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On CR-moment condition (**)

1 - Introduction

In C^n (coordinates $z_1, \ldots, z_n$), let γ be a closed compact oriented curve of class C^1 and let M be an embedded Riemann surface of $C^n \setminus \gamma$ whose boundary is γ, then for any holomorphic differential form ω of degree 1 we have $\int_\gamma \omega = \int_{bM} \omega = \int_M d\omega$ from Stokes formula, but $d\omega$ is a holomorphic differential form of degree 2, whose restriction to M is zero, then $\int_\gamma \omega = 0$. So a necessary condition for a real curve γ to be the boundary of an embedded Riemann surface in C^n is: for any holomorphic 1-form ω in C^n, $\int_\gamma \omega = 0$; this is equivalent to $\int_\gamma z^a dz_j = 0$ where $z^a = z_1^{a_1} \ldots z_n^{a_n}$, $a_1, \ldots, a_n \in N$. This condition is called the *moment condition.*

It is known, from J. Wermer (1958) [**10**] and many further generalizations (see [**5**]), that this condition is sufficient for a real curve γ to be the boundary of a holomorphic chain in C^n. This condition has a meaning on any complex analytic manifold (or space) X, but it is empty if X has no holomorphic 1-form different from zero e.g. CP^n.

It can be generalized as follows: let M be a compact $(2p-1)$-submanifold of X of class C^1, then the moment condition is:

For every $(p, p-1)$-differential form φ such that $d''\varphi = 0$ then $\int_M \varphi = 0$ [**6**].

If M satisfies the moment condition, it is maximally complex, i.e. the complex tangent space $H_z(M)$ to M at z has dimension $p-1$. If $p = 1$, this last condition is empty.

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Harvey and Lawson showed that the moment condition is a necessary and sufficient condition for a $(2p-1)$-submanifold M to be the boundary of a holomorphic chain in $CP^n \setminus CP^{n-q}$ with $q = p$. Maximal complexity is enough for $p \geq q+1$; the same is true for $p \geq 2$ in C^n [7], [6].

The aim of this talk is:

1) to define a moment condition in $E \simeq R \times C^{n-1} \subset C^n$ for a compact submanifold N and in a half-space of E for a closed relatively compact submanifold N, to get a necessary and sufficient condition for solving a boundary problem in low dimensions

2) to explain the relation between this condition in $CP^n \setminus CP^{n-q}$, in the simplest case $n=2$, $q=1$, and necessary and sufficient conditions to solve a boundary problem in CP^n [3], according to unpublished results of J. B. Poly [4].

We give only results and sketches of proofs; the detailed proofs will appear elsewhere.

Definitions.

A *q-subvariety of class C^k with negligible singularities in a Riemannian manifold X* is a closed set W of X which contains a closed set σ such that $\mathcal{H}^q(\sigma) = 0$ (q-dimensional Hausdorff measure) and such that $W \setminus \sigma$ is a closed oriented q-submanifold of class C^k of $X \setminus \sigma$, of locally finite q-dimensional volume. We denote also by W the integration current defined by W. W is said to be CR if $W \setminus \sigma$ is a CR submanifold.

A *q-chain V of class C^k of X* is a locally finite linear combination, with coefficients in Z, of integration currents $[W_j]$ where W_j is a q-subvariety of class C^k with negligible singularities. V is q-cycle if $dV = 0$. If every W_j is CR of CR-dim r, V is said to be of CR-dim r.

A *holomorphic p-chain T of the complex manifold X* is a $2p$-chain where W_j is a complex analytic set of complex dimension p.

2 - Case of a real hyperplane E of C^n

Let E be the real hyperplane $\{z \in C^n;\ \Im m\, z_1 = y_1 = 0\}$. Let N be a compact $(2p-2)$-subvariety of class C^1 with negligible singularities of C^n, contained in E, with CR-dim $N = p-2$. The integration current defined by N, and also denoted N, is supposed to be d-closed. We look for a maximally complex chain

M of $C^n \setminus N$, with $\operatorname{supp} M \subset\subset E$, of finite mass, such that $\dim M = 2p - 1$, having a simple extension to E still denoted M such that $dM = N$.

