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Essential extensions of a dimension module (**)

1 - Introduction

Let M be a *left R-module* and $d(M)$ denote the *Goldie (uniform) dimension* of M (that is, $d(M)$ is the number of components in a longest direct sum of submodules contained in M, and is ∞ if no such direct sum exists). M is called a *dimension module* if

$$d(A + B) = d(A) + d(B) - d(A \cap B)$$

holds for all submodules A and B of M. In [**1**], dimension modules have been shown to be modules which have no submodules of the form $X \oplus X/Y$ with Y an essential submodule of X. If M is a dimension module, then there are essential extensions of M which are dimension modules. We call them *essential dimension extensions* of M.

The present paper exhibits the relationship between the rational extension and the essential dimension extension of a given dimension module. Recall that a module P is called a *rational extension* of a module M, if M is a submodule of P and $\mathrm{Hom}_R(K/M, P) = 0$ for every between module $P \supseteq K \supseteq M$, or equivalently, if for any pair a, $b \in P$ and $b \neq 0$, there is an element $r \in R$ such that $ra \in M$ and $rb \neq 0$ ([3], Prop. 19.32).

Using torsion theories, we also characterize *τ-torsionfree modules*, which are dimension modules, where τ is a torsion theory on R-mod.

Throughout this paper, R is an associative ring with identity and modules are unitary left R-modules. If A, B are modules, then $A \leq B$ means that A is a

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submodule of B and $A \trianglelefteq B$ means that A is an *essential submodule* of B. $E(M)$ denotes the *injective hull* of M. The following notations are used frequently.

For any $x, y \in M$, $(Ry : x) = \{r \in R \mid rx \in Ry\}$. In particular, $(0 : x) = x^{\perp}$.

Lemma 1 ([1], Prop. 1 and Cor. 2). *The following conditions are equivalent for a module M:*

1. *M is a dimension module.*
2. *For every partial endomorphism f: $A \to M$ with $A \cap f(A) = 0$, $\ker f$ is closed in A.*
3. *M has no submodule of the form $X \oplus X/Y$ with Y a proper essential submodule of X.*

2 - Essential extensions of a dimension module

Let M be a dimension module. The injective hull $E(M)$ of M need not be a dimension module. For example, the Z_4-module $M = 2Z_4 \oplus 2Z_4$ is a dimension module, but $E(M) = Z_4 \oplus Z_4$ is not a dimension module since

$$E(M) \supseteq Z_4 \oplus 2Z_4 \cong Z_4 \oplus Z_4/2Z_4.$$

Lemma 2. *Let P be an essential extension of a given dimension module M. If $d(P) < \infty$, then P is a dimension module, if and only if we have*

$$A \cap M + B \cap M \trianglelefteq (A + B) \cap M$$

for all submodules A and B of P.

Proof. If $M \trianglelefteq P$ then $A \cap M \trianglelefteq A$. So $d(A) = d(A \cap M)$. Similarly, $d(B) = d(B \cap M)$ and $d(A \cap B) = d(A \cap B \cap M)$. Assume that we have $A \cap M + B \cap M \trianglelefteq (A + B) \cap M$ holds for all A, B of P. Then

$$d(A + B) = d((A + B) \cap M) = d(A \cap M + B \cap M)$$

$$= d(A \cap M) + d(B \cap M) - d(A \cap B \cap M) = d(A) + d(B) - d(A \cap B)$$

and so P is a dimension module.

Conversely, assume that P is a dimension module. Since we have $A \cap M + B \cap M \subseteq (A + B) \cap M$,

$$d(A + B) = d((A + B) \cap M) \geq d(A \cap M + B \cap M)$$

$$= d(A \cap M) + d(B \cap M) - d(A \cap B \cap M) = d(A + B).$$

It follows that $d((A + B) \cap M) = d(A \cap M + B \cap M)$, so we can write $A \cap M + B \cap M \trianglelefteq (A + B) \cap M$.

Lemma 3. *Let P be an essential extension of a given dimension module M and $d(P) < \infty$. Then the following conditions are equivalent:*

(1) *P is a dimension module.*

(2) *For any pair $x, y \in P$ and $0 \neq x + y \in M$, there is an $r \in R$ such that $r(x + y) \neq 0$, and $r(x + y) \in Ry$ or both $rx \in M$ and $ry \in M$ hold.*

(3) *For any pair $x, y \in P$ and $0 \neq x + y$, there is an $r \in R$ such that $r(x + y) \neq 0$, and $r(x + y) \in Ry$ or both $rx \in M$ and $ry \in M$ hold.*

Proof. It is immediate to prove that (3) $\Rightarrow$ (2).

