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Submanifolds and Gauss maps (**)

1 - Introduction

Many extrinsic properties of a riemannian submanifold (M, g) of $\boldsymbol{R}^n$ can be read in terms of the Gauss map of the submanifold itself.

A result in this framework is the following theorem of Ruh and Vilms ([9]):

The Gauss map is harmonic if and only if the mean curvature vector field of the submanifold is parallel in the normal bundle.

One can extend results of this kind to the Gauss maps relative to submanifolds of a space form or of an arbitrary riemannian manifold.

In this paper we will study some problems relative to riemannian submanifolds (M, g) of $\boldsymbol{R}^n$ satisfying a weaker property than that of Ruh-Vilms theorem. More precisely, given the Gauss map $\gamma\colon (M, g) \to (G(m, n), \Gamma)$ (where Γ is the canonical metric of the Grassmannian of the m-planes in $\boldsymbol{R}^n$) we will study and give examples of submanifolds M for which the tension field of γ, τ_γ, is non zero and orthogonal to the image manifold $\gamma(M)$.

Problems of this sort were first studied by B.Y. Chen and T. Nagano in a paper of 1984 ([3]). They assumed that the Gauss map γ was injective and they found conditions under which the identity map $i_M\colon (M, g) \to (M, \gamma^* \Gamma)$ is harmonic.

Sections 2 and 3 are devoted to some general properties of maps between riemannian manifolds and of the Gauss map, respectively.

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In section 4 the differential condition characterizing the submanifolds which are studied is determined. It is proved that it is equivalent to the condition of Chen and Nagano.

Section 5 and section 6 are devoted respectively to the study of the surfaces of $\boldsymbol{R}^n$ and the hypersurfaces of $\boldsymbol{R}^4$ which satisfy this condition.

2 - Second fundamental form and tension field of a map

Let $f\colon (M, g) \to (N, G)$ be a differentiable map between the riemannian manifolds M and N. The differential $\mathrm{d}f\colon TM \to f^{-1}TN$ is a 1-form with values in the bundle $f^{-1}TN$.

The covariant differential of df, $\nabla \mathrm{d}f$, called the *second fundamental form of the map f*, is the symmetric $f^{-1}TN$-valued 2-form defined explicitly by

$$(2.1) \qquad (\nabla \mathrm{d}f)(X, Y) = (\nabla_Y(\mathrm{d}f))(X) = \nabla_Y^{f^{-1}TN}(\mathrm{d}fX) - \mathrm{d}f(\nabla_Y^M X).$$

If $\nabla \mathrm{d}f = 0$, f is said *totally geodesic* (or *affine*) and is a map sending geodesic of M to geodesic of N, preserving the parametrization.

The *tension field* τ_f of f is the trace, with respect to the metric g, of $\nabla \mathrm{d}f$, i.e. $\tau_f = \nabla \mathrm{d}f(e_i, e_i)$, where $\{e_i\}$ denotes an orthonormal basis of (M, g). If $\tau_f = 0$ the map f is called *harmonic* (cfr. [4], [5], [6]).

The *energy density* of f is defined by

$$(2.2) \qquad e_f = \frac{1}{2}\,|\mathrm{d}f|^2 = \frac{1}{2}\,\mathrm{tr}_g f^* G$$

and the *energy of the map f* on a compact set $K \subset M$ is given by

$$(2.3) \qquad E_f = \int_K e_f \,\mathrm{d}v_g\,.$$

$\tau_f = 0$ is the Euler-Lagrange equation of the functional E, defined on the space of the differentiable maps of M into N.

In case f is a riemannian immersion of M into N, $\nabla \mathrm{d}f = h$ is the usual *second fundamental form of the immersion*, $\tau_f = mH$, where H is the *mean curvature vector* of the submanifold M and m is the dimension of M.

It will be useful to consider the *stress energy tensor* of f, defined by (cfr. [1])

$$(2.4) \qquad S_f = e_f g - f^* G\,.$$

$S_f = 0$, if and only if $m = 2$ and f is weakly conformal.