The case $p \geqslant 3$ has been studied previously [1], [2]. Consider the case $p = 2$. The projection method allows us to consider only the case $n = 3$, then $\operatorname{codim}_E M = 2$.

Let $j: E \hookrightarrow C^3$ be the canonical injection. We consider the type with respect to (z_2, z_3) and set $d''_E = \frac{\partial}{\partial \bar{z}_2} d\bar{z}_2 + \frac{\partial}{\partial \bar{z}_3} d\bar{z}_3$. Then, there exist rectifiable currents S and P of $E \setminus N$ and E, respectively, such that

$$M = j_{\#} S \qquad S = S^{1,1} + dx_1 \wedge (S^{1,0} + S^{0,1})$$

$$N = j_{\#} P \qquad P = P^{2,1} + P^{1,2} + dx_1 \wedge (P^{2,0} + P^{1,1} + P^{0,2})$$

and S having a simple extension to E, still denoted S, such that $P = dS$.

Let $\beta = dx_1 \wedge \varphi^{1,0} + \psi^{2,0}$ be a C^∞-differential form on E such that $d''_E \beta = 0$, then

$$\int_N \beta = \langle P, \beta \rangle = \langle dS, \beta \rangle = -\langle S, d\beta \rangle = -\langle S, d''_E \beta \rangle = 0 .$$

Definition. N (or P) satisfies the CR-*moment condition* if, for every β as above, such that $d''_E \beta = 0$, then

$$\int_N \beta = \langle P, \beta \rangle = 0 . \tag{1}$$

Remark that $\int_N \beta$ has a meaning for every $n \geqslant 3$.

Let $k: E \to R$ be the mapping defined by $(x_1, z_2, z_3) \mapsto x_1$ and $k^{-1}(x_1) = Q_{x_1} \simeq C^2$.

Proposition 1. *For every x_1 such that the slice $\nu = \langle P, k, x_1 \rangle$ is defined, ν is the direct image by i: $Q_{x_1} \hookrightarrow E$ of a 1-cycle π of Q_{x_1}, with compact support, satisfying the classical moment condition in Q_{x_1}.*

In C^2, let $\pi \in \mathcal{E}'^{\bullet}(C^2)$ (current with compact support), to solve

$$d'' s^{0,1} = \pi^{0,2} \tag{2}$$

with $\operatorname{supp} s^{0,1}$ compact, we check that the d″-cohomology class $[\pi^{0,2}]$ belongs to $H_c^{0,2}(C^2, C) \simeq H_c^2(C^2, \mathcal{O}) \simeq (H^0(C^2, \Omega^2))'$ (Serre duality).

Then equation (2) has a compact solution if and only if we have

$$\langle \pi^{0,2}, w\rangle = 0 \qquad \text{for every } w \in H^0(\boldsymbol{C}^2, \Omega^2). \tag{3}$$

Consider the (Cauchy) kernel $K^C(z_2, z_3) = \delta_0(z_3) \otimes \frac{1}{\pi z_2} \frac{\partial}{\partial \bar z_2}$ in $\boldsymbol{C}^2(z_2, z_3)$.

Lemma 1. *If $\pi_{0,2}$ satisfies* (3), *and if # means the convolution-contraction, then K^C # $\pi^{0,2}$ is the solution of* (2) *with compact support.*

Proposition 2. *If N is C^ω and if P satisfies the* CR*-moment condition, then* $\mathrm{d}''_E S^{0,1} = -P^{0,2}$ *has a solution with compact support which is C^ω in x_1.*

For the proof, in $Q_{x_1} \simeq \boldsymbol{C}^2$, consider the kernel K^C, use Lemma 1 and, in E, use the convolution kernel $K = \delta_0 \otimes K^C$.

