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Regular extension theorems for quaternionic functions (**)

Introduction

In this paper we develop some ideas contained in [1] and [2], in order to give trace theorems for quaternion regular functions. The theory of the regular functions of quaternion variable was founded and developed by R. Fueter in the decade following 1935, whereas the first mathematician who studied regular functions of several quaternion variables and established the first integral formulas was G. B. Rizza (cf. [8]). Recently this theory has been considerably extended; see, for instance, [3]-[6] and [9].

The problem, that we study in the present paper, is the following: consider a bounded connected open subset U of $\boldsymbol{H}^n$, with boundary of class C^1, such that $\boldsymbol{H}^n - \bar{U}$ is connected and let $f\colon \partial U \to \boldsymbol{H}$ be a continuous function. We want to determine some necessary and sufficient conditions on f, so that it may exist a function F continuous on $\boldsymbol{H}^n - U$, regular on $\boldsymbol{H}^n - \bar{U}$, which extends f. In Section 2 we solve this problem in the case $n = 1$, whereas in Section 3 we study the case $n > 1$, which differs, by Hartogs phenomenon, from the previous case. At last we give similar theorems also for complex holomorphic functions.

1 - Preliminaries

Let us denote by $\boldsymbol{H}$ the algebra of quaternions and by $i_0 = 1$, $i_1 = i$, $i_2 = j$, $i_3 = k$ its standard basis; hence if $q \in \boldsymbol{H}$ we can write $q = \sum_{0}^{3}{}_{\lambda}\, x_\lambda i_\lambda$, where $x_\lambda \in \boldsymbol{R}$. Let us consider now the space $\boldsymbol{H}^n$. If $q = (q_1, \ldots, q_n) \in \boldsymbol{H}^n$, where $q_h = \sum_{0}^{3}{}_{\lambda}\, x_\lambda^{(h)} i_\lambda$, we

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denote by $\bar{q}$ and $|q|$, respectively, the *conjugate* and the *euclidean norm* of q. In this paper $\boldsymbol{H}^n$ will be identified with $\boldsymbol{C}^{2n}$ by the map ψ_n: $\boldsymbol{C}^{2n} \to \boldsymbol{H}^n$, defined by

$$\psi_n(z_1^{(1)}, z_2^{(1)}, \ldots, z_1^{(n)}, z_2^{(n)}) = (z_1^{(1)} + jz_2^{(1)}, \ldots, z_1^{(n)} + jz_2^{(n)}).$$

Consider now an open subset $A \subseteq \boldsymbol{H}^n$ and a function F: $A \to \boldsymbol{H}$ of class C^1. We say that F is *left-regular* (or simply *regular*) if

$$\frac{\partial F}{\partial \bar{q}_h} = \sum_0^3{}_\lambda\, i_\lambda \frac{\partial F}{\partial x_\lambda^{(h)}} = 0 \qquad \text{in } A, \text{ for } h = 1, \ldots, n.$$

If the domain A of the regular function F is $\boldsymbol{H}^n$, we will say, in analogy with the complex case, that the function F is $\boldsymbol{H}$*-entire*. We recall that every regular function is *harmonic*. The *Bochner-Martinelli kernel* for regular functions is

$$\Omega_{q_0}(q) = \frac{(2n-1)!}{2\pi^{2n}} \sum_1^n{}_i\, G^i(q - q_0)\, \theta_1 \wedge \ldots \wedge \theta_{i-1} \wedge Dq_i \wedge \theta_{i+1} \wedge \ldots \wedge \theta_n$$

for q, $q_0 \in \boldsymbol{H}^n$, $q \neq q_0$, where $G^i(q) = \dfrac{\bar{q}_i}{|q|^{4n}}$, and Dq_i, θ_h are the $\boldsymbol{H}$-valued forms defined in [4].

If F is a regular function on an open subset $A \subseteq \boldsymbol{H}^n$ and U is a bounded open set with C^1-boundary such that $\bar{U} \subseteq A$, we have, for every $q_0 \in U$ (cf. [4])

$$F(q_0) = \int_{\partial U} \Omega_{q_0}(q)\, F(q). \tag{1}$$

If $n = 1$, this formula reduces to the classical Cauchy-Fueter formula (cf. [9])

$$F(q_0) = \frac{1}{2\pi^2} \int_{\partial U} G(q - q_0)\, Dq\, F(q) \tag{2}$$

where G is the regular function on $\boldsymbol{H} - \{0\}$ defined by $G(q) = \dfrac{\bar{q}}{|q|^4}$.

