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Complex landscapes (**)

0 - Introduction

Let M be a C^∞-smooth m-manifold and let $L(M)$ be the principal $GL(m, \boldsymbol{R})$-bundle of linear frames on M; a great deal of the geometry of M can be labelled as the study of *G-reductions* of (or, more in general, *G-structures* on) $L(M)$, where G is a Lie subgroup of $GL(M, \boldsymbol{R})$ (or, more in general, a representation $\rho: G \to GL(m, \boldsymbol{R})$ is given).

We are interested in complex geometry, therefore $m=2n$ and $G \subset GL(n, \boldsymbol{C})$.

$GL(n, \boldsymbol{C})$-reductions of $L(M)$ exist if and only if the $GL(n, \boldsymbol{C})$-bundle

$$W(M) = \frac{L(M)}{GL(n, \boldsymbol{C})}$$

admits global sections, or, equivalently, if and only if M is orientable and, given any Riemannian metric g on M, the $SO(2n)$-bundle

$$Z(M) = \frac{SO_g(M)}{U(n)}$$

admits global sections ($SO_g(M)$ being the principal $SO(2n)$-bundle of g-orthonormal, positively oriented linear frames on M).

So all the informations concerning the existence of $GL(n, \boldsymbol{C})$-reductions of $L(M)$ are encoded in the algebraic and geometric topology of M.

For example, if M is a compact connected 4-manifold, then $Z(M)$ admits

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global sections if and only if there exists $h \in H^2(M, \boldsymbol{Z})$ such that

$$h = w_2(M)(\mathrm{mod}\, 2) \qquad h^2 = p_1(M) + 2e(M).$$

(S^4, of course, does not meet these requirements.)

Definition 1. A section J of $W(M)$ is called a *complex structure* on M.

Clearly, given a complex structure J on M, $[u] \mapsto u \circ J_0 \circ u^{-1}$ (where $J_0 = [\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}]$) identifies J as a global section of $\mathrm{End}(TM)$ with $J^2 = -\mathrm{id}_{TM}$.

As usual, if the set of G-reductions is not empty, one looks further for some distinguished element, tipically critical points for some natural functional involving ideas as «*energy*», «*volume*», «*curvature*» or similar.

In the case of complex structures, we have the following:

a. J defines the bigraduation $\wedge^r (T^*M)^C = \bigoplus_{p+q=r} \wedge_J^{p,q} (T^*M)^C$.

b. Setting $\wedge^{p,q} = \wedge_J^{p,q}(T^*M)^C$, we have

$$\mathrm{d}\colon \wedge^{p,q} \to \wedge^{p+2,q-1} \oplus \wedge^{p+1,q} \oplus \wedge^{p,q+1} \oplus \wedge^{p-1,q+2}$$

and thus $\mathrm{d} = A_J + \partial_J + \bar\partial_J + \overline{A}_J$ with A_J and $\overline{A}_J$ zero order operators with $\overline{A}_J\alpha = \overline{A_J\bar\alpha}$ and $A_J(\alpha \wedge \beta) = A_J\alpha \wedge \beta + (-1)^{\deg\alpha} \alpha \wedge A_J\beta$.

c. A distinguished family of J's is therefore represented by those elements for which ∂_J (or, equivalently, $\bar\partial_J$) gives rise to a cohomology theory i.e. for which $\partial_J^2 = 0$.

d. It is easy to check that the following conditons are equivalent:

i. $\partial_J^2 = 0$ **ii.** $\mathrm{d} = \partial_J + \bar\partial_J$ **iii.** $A_J = 0$.

Moreover, $A_J\colon \wedge^{0,1} \to \wedge^{2,0}$ is given by: $A_J\alpha(X, Y) = \frac{1}{4}\alpha(N_J(X, Y))$ where:

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

is the *Nijenhuis tensor* of J, extended as antisymmetric form on $(TM)^C$ with values in $(TM)^C$, which is $\boldsymbol{C}$-bilinear with respect to the canonical complex structure and $\boldsymbol{C}$-biantilinear with respect to the extension of J to $(TM)^C$ (in fact, it satisfies $N_J(JX, Y) = N_J(X, JY) = -JN_J(X, Y)$) therefore: $A_J = 0 \Leftrightarrow N_J = 0$.

