.

[5] C. T. NG, *On midconvex functions with midconcave bounds*, Proc. Amer. Math. Soc. **102** (1988), 538-540.

[6] K. NIKODEM, *Problems and remarks*, Proceedings of the International Conference on functional equations and inequalities, Sielpia (Poland) 1984; Wyz. Szkola Ped. Kraków, Rocznik Nauk.-Dydakt. Prace Mat. **97** (1985).

[7] K. NIKODEM, *Continuity of K-convex set-valued functions*, Bull. Polish Acad. Sci. Math. **34** (1986), 392-399.

[8] K. NIKODEM, *Midpoint convex functions majorized by midpoint concave functions*, Aequationes Math. **32** (1987), 45-51.

[9] K. NIKODEM, *K-convex and K-concave set-valued functions*, Zeszyty Nauk. Politech. Lòdz. Mat. **559** (1989).

Summary

In this note we obtain a sufficient condition for a K-midconvex (K-midconcave) function, with values in a Banach space, to be the sum of a K-convex (K-concave) function and an additive function. This proposition contains theorems due to C. T. Ng, K. Nikodem and Z. Kominek.

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