If we denote by Ω and K the *Bochner-Martinelli kernels* in $\boldsymbol{H}^n$ and $\boldsymbol{C}^{2n}$ respectively, we have

$$\psi_n^* \Omega = (-1)^n [K + j\Phi] \tag{3}$$

where $\Phi = \Phi_{z_0}(z)$ is a complex $(4n-1)$-form, d-exact in $\boldsymbol{C}^{2n} - \{z_0\}$, for every $z_0 \in \boldsymbol{C}^{2n}$.

We conclude this introductive section with some notations. If m is a non-negative integer, we set

$$\sigma_m = \{\nu = (m_1, m_2, m_3) \,|\, m_1, m_2, m_3 \in N,\ \ m_1 + m_2 + m_3 = m\}.$$

It is clear that the cardinality of σ_m is $\frac{1}{2}(m+1)(m+2)$. If $\nu = (m_1, m_2, m_3) \in \sigma_m$ we define $G_\nu = \dfrac{\partial^m G}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}$ and the *basic polynomial* $P_\nu(q)$ as in [9]. We recall that G_ν and P_ν are *regular functions* for all ν.

2 - Regular extension on unbounded domains of *H*

Let U be a bounded connected open subset of $\boldsymbol{H}$, with C^1-boundary, such that $\boldsymbol{H} - \bar{U}$ is connected. Moreover suppose that the origin $O \in U$ and let $f\colon \partial U \to \boldsymbol{H}$ be a continuous function. We define

$$F^+(\xi) = \frac{1}{2\pi^2} \int_{\partial U} G(q - \xi) Dq\, f(q) \qquad \text{for } \xi \in \boldsymbol{H} - \bar{U}$$

$$F^-(\xi) = \frac{1}{2\pi^2} \int_{\partial U} G(q - \xi) Dq\, f(q) \qquad \text{for } \xi \in U.$$

F^+ and F^- are regular functions, because G is regular.

Suppose that there exists a continuous function $F\colon \boldsymbol{H} - U \to \boldsymbol{H}$ such that

$$(4) \qquad \frac{\partial F}{\partial \bar{q}} = 0 \quad \text{in } \boldsymbol{H} - \bar{U} \qquad\qquad F|_{\partial U} = f.$$

If $m \in N$ and $\nu = (m_1, m_2, m_3) \in \sigma_m$, we set

$$a_\nu = \int_{\partial U} G_\nu(q) Dq\, f(q).$$

Since U is bounded, there exists $R > 0$ such that $\bar{U} \subseteq \{|q| < R\}$. Then, if $r \geqslant R$, we have

$$a_\nu = \int_{|q| = r} G_\nu(q) Dq\, F(q)$$

because $G_\nu(q)\,Dq\,F(q)$ is a d-closed 3-form (cf. [4], p. 48). In [6] we established the following inequality for G_ν

$$|G_\nu(q)| \leqslant \frac{125}{e^4}\,\frac{(20e)^m\,\nu!}{|q|^{m+3}} \qquad \text{where } \nu! = m_1!\,m_2!\,m_3!\,. \tag{5}$$

Then

$$|a_\nu| \leqslant \int_{|q|=r} |G_\nu(q)|\,|F(q)| \leqslant \frac{125}{r^3 e^4}\left(\frac{20e}{r}\right)^m \nu! \int_{|q|=r} |F(q)|\,.$$

Thus we get

$$\sum_{\nu\in\sigma_m} \frac{|a_\nu|}{\nu!} \leqslant \left(\frac{125}{r^3 e^4}\int_{|q|=r} |F(q)|\right)\frac{(m+1)(m+2)}{2}\left(\frac{20e}{r}\right)^m$$

and then

$$\overline{\lim_{m\to+\infty}}\ \sqrt[m]{\sum_{\nu\in\sigma_m}\frac{|a_\nu|}{\nu!}} \leqslant \frac{20e}{r} \qquad \text{for every } r \geqslant R\,.$$

Thus we obtain

$$\lim_{m\to+\infty} \sqrt[m]{\sum_{\nu\in\sigma_m}\frac{|a_\nu|}{\nu!}} = 0\,. \tag{6}$$