e. It is well known ([5], [6]) that J is *integrable* (or it is a *holomorphic structure*) i.e. it is possible to find local complex coordinates $z_j = x_j + iy_j$, $1 \leqslant j \leqslant n$, in such a way that

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

if and only if $N_J \equiv 0$ and this gives birth to the entire galaxy of holomorphic geometry, but the existence of such a J looks like a bizarre condition, extremely hard to be detected.

f. On the other hand, everybody knows that holomorphic objects are outstandingly useful as exceptional patterns to describe our exceptional universe and thus the old question

what is a holomorphic manifold?

is still there.

g. Therefore, it is possible to consider this double edged scheme:

1. Extend to the complex case as many results as possible from holomorphic theory.

2. Investigate carefully how to pass from the generic the situation to the exceptional one.

[3] Provides a nice example of **1**; we want here briefly discuss some ideas connected with **2**.

1 - Totally non integrable complex structures

We start the following

Definition 2. J is *totally non integrable at* $p \in M$ if

$$S_J[p] = [\![\{N_J[p](X, Y) \mid X, Y \in T_pM\}]\!] = T_pM$$

i.e. if the elements of the form $N_J[p](X, Y)$ span the entire T_pM.

It is clear that the first case where it is possible to find totally non integable complex structures is $\dim_R M = 6$; from now on, we will confine ourselves to this case.

We need some linear and bilinear algebra. Let (V, J) be a 6-dimensional real vector space equipped with a complex structure; let g be a J-Hermitian scalar product on V and, as usual, set

$$h_g(v, w) = \frac{1}{2}(g(v, w) + ig(v, Jw)).$$

Let $\alpha \in \Lambda^{3,0}V^*$ with $|\alpha|_g = 1$; define $V_\alpha^{(g)}: V \times V \to V$ as

$$\alpha(v, w, u) = h_g(v, V_\alpha^{(g)}(w, u)).$$

Then clearly $V_\alpha^{(g)}$ is bilinear and satisfies

$$V_\alpha^{(g)}(v, w) = -V_\alpha^{(g)}(w, v) \tag{2.1}$$

$$V_\alpha^{(g)}(Jv, w) = V_\alpha^{(g)}(v, Jw) = -JV_\alpha^{(g)}(v, w). \tag{2.2}$$

Therefore $V_\alpha^{(g)}$ extends as antisymmetric form on V^C with values in V^C which is C-bilinear with respect to the canonical complex structure and C-biantilinear with respect to the extension of J to V^C. Note that, if $\mathfrak{B} = \{v_1, v_2, v_3\}$ is a h_g-unitary, α-special (i.e. $\alpha(v_1, v_2, v_3) = 1$) C-basis of V, then $V_\alpha^{(g)}(v_2, v_3) = v_1$, $V_\alpha^{(g)}(v_3, v_1) = v_2$, $V_\alpha^{(g)}(v_1, v_2) = v_3$.

Assume $\beta \in \Lambda^{3,0}V^*$ satisfies $|\beta|_g = 1$; then $\beta = e^{i\theta}\alpha$ and

$$V_\beta^{(g)} = V_{e^{i\theta}\alpha}^{(g)} = e^{-i\theta}V_\alpha^{(g)}.$$

Assume q is another J-Hermitian metric on V; then $q = L^*(g)$ for $L \in \mathrm{End}(V)$, L g-symmetric positive definite, satisfying $[L, J] = 0$; assume $\det L = 1$ (i.e. g and q determine the same volume form).

If $\alpha \in \Lambda^{3,0}V^*$ satisfies $|\alpha|_g = 1$, then $|\alpha|_{L^*(g)} = 1$ and

$$V_\alpha^{(L^*(g))} = L^{-2} \circ V_\alpha^{(g)}.$$

Let $N: V \times V \to V$ be a bilinear map satisfying (2.1) and (2.2). Given a J-Hermitian metric g and $\alpha \in \Lambda^{3,0}V^*$ with $|\alpha|_g = 1$, then there exists $R_\alpha^{(g)}(N) \in \mathrm{End}(\mathrm{V})$ such that $N = R_\alpha^{(g)}(N) \circ V_\alpha^{(g)}$. Clearly:

$$[R_\alpha^{(g)}(N), J] = 0 \tag{2.3}$$

$$R_{e^{i\theta}\alpha}^{(g)}(N) = e^{i\theta}R_\alpha^{(g)}(N) \tag{2.4}$$

$$R_\alpha^{(L^*(g))}(N) = R_\alpha^{(g)}(N) \circ L^2 \tag{2.5}$$

$$\frac{1}{3}\mathfrak{S}h_g(v, N(w, u)) = \overline{\mathrm{tr}\, R_\alpha^{(g)}(N)}\, \alpha(v, w, u). \tag{2.6}$$