Theorem 1. *Let N be a compact, C^ω,* CR*-subvariety with negligible singularities of $\boldsymbol{C}^n$, contained in E, such that* $\dim N = 2$, CR-$\dim N = 0$. *Assume that N satisfies the* CR*-moment condition and that:*

H. *There exists a closed subset τ of N such that $\mathcal{H}^2(\tau) = 0$ and, for every $z \in N \setminus \tau$, $N \setminus \tau$ is a submanifold transverse to the maximal complex affine subspace of E through z*

β. *Either N is smooth or N is the intersection of E and of a maximally complex subvariety with negligible singularities.*

Then there exists a unique C^ω maximally complex 3*-chain M of $\boldsymbol{C}^n \setminus N$,* $\operatorname{supp} M \subset\subset E$, *of finite mass and having a simple extension to $\boldsymbol{C}^n$, still denoted M, such that* $N = \mathrm{d}M$ *and M is foliated by holomorphic* 1*-chains.*

Proof. For $p = 2$, $n = 3$, analogous to the proof for $p \geqslant 3$, using Proposition 2. For $n > 3$, use the classical projection method.

3 - Case of a half-space of E with complex boundary

Definitions. Let $U'' = \{z \in \boldsymbol{C}^n; \Re e\, z_1 > 0\} \subset \boldsymbol{C}^n$ and $U = \{(x_1, z_2, \ldots, z_n) \in E, x_1 > 0\}$; $\partial U = \{z \in \boldsymbol{C}^n, z_1(z) = 0\} \simeq \boldsymbol{C}^{n-1}$. Let N be a CR-subvariety of U'', with negligible singularities, contained in U, defining a d-closed integration current denoted also N, of finite mass, and such that $N \subset\subset E$, $\dim N = 2p-2$, CR-$\dim N = p-2$. Assume $p = 2$, $n = 3$. We look for a chain M of $U'' \setminus N$, $\operatorname{supp} M \subset\subset E$, of finite mass, $\dim M = 3$, CR-dim $M = 1$, having a simple extension to U'', still denoted M, such that $dM = N$.

Let j be the restriction to U of the canonical injection $E \hookrightarrow \boldsymbol{C}^3$, then there exist well defined rectifiable currents P and S in U such that $N = j_\# P$, $M = j_\# S$, having the same expressions as in Section 2 and satisfying $dS = P$.

But these currents also act on the space $\mathcal{E}^\bullet_+(U)$ of differential forms φ of class C^∞ on U such that $\operatorname{supp}\varphi$ is contained in $\{x_1 > \delta\}$ for $\delta > 0$ small enough. For every 2-form $\beta = dx_1 \wedge \varphi^{1,0} + \psi^{2,0} \in \mathcal{E}^\bullet_+(U)$, the expression $\int_N \beta = \langle P, \beta\rangle$ makes sense and is equal to $\langle dS, \beta\rangle = -\langle S, d\beta\rangle = -\langle S, d''_E\beta\rangle$. If $d''_E\beta = 0$, then $\langle P, \beta\rangle = 0$.

Definition. N (or P) satisfies the CR-*moment condition* if

$$\text{(4)} \qquad \text{for every } \beta \in \mathcal{E}^\bullet_+(U), \text{ such that } d''_E\beta = 0, \text{ then } \int_N \beta = \langle P, \beta\rangle = 0\,.$$

A subvariety N with negligible singularities of U'' defined at the beginning of this section, and satisfying the CR-moment condition, has the properties described in Section 2, $\boldsymbol{C}^3$ beeing replaced by U'', E by U, and the compactness of N by the relative compactness in E.

4 - Case of a half-space of E with non complex boundary

Definitions. Let w'' be a real linear form on $\boldsymbol{C}^n$ and w its restriction to E. We assume that the real hyperplane $F'' = \{z \in \boldsymbol{C}^n \mid w''(z) = 0\}$ is different from E, then $F = F'' \cap E = \{\zeta \in E \mid w(\zeta) = 0\}$ is a real hyperplane of E; $\dim E = 2n-2$ and CR-dim $F = n-2$, in general. Let $W'' = \{z \in \boldsymbol{C}^n \mid w''(z) > 0\}$ and $W = \{\zeta \in E \mid w(\zeta) > 0\}$ be the half-spaces of $\boldsymbol{C}^n$ and E respectively. In what follows, we assume CR-dim $F = n-2$.

Let G be the maximal complex linear subspace of F, then $G \simeq \boldsymbol{C}^{n-2}$; G is contained in the maximal complex linear subspace H of E. Let $\boldsymbol{C}_{z_1}$ be the z_1-axis of $\boldsymbol{C}^n$ and $A = G \oplus \boldsymbol{C}_{z_1}$; then $E' = E \cap A$ is a real hyperspace of A and $F' = F \cap A = G \simeq \boldsymbol{C}^{n-2}$. $W' = \{\zeta \in E' \mid w(\zeta) > 0\}$ is a half-space of E' with complex boundary as in Section 3, for the dimension $n-1$.

