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A characterization of locally 3-symmetric spaces (**)

1 - Introduction

Locally 3-symmetric spaces are nice generalizations of locally symmetric spaces and have many remarkable properties. We refer to [2], [3], [6] for the basic results and for a lot of non-symmetric examples. We mention two important aspects of their geometry. They are *locally homogeneous* spaces and further, they also have an invariant quasi-Kähler structure. From these two properties it follows that such manifolds are equipped with a very special *homogeneous structure* [10]. For that reason we focus in this note on almost hermitian homogeneous structures and derive a new characterization of locally 3-symmetric spaces. It happens that this criterion is of a more practical use than the theoretical definitions, as has already been shown in [1]. Our main result generalizes at the same time Sato's criterion [8] for nearly Kähler locally 3-symmetric spaces, that is, locally 3-symmetric spaces with a naturally reductive canonical homogeneous structure.

2 - Locally 3-symmetric spaces

We start with some definitions and basic properties. Let (M, g) be a smooth, finite-dimensional, connected riemannian manifold with Levi Civita connection ∇ and riemannian curvature tensor R.

A family of *local cubic diffeomorphisms* is a differentiable function $p \mapsto s_p$

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which assigns to each $p \in M$ a diffeomorphism s_p on a neighborhood $\mathcal{U}_p$ of p such that $s_p^3 =$ identity and p is the unique fixed point of s_p.

Definition. A *locally 3-symmetric space* is a riemannian manifold (M, g) endowed with a family of local cubic diffeomorphisms $p \mapsto s_p$, $p \in M$, such that each s_p is a holomorphic isometry with respect to the canonical almost complex structure J determined by

$$S_p = (\mathrm{d}s_p)_p = -\frac{1}{2} I_p + \frac{\sqrt{3}}{2} J_p \tag{1}$$

where I_p is the identity endomorphism on the tangent space T_pM at p.

These manifolds are special elements of the broader class of *riemannian locally s-regular manifolds*. We refer to [2] for more information and note that they may be defined by using tensor conditions involving S and its covariant derivatives.

Locally 3-symmetric spaces are *quasi-Kähler manifolds* with respect to the almost complex structure J. More precisely, (M, g, J) is an almost hermitian manifold such that

$$(\nabla_X J) Y + (\nabla_{JX} J) JY = 0 \tag{2}$$

for all vector fields X, Y on M [3], [6]. Moreover, (M, g, J) is *locally homogeneous* which means that for all $p, q \in M$ there exists a holomorphic isometry f defined on a neighborhood of p and with $f(p) = q$. Such spaces may be characterized by using the following criterion of Sekigawa.

Proposition 1 [9]. *A connected almost hermitian manifold (M, g, J) is a locally homogeneous almost hermitian manifold if and only if there exists a $(1, 2)$-tensor field T such that*

$$\bar{\nabla} g = \bar{\nabla} R = \bar{\nabla} T = \bar{\nabla} J = 0 \tag{3}$$

where $\bar{\nabla} = \nabla - T$.

We note that the first three conditions in (3) express that T is a *homogeneous structure* on (M, g) (see [10] for more details). If also $\bar{\nabla} J = 0$, then T is called an *almost hermitian homogeneous structure*. Further, if in addition

$$T_{JX} Y = T_X JY = -JT_X Y \tag{4}$$

then T is said to be a *hermitian-homogeneous structure* on (M, g, J) and this space is then called a *hermitian-homogeneous space* [5].

These notions have been used in [7] to prove the following key result which generalizes that for the special case of nearly Kähler spaces in [8].

Proposition 2 [7]. *A locally 3-symmetric space is hermitian-homogeneous with respect to its canonical almost hermitian structure. Conversely, any hermitian-homogeneous almost hermitian manifold (M, g, J) is a locally 3-symmetric space with J as canonical almost complex structure.*

In the course of the proof of this proposition it is shown that the tensor field $\tilde{T}$ defined by

$$\tilde{T}_X Y = \frac{1}{2} J(\nabla_X J) Y \tag{5}$$

for all vector fields X, Y, is a hermitian-homogeneous structure on any locally 3-symmetric space.

