Riv. Mat. Univ. Parma (5) 2 (1993), 313-318

V. ANDRIANO and A. BACCIOTTI (*)

A topological criterion for global asymptotic stability (**)

1 - Introduction

The main goal of this paper is to present a criterion for checking global asymptotic stability of an equilibrium point of a dynamical system.

Let E be a locally compact metric space and consider a *dynamical system* π, that is map $\pi: E \times \boldsymbol{R} \to E$ such that:

ds1 $\pi(x, 0) = x$ for all $x \in E$

ds2 $\pi(\pi(x, t), s) = \pi(x, t + s)$ for all $x \in E$ and all t and s in $\boldsymbol{R}$

ds3 π is continuous.

An *equilibrium point* of π, is a point $\bar{x} \in E$ such that $\pi(\bar{x}, t) = \bar{x}$ for all $t \in \boldsymbol{R}$.

We say that an equilibrium point $\bar{x}$ is *globally asymptotically stable* for the system π if

as1 $\bar{x}$ is stable: for each neighborhood U of $\bar{x}$, there exists a neighborhood V of $\bar{x}$ such that, for all x in V and for all $t > 0$, $\pi(x, t) \in U$

as2 $\bar{x}$ is a global attractor: for all $x \in E$: $\pi(x, t) \to \bar{x}$ when $t \to +\infty$.

It is well known that the problem of checking global asymptotic stability is equivalent to the one of finding a *Liapunov function*. Precisely, we have the following result (see for example [5]).

(*) Dip. di Matem., Politecnico Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italia.

(**) Received April 23, 1993. AMS classification 58 F 10.

Theorem 1. *An equilibrium point* $\bar{x} \subset E$ *is globally asymptotically stable if and only if there exists a continuous uniformly unbounded function* $\phi(x)$ *defined on* E *such that*

i $\phi(\bar{x}) = 0$ *and* $\phi(x) > 0$ *for* $x \neq \bar{x}$

ii $\phi(\pi(x, t)) < \phi(x)$ *for* $x \neq \bar{x}$ *and* $t > 0$.

A continuous uniformly unbounded function ϕ which satisfies *i* and *ii* is often called a *strong Liapunov function*. In this note we give a criterion for global asymptotic stability without using Liapunov functions. Of course, it is just the case of remarking that by virtue of Theorem 1, our criterion actually implies the existence of a strong Liapunov function. However, it is usually difficult to find explicitly strong Liapunov functions. Thus, our criterion may be useful in some cases. As illustrative examples, we consider two applications to the stabilization of nonlinear systems.

2 - Main result

For a subset M of E, we use the notation

$$\operatorname{diam} M = \sup_{x,\, y \in M} d(x, y)$$

where d is the distance of E. Moreover we denote by $\gamma^+(x)$ the positive trajectory issuing from x, i.e., $\gamma^+(x) = \{y \in E\colon y = \pi(t, x),\ t \geqslant 0\}$.

Theorem 2. *Let* E *be a locally compact metric space, and* $\pi\colon E \times \boldsymbol{R} \to E$ *a dynamical system. Assume that there exists a family of sets* $\{M_i\}_{i \in \boldsymbol{R}^+}$ *with the following properties*:

a1 *for all* i, M_i *is a compact neighborhood of* $\bar{x}$

a2 *for all* $\varepsilon > 0$ *there exists* $j \in \boldsymbol{R}^+$ *such that* $\operatorname{diam} M_j < \varepsilon$

a3 $M_i \subseteq M_j$ *if* $i < j$

a4 *for all* $x \in E$ *there exists* $i \in \boldsymbol{R}^+$: $x \in \partial M_i$ *and* $x \notin M_j$ *for all* $j < i$

a5 *for all* i, *for all* $x \in M_i$: $\pi(t, x) \in M_i$ *for all* $t > 0$ *and there exists* $T > 0$ *such that* $\pi(T, x) \in \operatorname{int} M_i$.

Then $\bar{x}$ *is a globally stable attractor for the system* π.

Proof. The proof is accomplished in several steps.

From *a5* it follows that M_i is a positive invariant set (see for example [5]) for all $i \in \boldsymbol{R}^+$.

Stability. First observe that $\bigcap_i M_i = \{\bar{x}\}$ for *a1* and *a2*. Now, for all $\varepsilon > 0$ there exists j such that $M_j \subset B(\bar{x}, \varepsilon)$ (for *a2*) and from *a1* there exists δ: $\bar{x} \in B(\bar{x}, \delta) \subset M_j$. Thus, for all $p \in B(\bar{x}, \delta)$, for the positive trajectory from p, $\gamma^+(p)$, we have $\gamma^+(p) \subset M_j \subset B(\bar{x}, \varepsilon)$.