Conversely suppose that (6) holds true. If $\xi\in U$ and $|\xi| < \inf_{q\in\partial U}|q|$, from Proposition 10 of [9], we have

$$F^-(\xi) = \frac{1}{2\pi^2}\int_{\partial U} G(q-\xi)\,Dq\,f(q)$$

$$= \frac{1}{2\pi^2}\sum_0^{+\infty}{}_m \sum_{\nu\in\sigma_m} P_\nu(\xi)\int_{\partial U} G_\nu(q)\,Dq\,f(q) = \frac{1}{2\pi^2}\sum_0^{+\infty}{}_m \sum_{\nu\in\sigma_m} P_\nu(\xi)\,a_\nu\,.$$

Since $|P_\nu(\xi)| \leqslant \dfrac{|\xi|^m}{\nu!}$ for every $\nu\in\sigma_m$, from (6) we deduce that the radius of convergence of the series $\sum_0^{+\infty}{}_m \sum_{\nu\in\sigma_m} P_\nu(\xi)\,a_\nu$ is $+\infty$, that is this series converges for all $\xi\in\boldsymbol{H}$. Thus F^- can be regularly extended to $\boldsymbol{H}$. From Lemma 3 of [4], also F^+ can be continuously extended on $\boldsymbol{H}-U$, and on ∂U we have $F^+ - F^- = f$. Therefore the function $F = F^+ - F^-$ is a solution of the problem (4).

Thus we have proved

Theorem 1. *Let U be a bounded connected open subset of $\boldsymbol{H}$, with boundary of class C^1, such that $\boldsymbol{H} - \bar{U}$ is connected. Suppose $0 \in U$, and let $f\colon \partial U \to \boldsymbol{H}$ be a continuous function. Then there exists a continuous function $F\colon \boldsymbol{H} - U \to \boldsymbol{H}$, regular on $\boldsymbol{H} - \bar{U}$, which extends f, if and only if*

$$\lim_{m \to +\infty} \sqrt[m]{\sum_{\nu \in \sigma_m} \frac{|a_\nu|}{\nu!}} = 0 \qquad \text{where } a_\nu = \int_{\partial U} G_\nu(q) Dq\, f(q) \quad \text{for every } \nu .$$

We remark that, if the solution of problem (4) exists, it is unique. In fact, if $F\colon \boldsymbol{H} - U \to \boldsymbol{H}$ is a continuous function, regular on $\boldsymbol{H} - \bar{U}$, which is zero on ∂U, we can continuously extend F to a function v defined on all $\boldsymbol{H}$, setting v equal to zero in U. The function v (and hence also F) is necessarily identically zero, by the principle of identity. In fact v is regular on $\boldsymbol{H}$ from Proposition 12 of [6].

By the same methods, it is possible to prove analogs of Theorem 1 for holomorphic functions of one complex variable (cf. [1]). More precisely, we can show the following

Theorem 2. *Let U be a bounded open subset of $\boldsymbol{C}$, whose boundary is a Jordan curve of class C^1. Suppose $0 \in U$, and let $f\colon \partial U \to \boldsymbol{C}$ be a continuous function. Then there exists a continuous function $F\colon \boldsymbol{C} - U \to \boldsymbol{C}$, holomorphic on $\boldsymbol{C} - \bar{U}$, which extends f, if and only if*

$$\lim_{k \to +\infty} \sqrt[k]{|b_k|} = 0 \qquad \text{where } b_k = \int_{\partial U} \frac{f(\zeta)}{\zeta^{k+1}}\, \mathrm{d}\zeta \quad \text{for every } k \in \boldsymbol{N}.$$

Taking into account Theorem 9 of [4], Theorem 1 and Proposition 12 of [6], we can deduce the following

Theorem 3. *Let $U \subseteq \boldsymbol{H}$ and $f\colon \partial U \to \boldsymbol{H}$ be as in the statement of Theorem 1. Then f can be extended to a $\boldsymbol{H}$-entire function if and only if*

$$\int_{\partial U} P_\nu(q) Dq\, f(q) = 0 \quad \forall \nu \in \sigma_m,\ \forall m \in \boldsymbol{N} \qquad \text{and} \qquad \lim_{m \to +\infty} \sqrt[m]{\sum_{\nu \in \sigma_m} \frac{|a_\nu|}{\nu!}} = 0$$

where $a_\nu = \int_{\partial U} G_\nu(q) Dq\, f(q)$ *for every* ν.