Now we prove that (2) $\Rightarrow$ (1). By Lemma 2, it suffices to prove that $A \cap M + B \cap M \trianglelefteq (A + B) \cap M$ holds for all A, B of P. If $0 \neq a + b \in M$, where $a \in A$ and $b \in B$, then there exists an $r \in R$ such that $r(a + b) = ra + rb \neq 0$, and $r(a + b) \in Rb$ or both $ra \in M$ and $rb \in M$ hold. Thus $r(a + b) \in Rb$ implies that $r(a + b) \in B \cap M$ or $ra \in M$ implies that $ra \in A \cap M$ and so $rb \in B \cap M$.

Conversely, (1) $\Rightarrow$ (2). For any pair $a, b \in P$ and $0 \neq a + b \in M$, put $A = Ra$ and $B = Rb$. By the assumption, $A \cap M + B \cap M \trianglelefteq (A + B) \cap M$. Hence there exists $0 \neq r \in R$ such that $0 \neq r(a + b) \in A \cap M + B \cap M$. Then there are $s, t \in R$ such that $r(a + b) = sa + tb$ with $sa \in A \cap M$ and $tb \in B \cap M$. If $s(a + b) = 0$ then $r(a + b) = sa + tb = -sb + tb \in B \cap M$. If $s(a + b) \neq 0$ then we have $0 \neq s(a + b) = sa + sb \in M$ and so $sb \in N$ since $sa \in M$. Therefore condition (2) is satisfied.

Finally, (2) $\Rightarrow$ (3). For any $0 \neq a + b \in P$, since $M \triangleleft P$, there exists an $r \in R$ such that $0 \neq r(a + b) = ra + rb \in M$. The remainder of the proof follows from (2).

Corollary 1. *Let $d(M) < \infty$. Then M is a dimension module, if and only if there exists an essential dimension submodule A of M, such that, for any $x, y \in M$ and $x + y \neq 0$, there is an $r \in R$ such that $r(x + y) \neq 0$, $rx + ry \in Ry$ or both $rx \in A$ and $ry \in A$ hold.*

Now, it is worth recalling that every rational extension of M is an essential extension and any two maximal rational extensions of M are isomorphic ([3], Prop. 19.32). Moreover a module M is called *rational closed* in case there is no properly rational extension of M.

Lemma 4. *If M is a dimension module and $d(M) < \infty$, then every rational extension P of M is dimension.*

Proof. For any pair $x, y \in P$ and $0 \neq x + y \in M$, if $x \in M$, then $y \in M$. So we can assume that $x \notin M$ and $y \notin M$. If $(M : x) \subseteq (x + y)^{\perp}$, then f: $(Rx + M)/M \to P$ defined by $rx + M \mapsto r(x + y)$ is a nonzero R-homomorphism, which is a contradiction. Thus $(M : x) \nsubseteq (x + y)^{\perp}$, and then we have an $r \in (M : x)$ but $r \notin (x + y)^{\perp}$. Therefore $r(x + y) \neq 0$, $rx \in M$ and $ry = r(x + y) - rx \in M$. By Lemma 3, P is an essential dimension extension of M.

Remark. The converse of Lemma 4 is false, since an essential dimension extension of M need not be rational.

For example. Let F be a field and $R = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in F\right\}$. Then R is a left and right perfect ring and the Jacobson radical $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. So R is not a *V-ring* ([4], p. 356), and so there exists a simple R-module M such that $M \neq E(M)$. Clearly $E(M)$ is an essential dimension extension of M and is not a rational extension of M, since every simple module over a left and right perfect ring is rational closed ([5], Prop. 2.10).

Now we are ready to prove the main result of this section.

Theorem 1. *Let M be a dimension module and $d(M) < \infty$. Then a module P is a rational extension of M, if and only if the following conditions hold*

(1) *P is an essential dimension extension of M*
(2) *If $Rx \cap Ry \neq 0$ for $x, y \in P$ and $y \neq 0$, then $(M : x) \nsubseteq y^{\perp}$.*

Proof. If P is a rational extension of M, then (1), (2) hold obviously.