Note also that $\quad \mathrm{tr}\, R_\alpha^{(L^*(g))}(N) = h_g(R_\alpha^{(g)}(N), L^2).$

Let now M be a 6-dimensional oriented manifold equipped with a complex structure J; then, it is clear that, $|\det R_\alpha^{(g)}(N_J)|$ depends only on the volume form of g. Moreover:

$$J \textit{ is totally non integrable} \Leftrightarrow \det R_\alpha^{(g)}(N_J) \neq 0 .$$

Consider now the following

Definition 3. Let J be a complex structure on M; we say that J is *strongly totally non integrable* if it is totally non integrable at every point of M and there exists a J-Hermitian Riemannian structure g on M such that, if we set $\eta(X, Y) = g(X, JX)$, then we have that

$$q(X, Y, Z) = \mathfrak{S} h_g(X, N_J(Y, Z)) = (\mathrm{d}\eta)^{3,0}(X, Y, Z)$$

is everywhere different from zero.

It is easy to prove the following

Lemma 1. *Let M be a 6-dimensional oriented manifold equipped with a complex structure J. Let g be a J-Hermitian structure on M. Then:*

1. *J is strongly totally non integrable with respct to g if and only if, for any $\alpha \in \bigwedge^{3,0}(M)$ with $|\alpha|_g = 1$, we have* $\det R_\alpha^{(g)}(N_J) \neq 0$ *and* $\operatorname{tr} R_\alpha^{(g)}(N_J) \neq 0$.

2. *J corresponds to a totally real submanifold of the twistor space $Z_g(M)$ if and only if, for any $\alpha \in \bigwedge^{3,0}(M)$ with $|\alpha|_g = 1$, we have* $\det(R_\alpha^{(g)}(N_J) - (\operatorname{tr} R_\alpha^{(g)}(N_J))I) \neq 0$.

We leave as an interesting exercise to investigate the relations between total non integrability, strong total non integrability and being a totally real submanifold of the twistor space.

Examples.

a. Let $S^6 = \{x \in \Im m\, \boldsymbol{Cay} \mid |x| = 1\}$. For $p \in S^6$, $x \in T_pS^6$ set $J[p](x) = px$. Then J is a strongly totally non integrable complex structure. In fact

$$\frac{1}{2} N_J[p](x, y) = (px)\, y - p(xy)$$

and so, for every $p \in S^6$, there exists an orthonormal $\boldsymbol{C}$-basis $\{v_1, v_2, v_3\}$ of

T_pS^6 such that:

$$N_J[p](v_1, v_2) = -4Jv_3 \qquad N_J[p](v_2, v_3) = -4Jv_1 \qquad N_J[p](v_3, v_1) = -4Jv_2$$

and $$q[p] = (v_1^* - iJ^* v_1^*) \wedge (v_2^* - iJ^* v_2^*) \wedge (v_3^* - iJ^* v_3)$$

is a well defined never vanishing global section of $\wedge^{3,0} T^* S^6$.

(Just consider the standard basis of ***Cay***:

$$e_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad e_2 = \begin{bmatrix}i\\0\end{bmatrix} \quad e_3 = \begin{bmatrix}j\\0\end{bmatrix} \quad e_4 = \begin{bmatrix}k\\0\end{bmatrix}$$

$$e_5 = \begin{bmatrix}0\\1\end{bmatrix} \quad e_6 = \begin{bmatrix}0\\i\end{bmatrix} \quad e_7 = \begin{bmatrix}0\\j\end{bmatrix} \quad e_8 = \begin{bmatrix}0\\k\end{bmatrix}.$$

At $p = e_2$, choose $v_1 = e_3$, $v_2 = e_5$, $v_3 = e_7$, then use the fact that $S^6 = \dfrac{G_2}{SU(3)}$ and the action of G_2 is J-holomorphic).

b. Let (M, g) be an oriented Riemannian 4-manifold and let $Z(M) = \dfrac{SO_g(M)}{U(2)}$ be its twistor space. Then $\dim_R Z(M) = 6$ and the standard complex structure $\boldsymbol{J}$ on $Z(M)$ is never totally non integrable, in fact $N_{\boldsymbol{J}}$ is horizontal (and vertical valued) and so, at any $p \in Z(M)$, $\dim_R S_J[p] \leq 2$.

