Riv. Mat. Univ. Parma (5) 3 (1994), 67-75

EDOARDO VESENTINI (*)

Geodesics and Moebius transformations (**)

1 - Introduction

If X and X' are differentiable manifolds endowed with affine symmetric connections Γ and Γ' in their tangent bundles, an *affine collineation* of X onto X' is, by definition, a diffeomorphism F of X onto X' mapping every geodesic γ of Γ onto a geodesic γ' of Γ', and inducing an affine correspondence between affine parameters on γ and on γ'. If Ω and Ω' are the connection forms of Γ and Γ', F is an affine collineation if, and only if, $F^* \Omega = \Omega'$, where F^* is the linear mapping on differential forms induced by F. If, more in general, F maps the geodesics of Γ onto the geodesics of Γ' without any further restriction on its action on affine parameters, F is called a *projective collineation*, and Γ and Γ' are said to be projectively related.

Projective collineations were first investigated in 1921 by H. Weyl who, in a letter to F. Klein [17], noted that what he called a projective property (projective Beschaffenheit) of X—a property which he characterized in terms of parallel transport of directions—is preserved when the affine connection Γ is modified preserving the geodesic curves. H. Weyl's research seems to have been initially motivated by the theory of general relativity (1). From a historical point of view, it should be noted, however, that—as was pointed out by E. Bortolot-

---

(*) Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italia.
(**) Received October 6, 1994. AMS classification 32 M 15.
(1) *In der Relativitätstheorie haben projective und konforme Beschaffenheit eine unmittelbar anschauliche Bedeutung. Die erstere, die Beharrungstendenz der Weltrichtungen eines sich bewegenden materiellen Teilchens, welche ihm, wenn es in bestimmter Weltsrichtung losgelassen ist, eine bestimmte natürliche Bewegeng aufnötigt, its jene Einheit von Trägheit und Gravitation, welche Einstein an Stelle beider setzte, für die es aber bislang an einem suggestiven Namen mangelt* [17]. In the same article, after having shown that a pseudo-riemannian metric is completely determined by its projective and conformal geometries, Weyl notes that *Es geht aus diesem Satz hervor, dass allein durch die Beobachtung der natürlichen Bewegung materieller Teilchen und der Wirkuns-, insbesondere der Lichtsausbreitung die Weltmetrik festgelegt werden kann; Maßstäbe und Uhren sind nicht dazu erförderlich*; see also [18] and [19].

ti [4]—the first example of a projective collineation goes bach to 1865, when E. Beltrami [1] (see also [3], p. 310-313, 642-645) proved that the surfaces with constant curvature are the only ones which are projectively equivalent (at least locally) to the euclidean plane. Four years later, U. Dini [5] (see also [3] p. 313-315) characterized projectively equivalent riemannian surfaces, showing that (apart from trivial cases) they are all Liouville surfaces.

Assuming for the sake of simplicity that $X = X'$, according to H. Weyl [17] (see also [4] [6]) the identity map is a projective collineation if, and only if, there is a linear differential form φ on X such that, in terms of local coordinates $x_1, \ldots, x_n$ on X (where $n = \dim_{\boldsymbol{R}} X$), φ is expressed by $\varphi = \sum \varphi_j \, \mathrm{d}x_j$, and the coefficients Γ^i_{jk} and Γ'^i_{jk} are related by

$$\Gamma'^i_{jk} = \Gamma^i_{jk} + \delta^i_j \varphi_k + \delta^i_k \varphi_j$$

for all $i, j, k = 1, \ldots, n$. Affine parameters t and t' for Γ and Γ' are then related by the equation

$$t' = a \int e^{2\int \sum \varphi_j \mathrm{d}x_j} \, \mathrm{d}t + b$$

where a and b are real constants and the integrals are taken along the geodesic.

The fact that, if the identity map is an affine collineation, the affine parameters are preserved, up to an affine map, raises the question whether it is possible to define an intrinsic parameter on the geodesic γ which is essentially preserved under the action of a projective collineation. An answer to this question was provided by L. Berwald in [2] (see also [4] for further bibliographical references). If γ is a geodesic for the connection Γ, starting from an affine parameter on γ L. Berwald constructs a parameter p – which he calls a *normal projective parameter* – such that, if the same construction is performed on γ for a connection Γ', projectively related to Γ, then the corresponding normal projective parameter is related to p by a Moebius transformation on $\boldsymbol{R}$ (2).

In this lecture we shall see how the theory originating from H. Weyl's paper of 1921 sets the stage for somewhat similar developments in which *the differentiable manifold X and the geodesics γ are replaced by a domain B in a complex Banach space and by the complex geodesics for an invariant Finsler metric on B*. It is not unexpected that the rigidity of the complex geodesics (when they exist) leads to stronger conclusions.