The tensor S_f has a relevant geometrical meaning. It represents twice the gradient of $E_f(g)$, the latter considered as a functional on the metric g of M, for a fixed f. In fact one has

$$\frac{\mathrm{d}E_f(g(t))}{\mathrm{d}t}\Big|_{t=0} = \frac{1}{2}\int_K (S_f, \frac{\mathrm{d}g}{\mathrm{d}t}(0))\,\mathrm{d}v_g \,.$$

One can prove (cfr. [1])

$$\operatorname{div} S_f = -(\tau_f, \mathrm{d}f)_G \tag{2.5}$$

where $\operatorname{div} S_f(X) = (\nabla_{e_i} S_f)(e_i, X)$.

About the product of two differentiable maps $f\colon (M, g) \to (N, G)$ and $\phi\colon (N, G) \to (P, G')$ we have

$$\nabla \mathrm{d}(\phi f) = \mathrm{d}\phi(\nabla \mathrm{d}f) + \nabla \mathrm{d}\phi(\mathrm{d}f, \mathrm{d}f)\,. \tag{2.6}$$

It follows

$$\tau_{\phi f} = \mathrm{d}\phi \tau_f + \nabla \mathrm{d}\phi(\mathrm{d}fe_i, \mathrm{d}fe_i)\,. \tag{2.7}$$

3 - Gauss maps

Let (M, g) be an m-dimensional riemannian submanifold, isometrically immersed in $\boldsymbol{R}^n$ The Gauss map $\gamma\colon (M, g) \to G(m, n)$, Grassmannian of the m-planes of $\boldsymbol{R}^n$

$$G(m, n) = \frac{SO(n)}{SO(m) \times SO(n-m)}$$

associates with any point $x \in M$ the tangent space $T_x M$, translated into the origin of $\boldsymbol{R}^n$.

A tangent vector to $G(m, n)$ at a point $[\pi]$, corresponding to the m-plane π of $\boldsymbol{R}^n$,is a linear map of π in $\pi^\perp$. $G(m, n)$ has a canonical structure of symmetric space (cfr. e.g. [7], vol. II, ch. 11). If $\{e_i\}$ and $\{e_\alpha\}$ are orthonormal basis of π and $\pi^\perp$, the metric of $G(m, n)$ is the one with respect to which $\{e_i^* \otimes e_\alpha\}$ is an orthonormal basis.

One can verify that (cfr. [5], [9])

$$(\mathrm{d}\gamma(X))\,Y = h(X, Y) \qquad \text{for any } X, Y \in T_x M \tag{3.1}$$

where h is *the second fundamental form* of M in $\boldsymbol{R}^n$.

A simple computation shows that

$$((\nabla d\gamma)(X, Y))Z = (\nabla^\perp_Y h)(X, Z) = (\nabla^\perp_Z h)(X, Y) \tag{3.2}$$

which implies in particular

$$\tau_\gamma = m\nabla^\perp H \tag{3.3}$$

and hence the theorem of Ruh and Vilms.

Moreover one should note that

$$(\gamma^* \Gamma)(X, Y) = h(X, e_i)\cdot h(Y, e_i)\,. \tag{3.4}$$

Hence, using the Gauss equation,

$$(\gamma^* \Gamma)(X, Y) = mh(X, Y)\cdot H - \mathrm{Ric}_M(X, Y) \tag{3.5}$$

where the right hand side must be augmented by $c(m-1)g(X, Y)$ if M is immersed in a space form $N(c)$ of constant curvature c.