Conversely, for any pair $x, y \in P$ and $y \neq 0$, by (2), we can assume that $Rx \cap Ry = 0$. If $Ry \cap R(x + y) \neq 0$, then $r_1 y = r_2(x + y) \neq 0$ implies that

$r_2 x = (r_1 - r_2) y = 0$ and so $r_1 y = r_2 y \neq 0$. Thus we can assume that $Rx \cap Ry = 0$ and $Ry \cap R(x + y) = 0$. Since P is a dimension module, by Lemma 3 there is an $r \in R$ such that $r(x + y) = rx + ry \neq 0$, $rx \in M$ and $ry \in M$. If $ry \neq 0$, the proof ends. Thus we may assume that $ry = 0$. So

$$Rx \cap R(x + y) = (Rx : x + y)(x + y) = (Rx : y)(x + y) \neq 0 .$$

Let $r \in (Rx : y)$. If $r(x + y) = 0$, then $rx = ry = 0$ (note that $Rx \cap Ry = 0$) and so $r \in y^{\perp}$. If $r(x + y) \neq 0$, since $r(x + y) = r_1 x \in Rx$, $(r - r_1)x = -ry$, and so $ry = 0$. Hence $r \in y^{\perp}$ and $(Rx : y) = y^{\perp}$. Clearly $y^{\perp} \subseteq (Rx\colon y)$ and so $(Rx\colon x + y) = (Rx\colon y) = y^{\perp}$. Hence

$$Rx \cap R(x + y) = y^{\perp}(x + y) = y^{\perp} x \neq 0$$

and $f\colon R(x + y) \to Ry$ defined by $r(x + y) \mapsto ry$ is a nonzero R-homomorphism. From $R(x + y) \cap Ry = 0$ and Lemma 1 it follows that $\ker f = y^{\perp}(x + y) = y^{\perp} x$ is closed in $R(x + y)$. On the other hand, for every $0 \neq r(x + y) \in R(x + y)$, by Lemma 3 there is an $r_1 \in R$ such that $0 \neq r_1(rx + ry) \in Rx$ or $r_1 rx \in M$ and $r_1 ry \in M$. If the latter holds, as above, we may assume that $r_1 ry = 0$. Then $0 \neq r_1 r(x + y) \in Rx$ and so $R(x + y) \cap Rx = y^{\perp} x \trianglelefteq R(x + y)$. Hence $R(x + y) = y^{\perp} x = \ker f$, whence $f = 0$, a contradiction. This completes the proof.

3 - Torsion theories and dimension modules

In this section, we freely use terminology and notations of [4]. If M is a module, $X(M)$ is called the *(hereditary) torsion theory cogenerated by M.* Clearly $X(M) \geq \tau$ for any torsion theory τ, relative to which M is torsionfree.

If τ is a torsion theory, then $E_{\tau}(M)$ denotes the *τ-injective hull* of M.

There exists a torsion theory τ_G on R-mod defined by the condition that a left R-module M is τ_G-torsionfree, if and only if it is nonsingular. This torsion theory is called the *Goldie torsion theory.*

A module M is called *τ-full* if M is τ-torsionfree and every essential submodules of M is τ-dense in M. Every τ_G-torsionfree module is τ_G-full.

From [1] Prop. 5, τ_G-torsionfree **Z**-modules are dimension **Z**-modules. More generally, we have the following

Theorem 2. *If τ is a (hereditary) torsion theory, then every τ-torsionfree module is a dimension module, if and only if every τ-torsionfree module is τ-full.*

Proof. Assume that M is τ-torsionfree. If an essential submodule N of M is not τ-dense in M, then there is a τ-pure submodule T of M such that $M \supset T \supsetneq N$. So $M \oplus M/T$ is τ-torsionfree, and then is dimension, a contradiction.

Conversely, assume that a τ-torsionfree module M is not dimension, then M has a submodule of the form $X \oplus X/Y$ for some $Y \lhd X$ with $Y \neq X$. Since X is τ-full, Y is τ-dense in X, a contradiction.

By [4], Cor. 15.7, we have

Corollary 2. *If every τ-torsionfree module is dimension, then every τ-torsionfree module is τ-dense in its injective hull.*

We recall that τ is called *faithful* if R is τ-torsionfree.

Corollary 3. *If τ is faithful, then every τ-torsionfree module is a dimension module, if and only if $\tau = \tau_G = X(R)$.*

Proof. By ([4], Prop. 15.5), the lattice of all τ-pure left ideals of R is complemented. Let M be a nonzero left R-module and let g be an R-homomorphism from $E(M)/M$ into E, where E is a member of τ. If $x + M \in E(M)/M$ but $x + M \notin \ker g$, then $(M : x) \lhd R$, $(M : x) \neq R$ and so $(M : x)$ is τ-dense in R. Thus $R(x + M) \cong R/(M : x)$ is τ-torsion, this contradicts the fact that $g|_{R(x+M)}$: $R(x + M) \to E$ is nonzero. Now by [4] Prop. 10.14, $\tau_G \leq \tau \leq X(R)$, and so τ_G is faithful by [4], example (5.15), $\tau_G = \tau = X(R)$.