---

(2) In Berwald's paper the Moebius transformation arise from differential equations involving the Schwarzian derivative. A similar approach was developed for a Finsler metric positive constant curvature on a surface, by P. Funk in [8].

The results described in this preliminary report are far from being exhaustive and concern only some examples. Detailed proofs are to be found in [14], [15], [16].

2 - Complex geodesics

In the following, B will be a bounded domain in a complex Banach space $\mathcal{E}$, and d will be a distance on B which is contracted by all holomorphic maps of B into B, and defines the relative topology of B in $\mathcal{E}$.

If B' is a domain in a complex Banach space $\mathcal{E}'$, $\mathrm{Hol}(B, B')$ will stand for the set of all holomorphic maps of B into B'. Throughout the following, d will always be the *Carathéodory distance*, which is defined as follows [7]. Let Δ be the open unit disc of $\boldsymbol{C}$, and let ω be the Poincaré distance on Δ. Then, for $x, y \in \Delta$

$$d(x, y) = \sup \{\omega(f(x), f(y)) \mid f \in \mathrm{Hol}(B, \Delta)\}.$$

A similar construction yields the *Carathéodory differential metric* of B. If, for $\zeta \in \Delta$ and $\tau \in \boldsymbol{C}$, $\langle \tau \rangle_\zeta$ is the lenght of τ with respect to the Poincaré differential metric of Δ at the point ζ, then the lenght $\kappa(x; v)$ of a vector $v \in \mathcal{E}$ with respect to the Carathéodory differential metric of B at the point $x \in B$, is given by

$$\kappa(x; v) = \sup \{\langle \mathrm{d}f(x)\, v \rangle_{f(x)} \mid f \in \mathrm{Hol}(B, \Delta)\}.$$

A *complex geodesic* [12] (for the Carathéodory distance d) is a function $\phi \in \mathrm{Hol}(\Delta, B)$ such that for all ζ_1 and ζ_2 in Δ,

$$\omega(\zeta_1, \zeta_2) = d(\phi(\zeta_1), \phi(\zeta_2)). \tag{1}$$

The immage $\phi(\Delta) \subset B$ will be called the *support* of the complex geodesic ϕ. Since the Poincaré distance is complete, and ϕ is an isometry, $\phi(\Delta)$ is closed in B. If ψ is another complex geodesic having the same support as ϕ, there is a Moebius transformation M of Δ onto Δ such that $\psi = \phi \circ M$; and viceversa [13].

Proposition 1. *Let $\phi \in \mathrm{Hol}(\Delta, B)$. If there are two distinct points ζ_1, ζ_2 in Δ for which* (1) *holds, or, if there is some $\zeta \in \Delta$ such that*

$$\kappa(\phi(\zeta); \phi'(\zeta)) = \langle 1 \rangle_\zeta$$

then ϕ is a complex geodesic [13].

Complex geodesics may or may not exist. There are no complex geodesics, for instance, when B is a bounded non-simply connected domain in $\boldsymbol{C}$.

If $B = B(\mathcal{H})$ is the open unit ball of a complex Hilbert space $\mathcal{H}$, and if ϕ is a complex geodesic in $B(\mathcal{H})$, $\phi(\Delta)$ is the intersection of $B(\mathcal{H})$ with a complex affine line L of $\mathcal{H}$. Viceversa, if L is any complex affine line in $\mathcal{H}$, such that $L \cap B(\mathcal{H}) \neq \emptyset$, there is an affine map $\boldsymbol{C} \to \mathcal{H}$ whose image is L, and whose restriction to Δ is a complex geodesic in $B(\mathcal{H})$. As a consequence, for any two distinct points x, y in $B(\mathcal{H})$, there is a complex geodesic whose support contains x and y; furthermore, all these complex geodesics have the same support.

Let T be a compact Hausdorff space and let $B = B(T)$ be the open unit ball of the Banach space $C(T)$ of all complex-valued, continuous functions on T, with the uniform norm. The complex geodesics of $B(T)$ are described by the following proposition [**9**].