The relation (3.4) implies

$$e_\gamma = \frac{1}{2}|h|^2 = \frac{1}{2}h(e_i, e_j)\cdot h(e_i, e_j)\,.$$

Hence by (2.4)

$$S_\gamma = \frac{1}{2}|h|^2 g - \gamma^* \Gamma\,. \tag{3.6}$$

If the second fundamental form h is considered as an element of $L(TM, L(TM, T^\perp M))$, the *relative nullity index* ν of M is given by $\nu = \dim\ker h$. If $\nu = 0$, $\gamma^*\Gamma$ is a metric on M, called the *Gauss metric* of M. If $\nu > 0$, $\gamma^*\Gamma$ is a symmetric positive semidefinite 2-form on M.

4 - On a particular type of submanifold

Here we study the M dimensional submanifolds (M, g) of $\boldsymbol{R}^n$ such that the tension field τ_γ of the Gauss map γ is *orthogonal* to $\gamma(M)$.

By (3.3) τ_γ, at any point $x \in M$, is the linear map of $T_x M$ into $T_x^\perp M$ defined by

$$\tau_\gamma(X) = m\nabla^\perp_X H\,.$$

On the other hand, any vector $d\gamma(X)$ tangent to $\gamma(M)$ at $\gamma(x)$ is the linear map

$$(d\gamma(X))\,Y = h(X, Y).$$

Hence τ_γ is *orthogonal* to $\gamma(M)$, if and only if

$$h(e_i, X)\cdot \nabla^{\perp}_{e_i} H = 0 \qquad \text{for any } X. \tag{4.1}$$

By (2.5), (4.1) is equivalent to

$$\operatorname{div} S_\gamma = 0 \tag{4.2}$$

which can be verified also directly.

If γ is injective, (4.2) was equivalently formulated by Chen and Nagano ([3]).

In fact let us consider the following maps:

$$i_M\colon (M, g) \to (M, \gamma^* \Gamma)\,; \qquad \gamma'\colon (M, \gamma^*\Gamma) \to (G(m, n), \Gamma) \tag{4.3}$$

where $\gamma = \gamma' i_M$ and γ' is an *isometric immersion*. By (2.7)

$$\tau_\gamma = d\gamma'(\tau_{i_M}) + \operatorname{tr}_g h' \tag{4.4}$$

where h' is the second fundamental form of the submanifold $\gamma(M)$, which is isometrically immersed in $G(m, n)$.

(4.4) represents the splitting of τ_γ in its tangential and normal part with respect to $\gamma(M)$. Hence τ_γ is orthogonal to $\gamma(M)$ if and only if i_M is harmonic, or also if and only if the Gauss metric $\gamma^*\Gamma$ is harmonic with respect to the metric g of M.

In [3] it is shown that the harmonicity of i_M is expressed by the following differential condition

$$\nabla_g \operatorname{tr}_g \gamma^* \Gamma = 2 \operatorname{div}_g \gamma^* \Gamma \tag{4.5}$$

which is the exact translation of (4.2) to this setting.

In general, a symmetric 2-form T on M is said harmonic if it satisfies (4.5). For instance, as an easy consequence of the second Bianchi identity, the Ricci tensor of any Riemannian manifold (M, g) is harmonic with respect to g.

Remark that (4.1) has the same meaning for any submanifold of a space form, independently on the injectivity of the Gauss map.

A general result on the submanifolds M for which the tension field of the Gauss map τ_γ is orthogonal to $\gamma(M)$ is expressed by

Proposition 1. *If M is a compact orientable submanifold for which* (4.1) *holds, then the mean curvature vector H has constant norm.*

As a matter of fact let us prove that, for any vector field X on M

$$\int_M X(|H|^2)\, dv_g = \int_M 2(\nabla_X^\perp H \cdot H)\, dv_g = 0\,. \tag{4.6}$$

Let us consider the symmetric 2-form $B(X, Y) = h(X, Y)\cdot H$.