The gauge invariant construction of complex structures on $Z(M)$ [2] provides patterns for totally non integrable objects.

This construction uses $SO(2n)$-equivariant parallelizations of $SO_g(M)$ instead of connections and so J is defined as follows

$$J[P](X) = \begin{cases} P \circ X & \text{if } X \text{ is vertical} \\ (r_*^{-1} \circ (L \circ P \circ {}^tL) \circ r_*)(X) & \text{if } X \text{ is horizontal} \end{cases}$$

where L is a g-orthogonal section of $\mathrm{End}(TM)$: $L = \mathrm{id}_{TM}$ in the standard case.

We have now the following easy

Lemma 2. *Assume* $\dim_R M = 6$ *and* J *is a complex structure on* M *which is totally non integrable at* $x \in M$, *then at* $x \in M$ *we have*:

$$A_J : \wedge^{0,1} \to \wedge^{2,0} \quad \textit{and} \quad \overline{A}_J : \wedge^{1,0} \to \wedge^{0,2} \quad \textit{are bijective}$$

$$A_J : \wedge^{0,2} \to \wedge^{2,1} \quad \textit{and} \quad \overline{A}_J : \wedge^{2,0} \to \wedge^{1,2} \quad \textit{are injective}$$

$$A_J : \wedge^{1,1} \to \wedge^{3,0} \quad \textit{and} \quad \overline{A}_J : \wedge^{1,1} \to \wedge^{0,3} \quad \textit{are surjective}.$$

Note e.g. that, in the assumptions of Lemma 2, $C_J = A_J^{-1} : \wedge^{2,0} \rightarrow \wedge^{0,1}$ is given by $C_J \alpha(Z) = 4\alpha(X, Y)$ with $Z = N_J(X, Y)$.

This is a good definition because of the following

Lemma 3. *Let (M, J) be a complex manifold of complex dimension* 3 *and assume J is totally non integrable at $p \in M$. Then:*

a. *$Z, W, U, V \in T_p M$ satisfy $N_J[p](Z, W) = N_J[p](U, V) \neq 0$*

if and only if $[\![Z, W]\!] = [\![U, V]\!]$ *with* $\dim_C [\![Z, W]\!] = 2$

and $$U = aZ + bW, \qquad V = cZ + dW \quad \textit{with} \quad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1 .$$

b. $Z, W \in T_p M$ satisfy $N_J[p](Z, W) = 0$ if and only if $\dim_C [\![Z, W]\!] \leq 1$.

3 - The set of bundle complex structures

One of the most interesting features of the generic case is that totally non integrability can be exploited in order to provide a good degree of holomorphicity i.e. formally holomorphic connections in the sense of [1].

Let (M, J_M) be a complex manifold and let G be a complex (reductive) Lie group. Let $\pi: P = P(M, G) \rightarrow M$ be a principal G-bundle. Then for every $x \in M$, $\pi^{-1}(x)$ admits a complex structure J_s defined by means of the relation $J_s[u](X^*) = (iX)^*(u)$ i.e. J_s is the complex structure induced from G via the isomorphism $\pi^{-1}(x) \simeq G$ we get fixing a point $u \in \pi^{-1}(x)$; of course J_s is holomorphic.

Definiton 4. A *bundle complex structure* (*bucs*) J on P is a complex structure on P such that:

a. $\pi: P \rightarrow M$ is (J, J_M)-holomorphic

b. J induces J_s on the fibres

c. G acts J-holomorphically on P.