Proposition 2. *A function $\phi \in \mathrm{Hol}(\Delta, B(T))$ is a complex geodesic if, and only if, there exists $t \in T$ such that the function $\Delta \to \Delta$ defined by $\zeta \mapsto \phi(\zeta)(t)$ is a Moebius transformation of Δ onto Δ.*

In this case (if T is not reduced to one point) the structure of the set of all complex geodesics is quite different from that of $B(\mathcal{H})$. Although also in this case any two distinct points x and y of $B(T)$ belong to the support of some complex geodesic, the support is not necessarily unique. Uniqueness can be characterized in the following way, in terms of the set $E(T)$ of all complex extreme points of the closure $\overline{B(T)}$ of $B(T)$. This set consists of all $u \in C(T)$ such that $|u(t)| = 1$ for all $t \in T$, and turns out to coincide with the set of all real extreme points of $\overline{B(T)}$.

First of all, due to the fact that the group $\mathrm{Aut}(B(T))$ of all holomorphic automorphisms of $B(T)$ acts transitively, and that, for any $F \in \mathrm{Aut}(B(T))$, $F \circ \phi$ is a complex geodesic whenever ϕ is a complex geodesic, there is no restriction in assuming that one, say y, of the two points is the center 0 of $B(T)$. Then, the support containing 0 and $x \neq 0$ is unique if, and only if, $\frac{1}{\|x\|} x \in E(T)$, i.e., if and only if $|x(t)|$ is independent of $t \in T$. In that case the complex geodesic is expressed – up to a holomorphic automorphism of Δ – by the holomorphic map $\zeta \mapsto \frac{\zeta}{\|x\|} x$ of Δ into $B(T)$.

We shall denote by G the family of all complex geodesics $\zeta \mapsto \zeta u$ for $u \in E(T)$.

3 - The unit ball of a Hilbert space

Complex geodesics have found several applications in complex analysis, especially in exploring the structure of the sets of fixed points of holomorphic maps (see [12] also for a comprehensive report and for bibliographical references). In this lecture we shall see how the behavior along complex geodesics (in particular, the condition that supports are preserved) characterizes special classes of holomorphic maps. The research is still in a preliminary stage, and the results established so far concern only special types of homogeneous bounded domains: namely $B(\mathcal{H})$ and $B(T)$.

Let $F \in \mathrm{Hol}(B(\mathcal{H}), B(\mathcal{H}))$ and suppose that *F maps the support of any complex geodesic into the support of a complex geodesic.* Since $\mathrm{Aut}(B(\mathcal{H}))$ acts transitively on $B(\mathcal{H})$, there is no restriction in assuming that $F(0) = 0$.

For any $x \in B(\mathcal{H})$, with $x \neq 0$, let K be the intersection of $B(\mathcal{H})$ with the complex line through x. The complex geodesic $\zeta \mapsto \frac{\zeta}{\|x\|} x$ – whose support is the complex disc K – is (up to the action of an element of $\mathrm{Aut}(\Delta)$) the unique complex geodesic in $B(\mathcal{H})$ whose support contains 0 and x. For the same reason, $F(K)$ is contained in the intersection of $B(\mathcal{H})$ with $\mathrm{d}F(0)(\boldsymbol{C}x)$. This fact implies that the power series expansion of $F(x)$ is given by

$$F(x) = (1 + p_1(x) + p_2(x) + \dots)\, \mathrm{d}F(0)(x),$$

where $p_n\colon \mathcal{H} \to \boldsymbol{C}$ is a continuous homogeneous polynomial of degree $n = 1, 2, \dots$. If, for any $x \in B(\mathcal{H})$, with $x \neq 0$, the restriction of F to K is a scalar Moebius transformation, then $1 + p_1(x)\zeta + p_2(x)\zeta^2 + \dots$ is the power series expansion of the Moebius transformation $\zeta \mapsto \frac{\zeta}{1 - p_1(x)\zeta}$, and F itself is the (vector valued) Moebius transformation defined on $x \in B(\mathcal{H})$ by

$$F(x) = \frac{1}{1 - p_1(x)}\, \mathrm{d}F(0)(x).$$

Since conversely any Moebius transformation maps complex affine lines into complex affine lines, the following theorem holds [14].

Theorem 1. *The map $F \in \mathrm{Hol}(B(\mathcal{H}), B(\mathcal{H}))$ is a Moebius transformation if, and only if, it maps the support K of any complex geodesic into the support of a complex geodesic, and its restrinction to K defines a scalar Moebius transformation.*

If $F(0) = 0$ and if $F(K)$ is the intersection of $B(\mathcal{H})$ with a complex line, the Moebius transformation of K is (conjugated to) a rotation $\zeta \mapsto e^{i\theta}\zeta$ for some $\theta \in \boldsymbol{R}$. Hence $\|F(x)\| = \|x\|$ for all $x \in B(\mathcal{H})$, and therefore F is the restriction to $B(\mathcal{H})$ of a linear isometry $\mathcal{H} \to \mathcal{H}$ [7]. This kind of argument yields.

