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A variation on Schreier's space (**)

1 - Introduction and preliminaries

The Schreier's space [11] is one of the earliest examples of *pathological* Banach spaces, so frequent nowadays in the literature. It is the first example of a Banach space not having the weak Banach-Saks property, i.e., it contains a weakly null sequence without Banach-Saks subsequences, which is an abbreviated form of calling the subsequences having norm convergent arithmetic means. The importance of Schreier's space, however, has not decreased. On one hand, the notion of *admissible* set is at the basis of many other notorius examples: Barnstein's spaces and Tsirelson's space among others. On the other hand, new features of this space are uncovered every now and then [3], [8], [9].

In this note we construct a Banach space, inspired by Schreier's space, which shares some of its features, improves others, and is a new counterexample, simultaneously, to several questions previously solved with *ad hoc* constructions (see [2], [3], [6], [11]).

The properties shared by the space L are: its canonical basis is a weakly null sequence without Banach-Saks subsequences and the space has not the Dunford-Pettis property, although it is c_0-saturated as well as its quotients. Moreover, the questions addressed are, in cronological order: a space with the surjective Dunford-Pettis property [6] without the Dunford-Pettis property (the previous example was an *ad hoc* space constructed by Leung); that the Dunford-Pettis property is not a three space property (proved in [3] with a construction

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similar to the present one, but using Schreier's space) and a 2-Banach-Saks sequence without subsequences contained inside the range of vector measure (the previous example in [2] used a Lorentz sequence space constructed by Rakov [10]).

Let $A \subset N$ be a subset. The *range* of A is defined as

$$r(A) = \inf\{b - a \mid b, a \in A, a < b\}.$$

The set A is said to be *lunatic* if $1 + r(A) \geq \operatorname{card} A$. The space L is defined as *the completion of the space of finite sequences with respect to the norm*

$$\|x\|_L = \sup_{[A \text{ lunatic}]} \sum_{j \in A} |x_j|.$$

It is routine to verify that the canonical vectors (e_i) form an unconditional basis for L. Since L is a subspace of a certain $C(K)$ space with K countable, one sees that *a bounded sequence* (x^n) *of* L *is weakly convergent to* x *if and only if, for every* j, *the sequence of* j-th *coordinates* $(x_j^n)_{n \in N}$ *converges to* x_j.

2 - A weakly null sequence without Banach-Saks subsequences (Schreier's example [11])

Recall that a sequence having norm convergent arithmetic means is sometimes called a *Banach-Saks sequence.*

Lemma 1. *Any subsequence of* (e_n) *admits a subsequence which is not Banach-Saks.*

Proof. Let $E \in P_\infty(N)$. The subset $F \subset E$ inductively defined as: $n_1 \in E$ and, if $n_j \in E$ then $n_{j+1} \in E$ such that $n_{j+1} > n_j + j + 1$, satisfies the following: if A denotes the subset of F formed by the first N elements, then $A = U \cup V$, $\max U < \min V$, and V is lunatic. Therefore

$$\|\sum_{n \in F} e_n\| \geq \|\sum_{n \in A} e_n\| \geq \|\sum_{n \in V} e_n\| \geq \frac{N}{2}.$$

3 - The Dunford-Pettis property is not a three space property [3]

A property P is said to be a *three-space property* if, whenever a closed subspace Y of a Banach space X and the corresponding quotient X/Y have P, then X also has P. In [3], it was shown that the Dunford-Pettis and the hereditary Dunford-Pettis properties are not three-space. Using the same construction, here it is another example.

A Banach space X is said to have the *Dunford-Pettis property* (DPP) if any weakly compact operator $T: X \to Y$ transforms weakly compact sets of X into relatively compact sets of Y. Equivalently, given weakly null sequences (x_n) and (x_n^*) in X and X^*, respectively, $\lim \langle x_n^*, x_n \rangle = 0$. L_1 and $C(K)$ spaces are examples of spaces with the DDP. A Banach space X is said to have the *hereditary* Dunford-Pettis property ($\mathrm{DPP_h}$) if any closed subspace of X has the DPP. l_1 and c_0 are example of spaces having the $\mathrm{DPP_h}$. A profound characterization, due to Elton, of this property is: any normalized weakly null sequence admits a subsequence equivalent to the canonical basis of c_0. Using [3], it is enough to prove that any weakly null sequence (x_n) contains a subsequence (x_m) such that, for some $K > 0$,

$$\| \sum^N x_m \| \leqslant K .$$

Since L is a subspace of a $C(K)$ with K countable, every closed subspace of L contains a copy of c_0. Despite this fact, the sequence (e_n) does not admit subsequences equivalent to the canonical basis of c_0, and thus L has not the hereditary Dunford-Pettis property.

Now we prove

Proposition 1. *The space L does not have the Dunford-Pettis property.*

Proof. The unit vector sequence is weakly null both in L and in L^*. This second assertion immediately follows from the estimate

$$\| \sum_{k=1}^N e_{i_k}^* \|_{L^*} \leqslant \sqrt{N}$$

which is true because any set of cardinality N^2 can be decomposed into N disjoint lunatic sets.