We denote by $\mathcal{B}(P) = \mathcal{B}[J_M](P)$ the set of bucs on P. We have the following results

Lemma 4.

a. *Let* $J \in \mathcal{B}(P)$ *and let* $\omega \in \mathcal{C}(P)$; *then* $\omega^{0,1} \in \mathcal{C}^{0,1}(P, g, ad)$ *and, consequently* $\omega^{1,0} \in \mathcal{C}(P)$.

b. *For any* $\omega \in \mathcal{C}(P)$, *there exists one and only one* $J \in \mathcal{B}(P)$ *such that* ω *is of type* $(1, 0)$ *with respct to* J.

c. *Consequently, there exists a surjective map* $\chi: \mathcal{C}(P) \to \mathcal{B}(P)$. *Moreover,* χ *is* $\mathcal{G}(P)$*-equivariant.*

Set $\mathcal{C}_J^{1,0}(P) = \chi^{-1}(J)$.

Lemma 5. *Let* $J \in \mathcal{B}(P)$ *and let* $\omega \in \mathcal{C}_J^{1,0}(P)$; *then* $(D_\omega)^{0,1} = \bar{\partial}_J$ *and so* $D_\omega : \mathcal{C}^0(P) \to \mathcal{C}^1(P)$ *splits as* $D_\omega = \partial_\omega + \bar{\partial}_J$, *with* $\partial_\omega = (D_\omega)^{1,0}$.

More in general, we have that:

$$D_\omega : \mathcal{C}^{p,q}(P) \to \mathcal{C}^{p+2,q-1}(P) \oplus \mathcal{C}^{p+1,q}(P) \oplus \mathcal{C}^{p,q+1}(P) \oplus \mathcal{C}^{p-1,q+2}(P)$$

and it splits as: $$D_\omega = A_J + \partial_\omega + \bar{\partial}_J + \bar{A}_J$$

where $A_J(\alpha) = A_{J_M}(\alpha \circ \lambda_\omega)$ *with* λ_ω *horizontal lift with respect to* ω.

Note that, since α is horizontal $\alpha \circ \lambda_\omega$ and consequently $A_J(\alpha)$ are independent of ω.

Lemma 6. *Let* $J \in \mathcal{B}(P)$ *and let* $\omega \in \mathcal{C}_J^{1,0}(P)$; *then*

$$N_J = \lambda_\omega \circ N_{J_M} \circ \pi_* + 4(\Omega_\omega^{0,2})^* . \tag{3.1}$$

Consequently

a. N_J *is horizontal*

b. *For every* $a \in G$, $(R_a)^*(N_J) = (R_a)_* \circ N_J$, *i.e.*

$$N_J((R_a)_*(X), (R_a)_*(Y)) = (R_a)_*(N_J(X, Y)).$$

Lemma 7. *Let* K *be a compact maximal subgroup of* G *(and so* $K^C = G$*) and let* Q *be a* K*-reduction of* P*. Then, for every* $J \in \mathcal{B}(P)$, $\mathcal{C}_J^{1,0}(P) \cap \mathcal{C}(Q)$ *consists of a single element.*

Example. Let $(E, \widehat{J})$ be a complex vector bundle of complex rank r i.e. a real vector bundle of rank $2r$ equipped with a section $\widehat{J}$ of $\mathrm{End}(E)$ with $\widehat{J}^2 = -\mathrm{id}_E$ and let $P = C(E)$ be the principal $GL(r, \boldsymbol{C})$-bundle of complex frames on E (in such a way that $E = C(E) \times_{GL(r, C)} \boldsymbol{R}^{2r}$). Again the existence of such a $\widehat{J}$ is equivalent to the orientability of E plus the existence, given any Riemannian structure g on E, of global section of the $SO(2r)$-bundle

$$Z(E) = \frac{SO_g(E)}{U(r)}$$

($SO_g(E)$ being the principal $SO(2r)$-bundle of g-orthonormal, positively oriented linear frames on E).

Let h be a $\widehat{J}$-Hermitian structure on E; h can be described in the following way:

$$h \in \mathcal{C}^0(C(E), gl(r, C), \tau)$$

where $$h(u) = {}^t\overline{h(u)} > 0 \text{ for every } u \in C(E)$$

and $$\tau: GL(r, \boldsymbol{C}) \to \mathrm{Aut}(gl(r, \boldsymbol{C}))$$

is given by $$\tau(a)(X) = {}^t\overline{a}^{-1} X a^{-1}.$$

Then $U_h(E) = \{u \in C(E) \,|\, h(u) = I\}$ is the principal $U(r)$-bundle of h-unitary frames on E and, given $J \in \mathcal{B}(C(E))$, then $\omega = h^{-1} \partial_J h$ is the unique element of $\mathcal{C}_J^{1,0}(C(E)) \cap \mathcal{C}(U_h(E))$.