We have

$$(\operatorname{div} B)(X) = (\nabla_{e_i} B)(e_i, X) = e_i(B(e_i, X)) - B(\nabla_{e_i}^M e_i, X) - B(e_i, \nabla_{e_i}^M X)$$

$$= \nabla_{e_i}^\perp (h(e_i, X))\cdot H + h(e_i, X)\cdot \nabla_{e_i}^\perp H - h(\nabla_{e_i}^M e_i, X)\cdot H - h(e_i, \nabla_{e_i}^M X)\cdot H$$

whence, by (4.1) and the symmetry of h,

$$(\operatorname{div} B)(X) = (\nabla_{e_i}^\perp h)(e_i, X)\cdot H = m\nabla_X^\perp H\cdot H\,. \tag{4.7}$$

On the other hand, by the symmetry of h,

$$(\operatorname{div} B)(X) = \operatorname{div}(B(X))^\# \tag{4.8}$$

where $(B(X))^\#$ is the vector field associated to the 1-form $B(X)$ via the metric g of M.

(4.6) is a consequence of Green's Theorem and (4.7), (4.8). As X can be arbitrary chosen, Proposition 1 follows.

In the sequel examples and properties of the submanifolds for which (4.1) with $\nabla^\perp H \neq 0$ holds will be examined. Indeed a classification of the above is of course impossible. More precise informations on these submanifolds can be obtained as soon as suitable restrictions on the dimension and the codimension are made.

Thus, if $\dim M = 1$, (4.1) expresses that M has constant first curvature, hence also $\nabla^\perp H = 0$.

In codimension 1, let $H = fe_n$ be the mean curvature vector, where f is the mean curvature function. In this case $\nabla^\perp H \neq 0$ is equivalent to $\nabla f \neq 0$ and (4.1) can be written as $h(X, \nabla f) = 0$, whence the relative nullity index of M is greater or equal to 1. The integral curves of unitary vector field $\xi = \nabla f\cdot |\nabla f|^{-1}$ orthogonal to the level sets of f are geodesics of M and of $\boldsymbol{R}^n$ (or of the space form $N(c)$ into which M is immersed as an hypersurface).

One should remark that, as a consequence of the Codazzi equation, the relative nullity foliation $\mathcal{N}$ of a submanifold M of a space of constant curvature is totally geodesic in M and in $N(c)$.

5 - Surfaces of R^n

Let us suppose that M has dimension 2. It is well known that this dimension is peculiar in the theory of harmonic maps, because the tension field of any map defined on (M, g) depends only on the conformality class of the metric g.

This implies that in dimension 2 (4.1) is certainly satisfied if the Gauss map γ is conformal. By (3.5) and as $\mathrm{Ric}_M(X, Y) = Kg(X, Y)$ (K Gaussian curvature of M), this condition is equivalent to

$$h(H, Y)\cdot H = g(X, Y)|H|^2\,, \tag{5.1}$$

i.e. M is a pseudoumbilical surface (the normal section determined by H, if $H \neq 0$, is a umbilical section).

To see if this is the only possible case in which (4.1) holds, one has to take into account the topology of M.

Let $z = x + iy$ be a system of isothermal coordinates such that $g = \lambda(\mathrm{d}x^2 + \mathrm{d}y^2)$. Let us consider a symmetric 2-form T on M and its Hopf transform f_T, given by the complex valued function

$$f_T = \frac{T(\partial_x, \partial_x) - T(\partial_y, \partial_y)}{2} - iT(\partial_x, \partial_y)\,.$$

The tensor T is harmonic with respect to g, i.e. it satisfies (4.5), if and only if (cfr. also [3]) the function f_T is holomorphic and hence $f_T\,\mathrm{d}z^2$ is a holomorphic quadratic differential on M. Hence, if M has the topology of the sphere, the Gauss metric $T = \gamma^*\Gamma$ is harmonic with respect to g if and only if $f_T = 0$, i.e. γ is conformal.

Taking account of this result we give examples of surfaces which locally satisfy (4.1). Distinction will be made with respect to the codimension and the pseudoumbilicity.

a. $\operatorname{codim} M = 1$.

$\mathbf{a}_1$. If M is *pseudoumbilical*, then M is minimal or M is totally umbilical. In both cases $\nabla^\perp H = 0$.