3 - Entire extension of functions of several variables

In this section we will try to extend, to functions of several variables, the methods and the results of the previous section. Hence consider a bounded connected open subset U of $\boldsymbol{H}^n$ ($n > 1$), with boundary of class C^1, such that $\boldsymbol{H}^n - \bar{U}$ is connected. Moreover suppose that the origin $O \in U$. If $f\colon \partial U \to \boldsymbol{H}$ is a continuous function, we can define, as in the previous section

$$F^+(\xi) = \int_{\partial U} \Omega_\xi(q) f(q) \qquad \text{for } \xi \in \boldsymbol{H}^n - \bar{U}$$

$$F^-(\xi) = \int_{\partial U} \Omega_\xi(q) f(q) \qquad \text{for } \xi \in U.$$

Since $n > 1$, the functions F^+ and F^- are, in general, not regular, but only harmonic. The conditions which ensure the regularity of F^+ and F^-, are the so called *weak integral conditions of Cauchy-Riemann-Fueter* (cf. [4], p. 64). These conditions are necessary in order to have any regular extension of f and can be expressed in the following form (cf. [4])

$$\int_{\partial U} \frac{\partial \Omega_{q_0}}{\partial \bar{q}_h^0} f = 0 \qquad \forall h = 1, \ldots, n, \ \ \forall q_0 \notin \partial U. \tag{7}$$

Then, if f verifies (7), F^+ and F^- are regular functions on $\boldsymbol{H}^n - \bar{U}$ and U respectively, and then, by Hartogs phenomenon (cf. [4]), F^+ can be extended to a $\boldsymbol{H}$-entire function. It is easy to see that we have $\lim_{\xi \to \infty} F^+(\xi) = 0$ (cf. [4], p. 64), and then, by Liouville theorem, F^+ must vanish identically on $\boldsymbol{H}^n$. From Lemma 3 of [4], F^- can be continuously extended on $\bar{U}$, and, on ∂U, we get $-F^- = f$.

We will try now some conditions, which ensure the extensibility of F^- to a $\boldsymbol{H}$-entire function; in this case $-F^-$ will be a solution of the problem

$$\frac{\partial F}{\partial \bar{q}_h} = 0 \ \text{in } \boldsymbol{H}^n, \ \text{for } h = 1, \ldots, n \qquad F|_{\partial U} = f \tag{8}$$

which is, by Hartogs phenomenon, the generalization to $n > 1$ variables of the problem (4). We remark that the solution of (8), if it exists, is necessarily unique.

We fix now some notations.

If $\alpha = (\alpha_0^{(1)}, \alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \ldots, \alpha_0^{(n)}, \ldots, \alpha_3^{(n)}) \in \boldsymbol{N}^{4n}$ is a multi-index we set

$$\alpha! = \prod_{\substack{\lambda = 0, \ldots, 3 \\ h = 1, \ldots, n}} \alpha_\lambda^{(h)}! \qquad |\alpha| = \sum_{\substack{\lambda = 0, \ldots, 3 \\ h = 1, \ldots, n}} \alpha_\lambda^{(h)} \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_0^{(1)\alpha_0^{(1)}} \ldots \partial x_3^{(n)\alpha_3^{(n)}}} .$$

Moreover if $\xi = (\xi_1, \ldots, \xi_n) \in \boldsymbol{H}^n$, with $\xi_h = \sum_{0}^{3}{}_\lambda \xi_\lambda^{(h)} i_\lambda$, $\xi_\lambda^{(h)} \in \boldsymbol{R}$, we define the real number ξ^α setting

$$\xi^\alpha = \prod_{\substack{\lambda = 0, \ldots, 3 \\ h = 1, \ldots, n}} \xi_\lambda^{(h)\alpha_\lambda^{(h)}} .$$

Since the function $G^i(q) = \dfrac{\bar{q}_i}{|q|^{4n}}$ is harmonic in $\boldsymbol{H}^n - \{0\}$, there exists $\varepsilon > 0$ such that if $|\xi| < \varepsilon$ and $q \in \partial U$ we have

$$G^i(q - \xi) = \sum_{\alpha \in \boldsymbol{N}^{4n}} (-1)^{|\alpha|} \frac{D^\alpha G^i(q)}{\alpha!} \xi^\alpha$$

(cf. [7], p. 199). Then, if $|\xi| < \varepsilon$, we have

$$F^-(\xi) = \frac{(2n-1)!}{2\pi^{2n}} \sum_{\alpha \in \boldsymbol{N}^{4n}} (-1)^{|\alpha|} c_\alpha \xi^\alpha$$

where $c_\alpha = \dfrac{1}{\alpha!} \int_{\partial U} \sum_{1}^{n}{}_i D^\alpha G^i(q) \theta_1 \wedge \ldots \wedge Dq_i \wedge \ldots \wedge \theta_n f(q) \qquad \forall \alpha \in \boldsymbol{N}^{4n}$.