Theorem 2. *Let $F \in \mathrm{Hol}(B(\mathcal{H}), B(\mathcal{H}))$. If there is a point $x \in B(\mathcal{H})$ such that, for any complex geodesic whose support K contains x, $F(K)$ is the support of a complex geodesic, and if the restriction of F to K is conjugated to a holomorphic automorphism of Δ, then F is an isometry for the Carathéodory distance, and viceversa.*

4 - The unit ball of the space of continuous functions

In the case of the open unit ball $B = B(T)$, the set $E(T)$ plays a crucial role in the description of the semigroup $\mathrm{Iso}(B(T))$ of all elements of $\mathrm{Hol}(B(T), B(T))$ which are isometries for the Carathéodory distance. If A is a continuous linear operator in $C(T)$, for every $t \in T$ there exists a unique regular complex Borel μ_t on T such that

$$Ax(t) = (x, \mu_t) = \int x \, \mathrm{d}\mu_t \tag{2}$$

for all $x \in C(T)$. If

$$AE(T) \subset E(T), \tag{3}$$

then $|(u, \mu_t)| = 1$ for all $u \in E(T)$, and, as a consequence, $\|A\| = \|\mu_t\| = 1$. One shows [15] that μ_t is concentrated on one point. Hence there is $\tau(t) \in T$ and a complex constant $\alpha(t)$, with $|\alpha(t)| = 1$, such that

$$(x, \mu_t) = \alpha(t)\, x(\tau(t)) \tag{4}$$

for every $x \in C(T)$.

Example. Here is a proof of this fact in the case in which T is the unit circle $\boldsymbol{T} = \{e^{2\pi i t} \,|\, t \in [0, 1]\}$. If $\hat{\mu}_t(n)$ denotes the n-th Fourier coefficient of μ_t, then $|\hat{\mu}_t(n)| = 1$ for all $t \in \boldsymbol{T}$ and all $n \in \boldsymbol{Z}$. By Wiener's theorem [11]

$$\sum_{s \in [0, 1]} |\mu_t(\{s\})|^2 = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n = -N}^{N} |\hat{\mu}_t(n)|^2 = 1 .$$

Hence there is a sequence $\{s_j\}$ in $[0, 1]$ such that

$$\sum_j |\mu_t(\{s_j\})|^2 = 1 .$$

If the cardinality of the set of points s_j on which $\mu_t > 1$ were > 1, there would be some integer $n > 1$ such that

$$\sum_j |\mu_t(\{s_j\})| > 1 .$$

But, since $\|\mu_t\| = 1$, then

$$1 = |\mu_t|(T) \geqslant |\mu_t(T \setminus \bigcup_{j=1}^{n} s_j)| + \sum_{j=1}^{n} |\mu_t(\{s_j\})| > 1 .$$

Going back to the general case, it is easily seen that the functions α: $T \to \partial\Delta$ and τ: $T \to T$ are continuous. Then (2) and (4) yield [**15**].

Theorem 3. *If the continuous linear operator A in $C(T)$ satisfies* (3), *there is a continuous map τ: $T \to T$ and a function $\alpha \in E(T)$ such that*

$$Ax(t) = \alpha(t)\, x(\tau(t))$$

for all $t \in T$ and all $x \in C(T)$. Moreover, A is a linear isometry of $C(T)$ if, and only if, A is injective, and that happens, if and only if τ is surjective.

The isometry A is surjective if and only if τ is bijective, i.e. τ is a homeomorphism of T onto T. Hence Theorem 3 extends a classical result by Banach and Stone (see [**15**] also for bibliographical references) to non-surjective linear isometries of $C(T)$.

The inclusion (3) is a sufficient but not necessary condition for A to be contained in $\mathrm{Iso}(B(T))$, as examples show. On the other hand, (4) establishes a link between linear isometries of $C(T)$ and linear operators contained in $\mathrm{Iso}(B(T))$: Theorem 3 implies [**15**] that, if the continuous linear operator A satisfies (3) and if τ is surjective, then $A_{|B(T)} \in \mathrm{Iso}(B(T))$.

5 - Morbius transformations

Let $F \in \mathrm{Hol}(B(T), B(T))$. If $F(0) = 0$ and if, for every $u \in E(T)$ and some $\zeta \in \Delta \setminus \{0\}$,

$$\frac{1}{\zeta} F(\zeta u) \in E(T) \tag{5}$$

or if

$$\lim_{\zeta\to 0}\frac{1}{\zeta}F(\zeta u)\in E(T) \tag{6}$$

then the strong maximum principle, coupled with L. A. Harris' Schwarz lemma [**10**], implies that F is the restriction of $dF(0)$ to $B(T)$. Hence (5) coincides with (3) where $A = dF(0)$, and Theorem 3 yields the following result which extends to a class of elements of $\mathrm{Iso}(B(T))$ H. Cartan's linearization theorem.