Let $(u_2, \dots, u_{n-1})$ be complex coordinates of G, then $(z_1; u_2, \dots, u_{n-1})$ are complex coordinates of A and $F' = \{z_1 = 0\} \subset A$. Let $\boldsymbol{C}_n$ be a supplement of A in $\boldsymbol{C}^n$, with complex coordinate u_n and $h\colon \boldsymbol{C}^n \to \boldsymbol{C}_n$ defined by $z \mapsto u_n(z)$ the projection. Then $h^{-1}(0) = A$ and $h^{-1}(u_n) = A_{u_n} \simeq A \simeq \boldsymbol{C}^{n-1}$. We set $h_E = h|_E$, then $h_E^{-1}(u_n) = E'_{u_n} \simeq E \cap A_{u_n}$.

CR-moment condition. Let N be a C^1 subvariety of W'', with negligible singularities such that $N \subset W$, $N \subset\subset E$, $\dim N = 2p-2$, CR-dim $N = p-2$ with finite $(2p-2)$-dimensional volume. Moreover denote also by N the integration current defined by N and assume $\mathrm{d}N = 0$. Suppose $p = 3$, and that there exists a 5-chain M of $W'' \setminus N$, with $\operatorname{supp} M \subset\subset E$ such that CR-dim $M = 2$, and having a simple extension, still denoted M, such that $\mathrm{d}M = 0$. Moreover we assume $n = 4$.

Define currents P and S of W, as in section 3, acting on the space $\mathcal{E}^{\bullet}_{+}(W)$ of C^{∞} differential forms β of W, with $\operatorname{supp} \beta \subset \{\zeta \in E \mid w(\zeta) > \delta\}$ for $\delta > 0$ small enough depending on β. For every 4-form $\beta = \mathrm{d}x_1 \wedge \varphi^{2,1} + \psi^{3,1} \in \mathcal{E}^{\bullet}_{+}(W)$ we have $\int_N \beta = \langle P, \beta \rangle = -\langle S, \mathrm{d}''_E \beta \rangle$. If $\mathrm{d}''_E \beta = 0$, then $\langle P, \beta \rangle = 0$.

Definition. N (or P) satisfies the CR-*moment condition* if

(5) for every $\beta \in \mathcal{E}^{\bullet}_{+}(W)$, as above, such that $\mathrm{d}''_E \beta = 0$, then $\int_N \beta = \langle P, \beta \rangle = 0$.

$\int_N \beta$ has a meaning for every $n \geq 4$.

Proposition 3. *For every $u_4 \in \mathbf{C}$ such that $\mu = \langle P, h_E, u_4 \rangle$ is defined, μ is the direct image by $i\colon E'_{u_4} \to E$ of a 2-cycle ν satisfying the* CR-*moment condition in E'_{u_4}.*

Proposition 4. *In the half-space W of E, if N is C^{ω} and if the current P satisfies the* CR-*moment condition, then the equation $\mathrm{d}''_E S^{0,1} = -P^{0,2}$ with $\mathrm{d}''_E P^{0,2} = 0$ has a solution $U^{0,1}$ in W, C^{ω} in x_1, such that* $\operatorname{supp} U^{0,1} \subset\subset E$.

The proof uses the solution with compact support of the d''_E-equation in $E' = E'_{u_4}$ (Section 2, 3), where E' is of dimension 5, that is why, in this section, we have to assume $\dim E = 7$ and $\dim N = 4$.