$\{\bar{x}\}$ *is a global attractor*. Let $x \in E$ and j such that $x \in \partial M_j$. The positive trajectory $\gamma^+(x)$ is bounded, thus the limit set $\Gamma^+ x \neq \emptyset$. We have to prove that $\Gamma^+ x = \{\bar{x}\}$. Let $y \in \Gamma^+ x$, $y \neq \bar{x}$. Then there exists a sequence $\{t_n\} \subset \boldsymbol{R}^+$, $t_n \to +\infty$ such that $\pi(t_n, x) \to y$. For *a4* there exists i such that $y \in \partial M_i$ but $y \notin M_h$ for $h < i$.

Then we prove that $\gamma^+(x) \cap \operatorname{int} M_i = \emptyset$. Let $\pi(s, x) = x' \in \operatorname{int} M_i$. Let h be such that $x' \in \partial M_h$. According to *a5*, there exists $T > 0$ such that $\pi(T, x') = \pi(s + T, x) \in \operatorname{int} M_h$ and hence $\pi(t, x) \in M_h$ for every $t > s + T$. Since $t_n \to +\infty$, we have in particular that for some integer N_1, $\pi(t_n, x) \in M_h$ for all $n > N_1$. Now let U be a neighborhood of y such that $U \cap M_h = \emptyset$. Since $\pi(t_n, x) \to y$, for some N_2 we have $\pi(t_n, x) \in U$ for all $n > N_2$. In conclusion, for $n > \max\{N_1, N_2\}$ we must have $\pi(t_n, x) \in U \cap M_h$ and this a contradiction.

Now, consider the trajectory $\pi(t, y)$, and let T be such that $\pi(T, y) \in \operatorname{int} M_i$. For all neighborhood V of y, we have $\gamma^+(x) \cap V \neq \emptyset$. Thus, since π is continuous, $\gamma^+(x)$ must intersect each neighborhood of $\pi(T, y)$. So there exists $t > 0$ such that $\pi(t, x) \in \operatorname{int} M_i$. And this contradicts $\gamma^+(x) \cap \operatorname{int} M_i = \emptyset$.

Therefore we conclude that $y = \bar{x}$ and Theorem 2 is proved.

The following corollary is an obvious consequence of Theorem 2.

Corollary 1. *Let E be a locally compact metric space and let π be a dynamical system. Assume that there exists a family of compact sets $\{M_i\}_{i \in \boldsymbol{R}^+}$ with the following properties*:

c1 $\bigcap_{i > 0} M_i = \bar{x}$

c2 *for all* $i, j \in \boldsymbol{R}^+$ *with* $i < j$ $M_i \subset \operatorname{int} M_j$

c3 *for all* $x \in E$ *there exists* $j \in \boldsymbol{R}^+$ *such that* $x \in \partial M_j$

c4 *for all* $i \in \boldsymbol{R}^+$ *and for all* $x \in \partial M_i$, *x is an enter point for* M_i, *i.e. there exists* $T \neq 0$ *such that* $\pi(t, x) \in \operatorname{int} M_i$ *for all* $t \in]0, T]$.

Then $\bar{x}$ is a globally stable attractor for the system π.

Remark. Under the assumptions of Corollary 1, the family $\{M_i\}$ can be easily interpreted as the family of the level sets of a uniformly unbounded strong Liapunov function. Thus, Corollary 1 is also a consequence of Theorem 1.

On the contrary, under the assumptions of Theorem 2 there is not in general a strong Liapunov function, whose level sets coincide with the M_i's. Of course, strong Liapunov functions exist by virtue of Theorem 1, but none of them can be recovered from the family $\{M_i\}$. Relationship among attraction, esistence of Liapunov functions and families of nested sets is studied in [**6**], as well. However, in the present setting the construction of Theorem 4.1 of [**6**] can not be used to obtain a continuous Liapunov function.

3 - Piecewise continuous stabilization

In the control theory literature there are many examples of systems which can be stabilized at an equilibrium point by applying piecewise constant feedback laws. Consider for instance the two dimensional system

$$\dot{x} = -4y \qquad \dot{y} = 4x + 3ux . \tag{1}$$

A simple geometric argument shows that setting $u = u^* = -\operatorname{sgn} xy$, (1) gives rise to a continuous dynamical system π^* with a globally asymptotically stable equilibrium at the origin ([2]). However, apparently there is not an obvious strong Liapunov function which allows us to prove formally this statement. This can be done by applying Theorem 2. Indeed, there is a family of ellipses (for instance, the trajectories of (1) with $u = 1$) which have all the required properties with respect to the above defined dynamical system π^*.

More generally, the following statement is easily proved.

Theorem 3. *Let π_1 and π_2 be two dynamical systems on a complete metric space E, with a common equilibrium position $\bar{x}$. Let S be an unbounded open subset of E such that $\bar{x} \in \partial S$, and let $R = E \setminus S$. Assume that*

$$\pi^* = \pi_1 \quad \text{on } R \qquad \pi^* = \pi_2 \quad \text{on } S \tag{2}$$

still defines a dynamical system on E.