Finally, from Lemma 2 we obtain immediately the following

Corollary. *Let (M, J_M) be a totally non integrable complex manifold of real dimension* 6; *let G be a complex (reductive) Lie group and let $\pi: P = P(M, G) \to M$ be a principal G-bundle. Then for any $J \in \mathcal{B}(P)$, for any $\omega \in \mathcal{C}_J^{1,0}(P)$, we have:*

$$A_J : \mathcal{C}^{0,1} \to \mathcal{C}^{2,0} \quad \textit{and} \quad \overline{A}_J : \mathcal{C}^{1,0} \to \mathcal{C}^{0,2} \quad \textit{are bijective}$$

$$A_J : \mathcal{C}^{0,2} \to \mathcal{C}^{2,1} \quad \textit{and} \quad \overline{A}_J : \mathcal{C}^{2,0} \to \mathcal{C}^{1,2} \quad \textit{are injective}$$

$$A_J : \mathcal{C}^{1,1} \to \mathcal{C}^{3,0} \quad \textit{and} \quad \overline{A}_J : \mathcal{C}^{1,1} \to \mathcal{C}^{0,3} \quad \textit{are surjective}.$$

4 - Some special connections and some theoretical mathematics

Lemma 8. *Let (M, J_M) be a totally non integrable complex manifold of real dimension 6; let G be a complex (reductive) Lie group and let $\pi: P = P(M, G) \to M$ be a principal G-bundle. Then for any $J \in \mathcal{B}(P)$, for any $\omega \in \mathcal{C}_J^{1,0}(P)$, we have*

$$\partial_\omega \overline{A}^{-1} \Omega_\omega^{0,2} = \Omega_\omega^{2,0}. \tag{4.1}$$

Proof. Let $J \in \mathcal{B}(P)$. In order to clarify and simplify our notations, set

$$A = A_J : \mathcal{C}^{0,1} \to \mathcal{C}^{2,0} \quad \textit{and} \quad \overline{A} = \overline{A}_J : \mathcal{C}^{1,0} \to \mathcal{C}^{0,2}$$

$$F = A_J : \mathcal{C}^{0,2} \to \mathcal{C}^{2,1} \quad \textit{and} \quad \overline{F} = \overline{A}_J : \mathcal{C}^{2,0} \to \mathcal{C}^{1,2}$$

$$R = A_J : \mathcal{C}^{1,1} \to \mathcal{C}^{3,0} \quad \textit{and} \quad \overline{R} = \overline{A}_J : \mathcal{C}^{1,1} \to \mathcal{C}^{0,3}$$

$$C = A_J^{-1} \quad \textit{and} \quad \overline{C} = \overline{A}_J^{-1}.$$

For any $\omega \in \mathcal{C}_J^{1,0}(P)$, from $D_\omega \Omega_\omega = 0$ it follows:

$$0 = (D_\omega \Omega_\omega)^{3,0} = \partial_\omega \Omega_\omega^{2,0} + R(\Omega_\omega^{1,1})$$

$$0 = (D_\omega \Omega_\omega)^{2,1} = \partial_\omega \Omega_\omega^{1,1} + \overline{\partial}_\omega \Omega_\omega^{2,0} + F(\Omega_\omega^{0,2})$$

$$0 = (D_\omega \Omega_\omega)^{1,2} = \partial_\omega \Omega_\omega^{0,2} + \overline{\partial}_\omega \Omega_\omega^{1,1} + \overline{F}(\Omega_\omega^{2,0})$$

$$0 = (D_\omega \Omega_\omega)^{0,3} = \overline{\partial}_\omega \Omega_\omega^{0,2} + \overline{R}(\Omega_\omega^{1,1}).$$