Since $\sum_{\alpha \in \boldsymbol{N}^{4n}} |(-1)^{|\alpha|} c_\alpha \xi^\alpha| \leqslant \sum_{0}^{+\infty}{}_m \big(\sum_{|\alpha| = m} |c_\alpha|\big) |\xi|^m$ if

(9) $$\lim_{m \to +\infty} \sqrt[m]{\sum_{|\alpha| = m} |c_\alpha|} = 0$$

the series $\sum_{\alpha \in \boldsymbol{N}^{4n}} (-1)^{|\alpha|} c_\alpha \xi^\alpha$ converges for all $\xi \in \boldsymbol{H}^n$, and then F^- can be extended to $\boldsymbol{H}^n$ as real analytic function. Since this function is regular for $|\xi| < \varepsilon$, it is regular for all $\xi \in \boldsymbol{H}^n$. Then, if (9) holds true, the problem (8) has a (unique) solution.

Conversely, let us prove that, if (8) has a solution, (9) necessarily holds. First of all we need of a generalization of (5).

Proposition 1. *If* $\alpha \in N^{4n}$, *we have*

$$|D^\alpha G^i(q)| \leqslant \frac{(4n+1)^{4n-1}}{e^{4n}} \frac{(4n(4n+1)e)^{|\alpha|}\alpha!}{|q|^{|\alpha|+4n-1}} \quad \forall q \in H^n - \{0\}, \ \forall i = 1, \ldots, n.$$

Proof. Let $\alpha = (\alpha_0^{(1)}, \ldots, \alpha_3^{(n)}) \in N^{4n}$. Since G^i and its derivatives are harmonic functions, we have (cf. [7], p. 197), if $q \neq 0$,

$$|D^\alpha G^i(q)| \leqslant \frac{(4n)^{\alpha_0^{(1)}} e^{\alpha_0^{(1)}-1} \alpha_0^{(1)}!}{\left(\frac{|q|}{4n+1}\right)^{\alpha_0^{(1)}}} \cdot \max_{|\xi - q| \leqslant \frac{|q|}{4n+1}} |D^{(0,\alpha_1^{(1)},\ldots,\alpha_3^{(n)})} G^i(\xi)|$$

$$\leqslant \frac{(4n)^{\alpha_0^{(1)}+\alpha_1^{(1)}} e^{\alpha_0^{(1)}+\alpha_1^{(1)}-2} \alpha_0^{(1)}!\,\alpha_1^{(1)}!}{\left(\frac{|q|}{4n+1}\right)^{\alpha_0^{(1)}+\alpha_1^{(1)}}} \cdot \max_{|\xi - q| \leqslant \frac{2|q|}{4n+1}} |D^{(0,0,\alpha_2^{(1)},\ldots,\alpha_3^{(n)})} G^i(\xi)|$$

$$\leqslant \ldots \leqslant \frac{(4n)^{|\alpha|} e^{|\alpha|-4n} \alpha!}{\left(\frac{|q|}{4n+1}\right)^{|\alpha|}} \cdot \max_{|\xi - q| \leqslant \frac{4n|q|}{4n+1}} |G^i(\xi)| \leqslant \frac{(4n(4n+1)e)^{|\alpha|}\alpha!}{e^{4n}|q|^{|\alpha|+4n-1}} (4n+1)^{4n-1}.$$

Now suppose that F is a solution of (8). Since the form $\Omega_\xi(q)F(q)$ is d-closed in $H^n - \{\xi\}$ (cf. [4], p. 44), if $\xi \in U$ and $r > \max_{q \in \partial U} |q|$, from Stokes theorem, we obtain

$$\int_{\partial U} \Omega_\xi(q) f(q) = \int_{|q|=r} \Omega_\xi(q) F(q). \tag{10}$$