Let $\{M_i\}$ be a family of compact sets satisfying assumptions a1, a2, a3, a4 of Theorem 1. *Assume that for each i and each $x \in \partial M_i \cap R$, there exists $T > 0$ such that $\pi_1(T, x) \in \partial M_i \cap S$ and $\pi_1(t, x) \in \partial M_i$ for all $t < T$.*

Assume finally that each $x \in \partial M_i \cap S$ is an enter point for π_2. Then, $\bar{x}$ is a globally asymptotically stable equilibrium point for (2).

4 - Stabilization of the rigid body

In this section we are concerned with the global feedback stabilization of the angular velocity of a rigid body. We consider the particular case when there are two control torques acting on the body. The corresponding equations are (see [3] for references)

$$\dot{x}_1 = J_1 x_2 x_3 \qquad \dot{x}_2 = J_2 x_1 x_3 + u \qquad \dot{x}_3 = J_3 x_1 x_2 + v \tag{3}$$

where $$J_1 = \frac{I_2 - I_3}{I_1} \qquad J_2 = \frac{I_3 - I_1}{I_2} \qquad J_3 = \frac{I_1 - I_2}{I_3}$$

and we suppose that $I_3 > I_1 > I_2$ so $J_1 < 0$ and J_2, J_3 are positive. We recall that our problem is to find two smooth functions $u(x)$ and $v(x)$, $u(0) = v(0) = 0$, such that the origin becomes a globally stable equilibrium point for (3). Observe that the result of Aeyels and Szafranski [1] does not apply to system (3).

In [4] it is proved, using center manifold theory, that the linear feedback

$$u(x) = -x_2 \qquad v(x) = x_1 - x_3 \tag{4}$$

makes system (3) locally asymptotically stable at the origin.

In [3] it is proved that the same function actually is a global stabilizer, provided that (3) is symmetric, that is $J_1 = -J_2$ and $J_3 = 0$. We prove here that this last restriction can be removed. In other words, (4) is a global stabilizer for (3), for each choice of $J_1 < 0$ and $J_2, J_3 > 0$.

To this purpose, consider the family of compact sets:

$$M_c = \{(x_1, x_2, x_3) \,|\, V(x) = J_2 x_1^2 - J_1 x_2^2 \leq c^2 \text{ and } B(x) = x_3^2 \leq (J_3 K_c^2 + K_c + K_c^2)^2\}$$

where $c > 0$ and $K_c = \max\left(\frac{c}{\sqrt{-J_1}}, \frac{c}{\sqrt{J_2}}\right)$.

M_c's are compact sets that satisfies hypothesis *c1*, *c2*, *c3* of Corollary 1. We have to prove that the points on the surface are enter points.

On the cylinder we have $\dot{V}(x) = 2J_1 x_2^2 \leq 0$, and if there exists a trajectory $x(t) = (x_1(t), x_2(t), x_3(t))$ on the surface of the cylinder $V(x) = c^2$ we have: $x_2(t) = 0$, $x_1(t) = \frac{c}{\sqrt{J_2}}$, $x_3(t) = 0$ and $\left(\frac{c}{\sqrt{J_2}}, 0, 0\right)$ is an equilibrium point.

By direct substitution in (3) (4), we see that this is impossible.

On the bases we have $B(x) = (J_3 K_c^2 + K_c + K_c^2)^2$ and

$$\dot{B}(x) = 2x_3 \dot{x}_3 = 2x_3 (J_3 x_1 x_2 + x_1 - x_3) \leqslant -2K_c^2 (J_3 K_c^2 + K_c + K_c^2) < 0 .$$

Thus on the bases the points are enter points.

From Corollary 1 the origin is a globally asymptotically stable equilibrium point for system (3).

References

[1] D. AEYELS and M. SZAFRANSKI, *Comments on the stabilizability of the angular velocity of a rigid body*, Systems Control Lett. **10** (1988), 35-39.

[2] A. ANDREINI and A. BACCIOTTI, *Stabilization by piecewise constant feedback*, Proceedings of ECC '91, Hermes, Paris 1991.

[3] V. ANDRIANO, *Global feedback stabilization of the angular velocity of a symmetric rigid body*, Systems Control Lett. **20** (1993), 361-364.

[4] A. BACCIOTTI, *Local stabilizability of nonlinear control systems*, Advances in Math. for Appl. Sci. 8, World Scientific, Singapore 1991.

[5] N. P. BHATIA and G. P. SZEGO, *Stability theory of dynamical systems*, Springer, Berlin 1970.

[6] P. SEIBERT, *Attractors in general systems*, Differential Equations, S. AHMAD, M. KEENER and A. C. LAZER Editors, Academic Press, New York 1980.

Sunto

Questa nota fornisce un criterio di stabilità asintotica globale di punti di equilibrio di un sistema dinamico. Il risultato viene applicato ad alcuni problemi di stabilizzazione.

* * *