From $D_\omega^2 = e(\Omega_\omega)$ it follows:

$$\partial_\omega^2 + F\overline{\partial}_\omega + \overline{\partial}_\omega A = e(\Omega_\omega^{2,0}) : \mathcal{C}^{0,1} \to \mathcal{C}^{2,1}$$

$$\partial_\omega^2 + R\overline{\partial}_\omega = e(\Omega_\omega^{2,0}) : \mathcal{C}^{1,0} \to \mathcal{C}^{3,0}$$

$$\overline{\partial}_\omega^2 + \overline{F}\partial_\omega + \partial_\omega \overline{A} = e(\Omega_\omega^{0,2}) : \mathcal{C}^{1,0} \to \mathcal{C}^{1,2}$$

$$\overline{\partial}_\omega^2 + \overline{R}\partial_\omega = e(\Omega_\omega^{2,0}) : \mathcal{C}^{0,1} \to \mathcal{C}^{0,3}$$

$$\partial_\omega \overline{\partial}_\omega + \overline{\partial}_\omega \partial_\omega + F\overline{A} = e(\Omega_\omega^{1,1}) : \mathcal{C}^{1,0} \to \mathcal{C}^{2,1}$$

$$\overline{\partial}_\omega \partial_\omega + \partial_\omega \overline{\partial}_\omega + \overline{F}A = e(\Omega_\omega^{1,1}) : \mathcal{C}^{0,1} \to \mathcal{C}^{1,2}$$

$$0 = R\partial_\omega + \partial_\omega A : \mathcal{C}^{0,1} \to \mathcal{C}^{3,0}$$

$$0 = \overline{R}\overline{\partial}_\omega + \overline{\partial}_\omega \overline{A} : \mathcal{C}^{1,0} \to \mathcal{C}^{0,3}.$$

For a.e. $\omega \in \mathcal{C}_J^{1,0}(P)$ we have:

1. $\Omega_\omega^{1,1} \notin \operatorname{Ker} \overline{R}$

2. $[\![\bar\partial_\omega(\bar\partial_\omega \overline{C}\Omega_\omega^{0,2} - \Omega_\omega^{1,1}),\ e(\Omega_\omega^{0,2})\overline{C}\Omega_\omega^{0,2}]\!] \cap \operatorname{Im} \overline{F} = \{0\}$.

For any such ω, choose $\overline{S}: \mathcal{E}^{0,3} \to \mathcal{E}^{1,1}$ and $\overline{G}: \mathcal{E}^{1,2} \to \mathcal{E}^{2,0}$ in such a way that

a. $\overline{R}\,\overline{S} = \mathrm{id}_{\mathcal{E}^{2,0}}$

b. $\overline{S}\,\overline{R}(\Omega_\omega^{1,1}) = \Omega_\omega^{1,1}$

c. $\overline{G}\,\overline{F} = \mathrm{id}_{\mathcal{E}^{2,0}}$

d. $\overline{G}(\bar\partial_\omega(\bar\partial_\omega \overline{C}\Omega_\omega^{0,2} - \Omega_\omega^{1,1})) = 0 = \overline{G}(e(\Omega_\omega^{0,2})\overline{C}\Omega_\omega^{0,2})$.

Now: $$\partial_\omega = e(\Omega_\omega^{0,2})\overline{C} - \bar\partial_\omega^2 \overline{C} - \overline{F}\Omega_\omega \overline{C}: \mathcal{E}^{0,2} \to \mathcal{E}^{1,1}$$

and so $$\partial_\omega \overline{C} = \overline{G}e(\Omega_\omega^{0,2})\overline{C} - \overline{G}\bar\partial_\omega^2 \overline{C} - \overline{G}\partial_\omega : \mathcal{E}^{0,2} \to \mathcal{E}^{2,0}.$$

In particular $$\partial_\omega \overline{C}\Omega_\omega^{0,2} = -\overline{G}\bar\partial_\omega^2 \overline{C}\Omega_\omega^{0,2} - \overline{G}\partial_\omega \Omega_\omega^{0,2}.$$

Finally

α. $\overline{G}\partial_\omega \Omega_\omega^{0,2} = \overline{G}\bar\partial_\omega \Omega_\omega^{1,1} + \Omega_\omega^{2,0}$