$\mathbf{a}_2$. If M is *not pseudoumbilical* and locally satisfies (4.1), then M has necessarily relative nullity index equal to 1 and the nullity vector field coincides with the gradient ∇f of the mean curvature function. We prove

Proposition 2. *The surfaces of a space form $N^3(c)$ which locally satisfy (4.1) with $\nabla^{\perp} H \neq 0$ are ruled surfaces by geodesics intersecting orthogonally a plane curve L of constant curvature in $N^3(c)$. In particular, if $c = 0$, they are round cones.*

Proof. Let us choose a Darboux frame $\{e_1, e_2, e_3\}$ on M so that e_3 is normal to M and e_2 is the nullity versor. Then

$$h(e_1, e_1) = 2fe_3 \qquad h(e_1, e_2) = h(e_2, e_2) = 0 \qquad e_2 = \nabla f \cdot |\nabla f|^{-1} .$$

This implies in particular that the Gaussian curvature of M is c, hence (cfr. e.g. [**10**], Chap. 7, Prop. 34) M is locally a ruled surface of $N^3(c)$, i.e. $M = \exp_{L(s)}(tV(s))$.

On the other hand, in this setting, if $\{\theta^1, \theta^2\}$ is the coframe on M dual to $\{e_1, e_2\}$, we have

$$\omega_1^3 = 2f\theta^1 \qquad \omega_2^3 = 0 \qquad e_1(f) = 0 .$$

Applying the Codazzi equations we get

$$\omega_1^2 = \alpha\theta^1 \qquad e_2(f) = \alpha f$$

and by the Gauss equation $e_2(\alpha) = \alpha^2 + c$.

Moreover, as $[e_1, e_2] = -\alpha e_1$, it follows $0 = [e_1, e_2] f = e_1(\alpha) f$, which implies $e_1(\alpha) = 0$.

Proposition 2 is a consequence of the following

Lemma. *Any integral curve L_1 of the vector field e_1 is a plane, constant curvature curve of $N^3(c)$. The plane of L_1 has a constant angle φ with the integral curves (geodesics) of the vector field e_2 based at points of L_1. The angle φ is independent of L_1 if and only if $c = 0$.*

Proof of the Lemma. As a consequence of the conditions written above

$$\nabla^N_{e_1} e_1 = \alpha e_2 + 2fe_3 \qquad \nabla^N_{e_1} \nabla^N_{e_1} e_1 = -(\alpha^2 + 4f^2) e_1$$

which implies that e_1, $\nabla^N_{e_1} e_1$ and $\nabla^N_{e_1}\nabla^N_{e_1} e_1$ lie on a same plane and moreover

$$|\nabla^N_{e_1} e_1|^2 = \alpha^2 + 4f^2$$

which is constant along any curve L_1.

The angle between e_2 and the normal to the plane of L_1 is $\frac{\pi}{2} - \varphi$ and

$$\cos\left(\frac{\pi}{2} - \varphi\right) = \sin\varphi = \frac{2f}{(\alpha^2 + 4f^2)^{\frac{1}{2}}}$$

which implies

$$e_2(\sin\varphi) = \frac{-2\alpha fc}{(\alpha^2 + 4f^2)^{\frac{3}{2}}} .$$

This proves the Lemma.

Remark that, if $c \geqslant 0$, the surfaces as above have a singularity (the vertex of the cone and its analogue). If $c < 0$ those surfaces may also be non singular.

For example, we consider the hyperbolic space $H^3(-1)$ represented by the semihyperboloid of $\boldsymbol{R}^{1,3}$ (with the Lorentz metric)

$$-x^2 + y^2 + z^2 + w^2 = -1 \qquad\qquad x > 0$$

the central projection of a round cone belonging to the hyperplane π: $x = 1$ (identified with the euclidean space $\boldsymbol{R}^3$) and with axis through $(1, 0, 0, 0)$ is a surface M of $H^3(-1)$ satisfying (4.1). If the vertex of the cone is external to the unit ball $B^3(1)$ of π, M has no singular point.

b. codim $M = 2$.