As a second step, we prove that *any normalized disjoint sequence of blocks (u_k) such that $\|u_k\|_\infty \to 0$ contains a subsequence equivalent to the canonical basis of c_0.*

Let B_n be the support of u_n. Since the blocks u_n are normalized and, at the same time, their sup norm goes to 0, the length of B_n necessarily increases to infinity. Fix now u_1 and u_2. A lunatic set having nonempty intersection with both finity. Fix now u_1 and u_2. A lunatic set having nonempty intersection with both B_1 and B_2 can have, at most, $d_{1,2} = \max B_2 - \min B_1$ elements. Choose then u_j such that $\operatorname{card} B_j > (d_{1,2})^2$ and

$$\|u_j\|_\infty < \min\{|u_1(k)|, |u_2(k)| : k \in N \text{ (minimum over the non-vanishing terms)}\}.$$

Without loss of generality, it can be assumed that this block is u_3. Iterating this process with $\operatorname{card} B_j > (\max\{d_{1,2}, d_{2,3}, d_{1,3}\})^2$, etc., and denoting by (u_m) the final sequence, one has

$$\|\sum^N u_m\| \leq 3 .$$

Since if A is a lunatic set which cuts the support of two different blocks, say, u_i, u_j, then A cannot have more than $d_{i,j}$ elements. A lunatic set with $d_{i,j}$ elements can be placed into B_{j+1}, and since all elements of u_{j+1} are greater than the elements of any other u_k, $k > j+1$, this gives the maximum value that can be reached.

The third step is to define an operator $T\colon l_1 \bigoplus_1 L \to c_0$ by means of the formula $T(y, x) = q(y) + i(x)$ where $q\colon l_1 \to c_0$ is a quotient map, and i denotes canonical inclusion. Obviously T is a quotient map. We only need to verify that $\operatorname{Ker} T$ has $\mathrm{DPP_h}$. To this end, let (y^n, x^n) be a weakly null sequence in $\operatorname{Ker} T$. Since $T(y^n, x^n) = 0$ and (y^n) is norm null, one sees that also $\|x^n\|_\infty \to 0$. If $\|x^n\|_L \to 0$, then the proof ends. If not, the calculations of the second step apply.

4 - Sequences which do not lie inside the range of a vector measure

This problem was treated in [2]. In [1], Ananatharaman and Diestel proved that every weakly-2-summable sequence, i.e., every sequence (x_n) satisfying an estimate

$$\|\sum^N \alpha_n x_n\| \leq C\|(\alpha_n)\|_{l_2}$$

is contained inside the range of a vector valued countably additive vector measure, and the question was raised if weakly null sequences inside the range of a vector measure must contain weakly-2-summable subsequences. A counterexample to this question was given in [2].

A sequence (x_n) is said to be *2-Banach-Saks* (following [5]) if it satisfies an estimate

$$\| \sum^{N} x_n \| \leqslant C\sqrt{N}\,.$$

It is clear that every weakly-2-summable sequence is 2-Banach-Saks (see [4] for a thorough treatment of the relationships between weakly-2-summable and 2-Banach-Saks sequences). On the other hand, 2-Banach-Saks sequences seem to be very close of weakly-2-summable sequences; for this reason, the following improvement of [2] is of some interest.

Proposition 2. *The canonical basis of the space L is a 2-Banach-Saks sequence such that no subsequence of it is contained inside the range of any countably additive vector measure.*

Proof. To prove the first assertion simply observe that a lunatic set having k elements must contain some element greater than k^2. Therefore

$$\| \sum_{k=1}^{k=N} e_k \|_L \leqslant \sqrt{N}\,.$$

To verify the second, note that the lunatic sequence forms an unconditional basis and does not contain weakly-2-summable subsequences (since every subsequence contains a subsequence which is not Banach-Saks). The result follows now from [1], where it is proved that an unconditional basic sequence inside the range of a vector measure must be weakly-2-summable.

5 - A space with the surjective Dunford-Pettis property without the Dunford-Pettis property

The surjective Dunford-Pettis property was introduced by Leung [6] as follows: X has sDDP if every weakly compact operator $T\colon X \to Y$ is completely continuous. We have already shown that L does not have the classical Dunford-Pettis property. It does have the sDDP, since

Lemma 2. *The space L does not admit infinite dimensional reflexive quotients.*

The proof of Lemma 2 follows closely [8], where only the estimate $m \geqslant 4\delta$ should be replaced by $\sqrt{m} \geqslant 4\delta$.

Remark. Observe that being hereditarily c_0 is not enough to guarantee the absence of reflexive quotients: Leung [7] has constructed an hereditarily c_0 Banach space having l_2 as a quotient.

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Sommario

In questo articolo, è costruito uno spazio di Banach L che ha simultaneamente le seguenti proprietà: la sua base canonica è una successione debolmente nulla 2-Banach-Saks che non ha né sucessione estratta Banach-Saks né successione estratta contenuta nel rango di una misura; lo spazio L ha la proprietà Dunford-Pettis suriettiva ma non la classica proprietà Dunford-Pettis; tutti i quozienti dello spazio sono c_0-saturati. Lo spazio L è una variazione dello spazio di Schreier.

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