If we derive (10) with respect to ξ and we set $\xi = 0$, for any $\alpha \in N^{4n}$ we obtain

$$\int_{\partial U} \sum_1^n{}_i D^\alpha G^i(q)\,\theta_1 \wedge \ldots \wedge Dq_i \wedge \ldots \wedge \theta_n f(q) = \int_{|q|=r} \sum_1^n{}_i D^\alpha G^i(q)\,\theta_1 \wedge \ldots \wedge Dq_i \wedge \ldots \wedge \theta_n F(q)$$

that is $\quad c_\alpha = \dfrac{1}{\alpha!} \displaystyle\int_{|q|=r} \sum_1^n{}_i D^\alpha G^i(q)\,\theta_1 \wedge \ldots \wedge Dq_i \wedge \ldots \wedge \theta_n F(q) \quad \forall \alpha \in N^{4n}.$

From this equality and from Proposition 1, for any $\alpha \in N^{4n}$ we get

$$|c_\alpha| \leqslant \frac{1}{\alpha!} \int_{|q|=r} \sum_1^n{}_i |D^\alpha G^i(q)|\,|F(q)|$$
$$\leqslant \left[\frac{n(4n+1)^{4n-1}}{e^{4n} r^{4n-1}} \int_{|q|=r} |F(q)|\right] \left(\frac{4n(4n+1)e}{r}\right)^{|\alpha|} \tag{11}$$

and hence

$$\sqrt[m]{\sum_{|\alpha|=m} |c_\alpha|} \leqslant \sqrt[m]{K} \sqrt[m]{(m+1)^{4n}} \frac{4n(4n+1)e}{r} \qquad \forall m \in \boldsymbol{N}$$

where K is the expression in square brackets in (11).

We deduce that

$$\overline{\lim_{m \to +\infty}} \sqrt[m]{\sum_{|\alpha|=m} |c_\alpha|} \leqslant \frac{4n(4n+1)e}{r}.$$

Since r is an arbitrary number greater than $\max_{q \in \partial U} |q|$, we obtain (9).

Thus we have proved

Theorem 4. *Let U be a bounded connected open subset of $\boldsymbol{H}^n$ $(n > 1)$, with boundary of class C^1, such that $\boldsymbol{H}^n - \bar{U}$ is connected. Suppose $O \in U$, and let f: $\partial U \to \boldsymbol{H}$ be a continuous function. Then f can be extended to a $\boldsymbol{H}$-entire function, if and only if it satisfies the weak integral conditions of Cauchy-Riemann-Fueter and* $\lim_{m \to +\infty} \sqrt[m]{\sum_{|\alpha|=m} |c_\alpha|} = 0$, *where*

$$c_\alpha = \frac{1}{\alpha!} \int_{\partial U} \sum_1^n {}_i D^\alpha G^i(q)\, \theta_1 \wedge \ldots \wedge Dq_i \wedge \ldots \wedge \theta_n f(q) \qquad \forall \alpha \in \boldsymbol{N}^{4n}.$$

Remark. If ∂U and f are of class C^4, the weak integral conditions of Cauchy-Riemann-Fueter in the statement of Theorem 4 can be substituted by the condition of admissibility (see [6] for the definition). This last condition, for a function of class C^6, is equivalent to the trace differential condition we introduced in [5].

By the same methods, we can prove analogs of Theorem 4 for complex holomorphic functions. The statement of this theorem becomes

Theorem 5. *Let U be a bounded connected open subset of $\boldsymbol{C}^n$ $(n > 1)$, with boundary of class C^1, such that $\boldsymbol{C}^n - \bar{U}$ is connected. Suppose $O \in U$, and let f: $\partial U \to \boldsymbol{C}$ be a C^1-function. Then f can be extended to an entire function, if and* *only if f is a CR-function and* $\lim_{m \to +\infty} \sqrt[m]{\sum_{|\alpha| = m} |d_\alpha|} = 0$, *where*

$$d_\alpha = \frac{1}{\alpha!} \int_{\partial U} \sum_{1}^{n} {}_i \frac{\partial^{|\alpha|} H^i}{\partial z^\alpha}(z) f(z)\, dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_i} \wedge \ldots \wedge d\bar{z}_n \qquad \forall \alpha \in N^n$$

and $H^i(z) = (-1)^{i-1} \dfrac{\bar{z}_i}{|z|^{2n}}$.

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Sommario

Si dimostrano alcuni risultati relativi alle tracce delle funzioni quaternionali, regolari nel senso di Fueter, ed alle tracce delle funzioni complesse olomorfe.

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