$\boldsymbol{b}_1$. *M is pseudoumbilical.* In codimension 2, if M is pseudoumbilical and $|H| =$ constant, then necessarily $\nabla^\perp H = 0$ (cfr. [2]. Prop. 2.4, pag. 179). Hence, in this case, a surface with conformal Gauss map and $\nabla^\perp H \neq 0$ must have mean curvature vector with non constant norm.

An example is given by

$$e^u(\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v)$$

whose metric is $g = e^{2u}(2du^2 + dv^2)$, while the Gauss metric is $\gamma^*\Gamma = e^{-2u} g$.

$\mathbf{b}_2$. *M not pseudoumbilical with* $|H| =$ constant. We prove

Proposition 3. *The non pseudoumbilical surfaces with* $|H|$ = constant, $\nabla^\perp H \neq 0$ *satisfying* (4.1) *are flat and are the Riemannian product of a constant curvature curve and a straight line.*

Proof. Let us choose a Darboux frame $\{e_1, e_2, e_3, e_4\}$ such that e_3 is parallel to the mean curvature vector and the Weingarten operator with respect to it has the diagonal form

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Then
$$A_4 = \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix}$$

and $H = \dfrac{1}{2}(\alpha+\beta)\, e_3$, $\alpha+\beta$ = constant, $\alpha \neq \beta$.

Set $\omega_3^4 = l\theta^1 + m\theta^2$,

$$\nabla_{e_1}^\perp H = \frac{1}{2}(\alpha+\beta)\, l e_4 \qquad \nabla_{e_2}^\perp H = \frac{1}{2}(\alpha+\beta)\, m e_4$$

with l, m never both zero. By (4.1) we get

$$\lambda l + \mu m = 0 \qquad \mu l - \lambda m = 0$$

which implies $\lambda = \mu = 0$, that is the first normal space has rank 1.

Differentiating $\omega_1^4 = 0$, $\omega_2^4 = 0$, one gets $m\alpha = 0$, $l\beta = 0$. Supposed for instance $\beta = 0$, it follows $\alpha \neq 0$, $m = 0$, hence $\omega_1^3 = \alpha\theta^1$, $\omega_2^3 = 0$. As $d\omega_1^3 = 0$, one gets $\alpha(-\omega_2^1 \wedge \theta^2) = 0$, hence $\omega_2^1 = \rho\theta^2$. Differentiating ω_2^3 one obtains $d\omega_2^3 + \omega_1^3 \wedge \omega_2^1 = 0$, whence $\rho = 0$, which proves that M is flat.

One has moreover $\nabla_{e_2}^{R^4} e_2 = 0$, hence the integral curves of e_2 are straight lines, while $\nabla_{e_1}^{R^4} e_1 = \alpha e_3$ shows that the integral curves of e_1 have constant curvature.

An explicit example of such surface is the *riemannian product of a circular helix and a line*

$$(r\cos\varphi,\ r\sin\varphi,\ h\varphi,\ t).$$

b$_3$. *M non pseudoumbilical with* $|H|$ *non constant.* An example of this kind satisfing (4.1) with $\nabla^\perp H \neq 0$ is given by

$$(x(s),\ y(s),\ \operatorname{ch} s\cos v,\ \operatorname{ch} s\sin v)$$

where $(x(s), y(s))$ is a curve of $\boldsymbol{R}^2$ parametrized with arc length and having the function ch s as curvature. For such a surface one has

$$g = \mathrm{ch}^2 s(\mathrm{d}s^2 + \mathrm{d}v^2)$$

with Gaussian curvature $$K = -\frac{1}{\mathrm{ch}^4(s)}$$

and Gauss metric $$\gamma^* \Gamma = \frac{1+\mathrm{ch}^2 s}{\mathrm{ch}^2 s}\mathrm{d}s^2 + \frac{1}{\mathrm{ch}^2 s}\mathrm{d}v^2 .$$

Moreover it is not difficult to prove the following

Proposition 4. *If M is a surface in $\boldsymbol{R}^4$ contained in the sphere S^3, it satisfies* (4.1) *as a surface of $\boldsymbol{R}^4$ if and only if it satisfies the same condition in S^3.*

c. codim $M > 2$.