Conversely, since every τ-torsionfree module is nonsingular, by [1] Prop. 4, every τ-torsionfree module is dimension.

A left ideal I of R is called *τ-critical* if R/I is τ-cocritical (i.e. R/I is τ-torsionfree and every proper homomorphic image of R/I is τ-torsion).

Theorem 3. *Let R be a commutative ring. If any ascending chain of essential τ-pure ideals of R terminates after finitely many steps, then the following conditions are equivalent:*

(1) $\tau \geq \tau_G$.

(2) *Every τ-torsionfree module is dimension and there is no essential τ-critical ideal of R.*

Proof. As a consequence of [1], Prop. 4 we remark that (1)$\Rightarrow$(2).

Now we prove that (2)$\Rightarrow$(1).If $\tau \not\leq \tau_G$, then there is a τ-torsionfree module M which is not τ_G-torsionfree. Hence we have a $0 \neq x \in M$ such that $x^{\perp} \lhd R$, and then $R/x^{\perp} \cong Rx$ implies $x^{\perp}$ is an essential τ-pure ideal of R. If $x^{\perp}$ is not a prime ideal of R then there are $r_1, r_2 \in R - x^{\perp}$ such that $r_1 r_2 \in x^{\perp}$. So $x^{\perp} \subset (r_1 x)^{\perp}$, but $x^{\perp} \neq (r_1 x)^{\perp}$. Now $R/(r_1 x)^{\perp} = R(r_1 x) \subseteq Rx$ implies that $(r_1 x)^{\perp}$ is also an essential τ-pure ideal of R. Assume that we have the proper inclusions

$$x^{\perp} \subset (r_1 x)^{\perp} \subset (r_2 r_1 x)^{\perp} \subset \dots \subset (r_n r_{n-1} \dots r_1 x)^{\perp}.$$

If $(r_n r_{n+1} \dots r_1 x)^{\perp}$ is not prime of R, as above, we have an essential τ-pure ideal $(r_{n+1} r_n \dots r_1 x)^{\perp}$ of R. So by assumption, there is an integer n such that $(r_n r_{n-1} \dots r_1 x)^{\perp}$ is prime of R. Write $y = r_n r_{n-1} \dots r_1 x$. Assume that N is a ideal of R with $y^{\perp} \subset N \subset R$. If $N/y^{\perp}$ is τ-dense in $R/y^{\perp}$, the proof ends. If $N/y^{\perp}$ is not τ-dense in $R/y^{\perp}$, we can assume that $N/y^{\perp}$ is τ-pure of $R/y^{\perp}$. Then $N/y^{\perp}$ is an essential ideal of $R/y^{\perp}$, since $y^{\perp}$ is a prime ideal of R. Now $R/y^{\perp} \oplus (R/y^{\perp})/(N/y^{\perp})$ is dimension by assumption, a contradiction. Hence $y^{\perp}$ is an essential τ-critical ideal of R, which is a contradiction.

Let M be a dimension module. Then there are essential dimension extensions of M ([1], Th. 8). We do not know if two maximal essential dimension extensions of a given dimension module M are isomorphic. However we have

Theorem 4. *Let N be a dimension module and $d(N) < \infty$. If M is a maximal essential dimension extension of N in $E(N)$ and all maximal essential dimension extensions of N are isomorphic, then M/N is $X(N)$-torsionfree, if and only if $N = E_{X(N)}(N)$, where $X(N)$ denotes the torsion theory cogenerated by N and $E_{X(N)}(N)$ denotes the $X(N)$-injective hull of N.*

Proof. If $N = E_{X(N)}(N)$, then $E(N)/N$ is $X(N)$-torsionfree (see [4], p. 84), and so M/N is $X(N)$-torsionfree.

Conversely, let M/N be $X(N)$-torsionfree. $E_{X(N)}(N)$ is a maximal rational extension of N by [5], Prop. 1.4, and so is an essential dimension extension of N. If G is a maximal essential dimension extension of N containing $E_{X(N)}(N)$, then $G \cong M$ and so there is a maximal rational extension of N in M. But by [3], Prop. 19.32, $E_{X(N)}(N)$ is a unique maximal rational extension of N in $E(N)$. Hence $E_{X(N)}(N) \subseteq M$. Now from [4] (Prop. 9.9 and Prop. 9.6), we have $N = E_{X(N)}(N) \cap M = E_{X(N)}(N)$.

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Summary

The present paper exhibits the relationship between the rational extension and the essential dimension extension of a given dimension module. Using torsion theories, we also characterize τ-torsionfree modules which are dimension modules.

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