Theorem 4. *If $F(0) = 0$, if, for every $u \in E(T)$, either* (5) *holds for some $\zeta \in \Delta\setminus\{0\}$, or* (6) *is satisfied, and if moreover $dF(0)$ is injective, then $F = dF(0)_{|B(T)} \in \mathrm{Iso}(B(T))$.*

In [**16**] we dealt with the weaker hypothesis whereby F maps the support of any element of G *into* the support of element of G establishing a theorem similar to Theorem 1.

Theorem 5. *If $F(0) = 0$, if $dF(0)$ is injective and if, for every $u \in E(T)$, there exists a neighborhood V of 0 in Δ such that:*

$F(\zeta u)$ is collinear to some point in $E(T)$ for all $\zeta \in V$
the restriction of F to Vu is a scalar Moebius transformation,

then there is a complex regular Borel measure ρ on T, with

$$\|\rho\| < 1, \tag{7}$$

such that
$$F(x) = \frac{1}{1-(x,\rho)}\,dF(0)x \qquad \textit{for all } x \in B(T).$$

Viceversa, let R be an injective continuous operator on $C(T)$ with $\|R\| \leq 1$, such that $|Ru(t)|$ is independent of $t \in T$ for all $u \in E(T)$, and let ρ be a complex, regular Borel measure on T satisfying (7). The Moebius transformation

$$F\colon x \mapsto \frac{1}{1-(x,\rho)}Rx$$

defines a holomorphic map of $B(T)$ into $C(T)$. If $F(B(T)) \subset B(T)$, F satisfies all the hypotheses of Theorem 5, and in particular maps the support of any element in G into the support of an element of G.

References

[1] E. Beltrami, *Risoluzione del problema: riportare i punti di una superficie sopra un piano, in modo che linee geodetiche vengano rappresentate da linee rette*, Ann. Mat. Pura Appl. 7 (1865), 185-204. *Opere matematiche* I, Hoepli, Milano 1902, 262-280.

[2] L. Berwald, *On the projective geometry of paths*, Ann. of Math. 37 (1936), 879-898.

[3] L. Bianchi, *Lezioni di geometria differenziale* I, Zanichelli, Bologna 1927.

[4] E. Bortolotti, *Spazi a connessione proiettiva*, Cremonese, Roma 1941.

[5] U. Dini, *Sopra un problema che si presenta nella teoria generale delle rappresentazioni geografiche di una superfice su di un'altra*, Ann. Mat. Pura Appl. 3 (1869-70), 269-293. *Opere* I, Cremonese, Roma 1953, 600-627.

[6] L. P. Eisenhart, *Non-Riemannian Geometry*, Amer. Math. Soc. Colloq. Publ. 8, AMS Publ. New York 1927.

[7] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam 1980.

[8] P. Funk, *Über zweidimensionale Finslersche Räume insbesondere über solche mit geradlinige Extremalen und positiver Konstanter Krümmung*, Math, Z. 40 (1936), 86-93.

[9] G. Gentili, *On complex geodesics of balanced convex domains*, Ann. Mat. Pura Appl. **144** (1986), 113-130.

[10] L. A. Harris, *Bounded Symmetric Homogeneous Domains in Infinite Dimensional Spaces*, Lecture Notes in Math., **634**, Springer, Berlin 1974.

[11] Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York 1968.

[12] P. Mazet et J. P. Vigué, *Points fixes d'une application holomorphe d'un domaine borné dans lui même*, Acta Math. **166** (1991), 1-26.

[13] E. Vesentini, *Complex geodesics*, Compositio Math. 44 (1981), 375-394.

[14] E. Vesentini, *Semigroups of linear contractions for an indefinite metric*, Atti Accad. Naz. Lincei, Memorie 2 (1994), 53-83.

[15] E. Vesentini, *On the Banach-Stone theorem*, Adv. in Math., to appear.

[16] E. Vesentini, *Holomorphic projective mappings*, to appear.

[17] H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projectiven und konformen Auffassung*, Göttinger Nachrichten (1921), 99-112, Gesammelte Abhandlungen 2, Springer, Berlin 1968, 195-207.

[18] H. Weyl, *Raum - Zeit - Materie*, Springer, Berlin 1923.

[19] H. Weyl, *Mathematische Analyse des Raumproblems*, Springer, Berlin 1923.

***