β. $$-\overline{G}\bar\partial_\omega^2 \overline{C}\Omega_\omega^{0,2} = -\overline{G}\bar\partial_\omega(\bar\partial_\omega \overline{C}\Omega_\omega^{0,2}) = \overline{G}\bar\partial_\omega(\overline{S}\bar\partial_\omega \Omega_\omega^{0,2} - \bar\partial_\omega \overline{C}\Omega_\omega^{0,2} - \overline{S}\bar\partial_\omega \Omega_\omega^{0,2})$$
$$= -\overline{G}\bar\partial_\omega \overline{S}\,\overline{R}\Omega_\omega^{1,1} - \overline{G}\bar\partial_\omega(\bar\partial_\omega \overline{C}\Omega_\omega^{0,2} + \overline{S}\bar\partial_\omega \Omega_\omega^{0,2})$$
$$= -\overline{G}\bar\partial_\omega \Omega_\omega^{1,1} - \overline{G}\bar\partial_\omega(\bar\partial_\omega \overline{C}\Omega_\omega^{0,2} - \Omega_\omega^{1,1}) = -\overline{G}\bar\partial_\omega \Omega_\omega^{1,1}$$

i.e. $$\partial_\omega \overline{A}_J^{-1}\Omega_\omega^{0,2} = \Omega_\omega^{2,0}.$$

Since (4.1) is clearly a closed condition, the proof is complete.

Proposition 1. *Let (M, J_M) be a totally non integrable complex manifold of real dimension 6; let G be a complex (reductive) Lie group and let $\pi: P = P(M, G) \to M$ be a principal G-bundle. Then for any $J \in \mathcal{B}(P)$ there exists exactly one $\omega \in \mathcal{C}_J^{1,0}(P)$ such that $\Omega_\omega^{0,2} = \Omega_\omega^{2,0} = 0$.*

Proof. For every $u \in P$, we have:

a. $T_u P = S_J[u] \oplus W_u$.

b. For every $a \in G$, $S_J[ua] = (R_a)_*(S_J[u])$.

c. $S_J[u]$ is $J[u]$-invariant.

Therefore $u \mapsto S_J[u]$ defines a connection, whose connection form ω belongs to $\mathcal{C}_J^{1,0}(P)$. Moreover, because of (3.1), we have $\Omega_\omega^{0,2} = 0$. The rest follows immediately from Lemma 8.

We want to perform some theoretical mathematics in the sense of [4].

From (3.1), it follows that, if J_M is integrable, then N_J is vertical valued, $\Omega_\omega^{0,2}$ is independent of the choice of $\omega \in \mathcal{C}_J^{1,0}(P)$ and $N_J = 0 \Leftrightarrow \Omega_\omega^{0,2} = 0$.

Therefore, assume we have:

1. a holomorphic 3-manifold (M, J_M).

2. a complex (reductive) Lie group G.

3. a principal G-bundle $\pi: P \to M$.

We want to investigate the existence of $J \in \mathcal{B}(P)$ with $N_J = 0$. We can imagine the following steps:

a. Perturb J_M as $(J_M^\varepsilon)_{\varepsilon > 0}$ with J_M^ε totally non integrable and $J_M^\varepsilon \to J_M$ as $\varepsilon \to 0$.

b. Therefore, by Proposition 1, for any $J \in \mathcal{B}_\varepsilon(P) = \mathcal{B}[J_M^\varepsilon](P)$ there exists a unique $\omega_J \in \mathcal{C}_J^{1,0}(P)$ with $\Omega_{\omega_J}^{0,2} = 0$.

c. Use some compactness argument in $\mathcal{C}(P)$ in order to find

$$(J_{\varepsilon_n}, \omega_{J_{\varepsilon_n}}) \to (J, \omega) \in \mathcal{B}(P) \times \mathcal{C}_J^{1,0}(P).$$

Clearly $\Omega_\omega^{0,2} = 0$.

In steps **a** and **b** one expects:

α. some obstructions e.g. in the Chern classes of M and P

β. some bubbling phenomena.

A similar pattern can be recognized in Taubes' construction of antiselfdual structures on compact 4-manifolds [7]. Start from $(M^4; [g])$, consider $(Z(M), \boldsymbol{J}_g)$

and perturb J in a totally non integrable way as J_ε. When you force J_ε to converge to an integrable $J_{\bar{g}}$, some infinite energy appears and you have to blow M up in a finite number of points.

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