Now we come to our main result. Therefore, let (M, g, J) be an almost hermitian manifold. Define a tensor field S of type (1,1) by

$$S = -\frac{1}{2} I + \frac{\sqrt{3}}{2} J . \tag{6}$$

Then $I - S$ is non-singular, g is S-invariant (that is, $g(SX, SY) = g(X, Y)$) and $S^3 = I$. Next, put

$$T_X Y = (\nabla_{(I-S)^{-1}X} S)(S^{-1} Y) \tag{7}$$

for all vector fields X, Y. Then we have

Lemma 1. *The tensor field T defined by* (6) *and* (7) *has the following expression*

$$T_X Y = \frac{1}{2} J(\nabla_X J) Y + \frac{1}{4} JS\{(\nabla_X J) Y + (\nabla_{JX} J) JY\} . \tag{8}$$

Proof. (8) follows at once from (7) by using (6) and

$$S^{-1} = -\frac{1}{2} I - \frac{\sqrt{3}}{2} J \qquad (I - S)^{-1} = \frac{1}{2} I + \frac{\sqrt{3}}{6} J .$$

Then we have

Theorem 1. *An almost hermitian manifold* (M, g, J) *admits the tensor field* T *given by* (7) *as an almost hermitian homogeneous structure if and only if* (M, g, J) *is a locally* 3-*symmetric space with* J *as canonical almost complex structure. In this case* $T = \tilde{T}$.

Proof. First, let (M, g, J) be a locally 3-symmetric space with J as canonical almost complex structure. Then the result follows directly from the information given above. Conversely, let T be given by (7). Then Lemma 1 and

$$(\nabla_X J) JY = -J(\nabla_X J) Y$$

yield, with $\bar{\nabla} = \nabla - T$,

$$(\bar{\nabla}_X J) Y = -\frac{1}{2} S\{(\nabla_X J) Y + (\nabla_{JX} J) JY\}.$$

Hence, $\bar{\nabla} J = 0$ if and only if (M, g, J) is a quasi-Kähler manifold. In this case $T = \tilde{T}$ and moreover, T satisfies (4). So, if T is an almost hermitian homogeneous structure, the result follows at once from Proposition 2.

Remarks.

A. As mentioned in **1**, Theorem 1 generalizes Sato's result ([8], p. 141), where it was assumed that (M, g, J) is a *nearly Kähler manifold*, that is

$$(\nabla_X J) Y + (\nabla_Y J) X = 0$$

for all X, Y. Such a space is necessarily quasi-kählerian. In Sato's case, T is a naturally reductive structure [**10**], or equivalently, $\bar{\nabla} = \nabla - T$ and ∇ have the same geodesics (they are projectively related).

B. For a quasi-Kähler manifold, (5) determines the connection $\bar{\nabla} = \nabla - T$ which is precisely the *characteristic connection* of the almost hermitian quasi-Kähler manifold (M, g, J) [4].

We finish this note with the following generalization of [8], Proposition 2(i).

Proposition 3. *Let* (M, g, J) *be an almost hermitian space with almost hermitian homogeneous structure* T. *Then* $T = \tilde{T}$ *if and only if*

$$T_X J = -J T_X \tag{9}$$

for all vector fields X.

Proof. Put $T = \tilde{T} + Q$. Then a straightforward computation yields

$$(\bar{\nabla}_X J)Y = -Q_X JY + JQ_X Y = 0$$

which leads to

$$T_X JY + JT_X Y = 2JQ_X Y.$$

Hence (9) holds if and only if $Q = 0$.

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Sommario

Gli spazi localmente 3-*simmetrici sono caratterizzati mediante le strutture quasi hermitiane omogenee.*

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