In this case, the conditions determined by (4.1) are weak. Hence it is possible to give various examples. Also because in relationship with what said above, we mention the surface *product of a circular helix and a circle*

$$P(u, v) = (a \cos u,\ a \sin u,\ u,\ b \cos v,\ b \sin v), \qquad a,\ b,\ \text{constant}.$$

One can verify that

$$|H|^2 = \frac{1}{4}\Big(\frac{a^2}{(1+a^2)^2} + \frac{1}{b^2}\Big) = \text{constant}$$

$\nabla^\perp H \neq 0$ and that (4.1) holds. This surface is flat and it is pseudoumbilical if and only if $b^2 = \dfrac{(1+a^2)^2}{a^2}$.

6 - Hypersurfaces of $\boldsymbol{R}^4$

The study of the hypersurfaces, which locally satisfy (4.1) with $\nabla^\perp H \neq 0$, can be made in a similar way as the one of the surfaces of $N^3(c)$ in Section 5.

As the relative nullity index of M is greater or equal to 1, we may choose an orthonormal frame $\{e_1, e_2, e_3\}$ on M such that the Weingarten operator has the following diagonal form

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.1}$$

The integral curves of e_3 (unitary vector of the gradient of the mean curvature function) are geodesics of M and $\boldsymbol{R}^4$, i.e. straight lines. Moreover

$$e_1(\lambda + \mu) = e_2(\lambda + \mu) = 0 \qquad e_3(\lambda + \mu) \neq 0$$

e_1 and e_2 being vector fields tangent to the level sets on M of the mean curvature function.

One has the following cases:

1 $\lambda \neq 0$, $\mu = 0$, thus the relative nullity index is 2.

2 $\lambda = \mu \neq 0$.

3 $\lambda \neq \mu \neq 0 \neq \lambda$.

By Codazzi and Gauss equations, in case **1** and **2** one gets easily a characterization of the three dimensional manifolds M of these kinds satisfying (4.1) with $\nabla^{\perp} H \neq 0$.

In case **1** one proves that these manifolds are the *Riemannian product of a circular cone and a straight line.*

The manifolds of type **2**, satisfying (4.1), are *cones which project from a point a sphere* S^2.

In both cases we have three dimensional conformally flat manifolds.

The most interesting and difficult case to be studied is **3**. Taking into account the conditions implied by the Gauss and Codazzi equations and some known results on three dimensional conformally flat manifolds (cfr. [8], Prop. 3. page 84), one gets that the hypersurfaces of type **3** satisfying (4.1) are conformally flat if and only if the foliation normal to the relative nullity foliation is umbilical, i.e. the level surfaces on M of the mean curvature function are umbilical.

If one projects from the origin of $\boldsymbol{R}^4$ the torus of S^3

$$(\cos u, \sin u, a \cos v, a \sin v)$$

with a constant $\neq 1$, one obtains a hypersurface of $\boldsymbol{R}^4$ verifying (4.1) with $\nabla^{\perp} H \neq 0$ and conformally flat.

One gets an analogous result by projecting from O any not umbilical surface of S^3 with constant mean curvature.

On the other hand, if M is the ruled manifold given by the straight lines intersecting the circles

$$\gamma_1: (\cos s, \sin s, 0, 0), \qquad \gamma_2: (0, 0, a\cos t, a\sin t)$$

then also this hypersurface (of which γ_1 and γ_2 are the focal manifolds) satisfies (4.1), but is not conformally flat.

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