Theorem 2. *Let N be a subvariety with negligible singularities of class C^{ω} of W, with* $\dim N = 4$, CR-dim $N = 1$, *finite* 4-*dimensional volume, $N \subset\subset E$, satisfying the* CR-*moment condition, condition* H *and condition* β, *as in Theorem* 1. *Then there exists a unique C^{ω} maximally complex* 5-*chain M in $W'' \setminus N$, of finite mass, such that* $\operatorname{supp} M \subset W$, $\operatorname{supp} M \subset\subset E$, *and having a simple extension still denoted M in W'', satisfying*:

i $\mathrm{d}M = N$

ii *M is foliated by holomorphic* 1-*chains.*

Proof. Thanks to the projection method, it is enough to consider the case $p = n - 1 = 3$. Using Proposition 4, we compute the coefficients as in [2], 3.5. The proof ends as for Theorem 6.9 of [1] for $p = n - 1$. The unicity of M and the existence of the foliation result from the slicing relative to the projections $k\colon E \to \mathbf{R}$ and h_E.

Corollary 3.1 of [2] can be extended for $p = 3$, N satisfying the CR-moment condition.

5 - Boundary problem in CP^2 and moment condition in $CP^2 \setminus CP^1$.

Let γ be a closed, oriented curve of class C^2 (and the integration current defined by the curve) in CP^2. We look for a holomorphic 1-chain S in $CP^2 \setminus \gamma$ such that exists a simple extension of S, still denoted S, to CP^2 satisfying $\gamma = bS$.

Let (w_0, w_1, w_2) be homogeneous coordinates in CP^2, chosen in such a way that $\gamma \cap \{w_0 = 0\} = \emptyset$; $z_j = \dfrac{w_j}{w_0}$, $j = 1, 2$, be affine coordinates in $CP^2 \setminus \{w_0 = 0\} \simeq \mathbf{C}^2$, $\tilde{g} = w_2 - \xi w_0 - \eta w_1$ and $g = \dfrac{\tilde{g}}{w_0} = z_2 - \xi - \eta z_1$. Let $D(\xi, \eta)$ be the projective line $\tilde{g} = 0$; when $(\xi, \eta) \in \mathbf{C}^2$, $D(\xi, \eta)$ describes a Zariski open set of $(CP^2)'$.

In $CP^2 \setminus \{w_0 = 0\} \simeq \mathbf{C}^2$, consider the affine lines, $D'(\xi, \eta) = D(\xi, \eta) \cap \mathbf{C}^2$.

Lemma 2. *Let Σ be a Riemann surface embedded into an open set of $\mathbf{C}^2$. Let $(\xi^*, \eta^*) \in \mathbf{C}^2$ such that $D'(\xi^*, \eta^*) \cap \Sigma$ is a finite set, for (ξ, η) in a small enough neighborhood of (ξ^*, η^*), then:*

$D'(\xi,\eta) \cap \Sigma$ is a finite set with fixed number of points $(f_j(\xi,\eta), \xi + \eta f_j(\xi,\eta))$, $j = 1, \ldots, N$

f_j is holomorphic and satisfies

$$f_j \frac{\partial f_j}{\partial \xi} = \frac{\partial f_j}{\partial \eta} \qquad j = 1, \ldots, N. \tag{6}$$

Conversely, if, in a neighborhood of (ξ^*, η^*), $f(\xi, \eta)$ is holomorphic and satisfies (6), then the point $(f(\xi, \eta), \xi + \eta f(\xi, \eta))$ generates a Riemann surface embedded into an open set of $\mathbf{C}^2$.

For the proof, see [3], Lemme 2.3 and [9], Theorem 1.

Lemma 3. *Let γ' be a compact oriented curve of class C^2 in $\boldsymbol{C}^2$, then the following properties are equivalent:*

1. $\int_{\gamma'} z_1 \frac{\mathrm{d}g}{g} = 0$

2. *γ' satisfies the moment condition in $\boldsymbol{C}^2$.*

For the proof, see [3], Corollaire 1.3 and [4] Section 4, for a different proof.

According to ideas from J. B. Poly [4], Section 3, we can reduce the proof of the following theorem ([3], Theorem 1.2) to Wermer's theorem.

Theorem 3. *Under the hypotheses and notations at the beginning of this section, the following conditions are equivalent:*

i *there exists a holomorphic* 1*-chain S such that $\gamma = bS$.*

ii *there exist $(\xi^*, \eta^*) \in \boldsymbol{C}^2$, holomorphic functions $f_j(\xi, \eta)$, $j = 1, \dots, N$ on a neighborhood of (ξ^*, η^*) and constants $\varepsilon_j = \pm 1$ satisfying:*

$$f_j \frac{\partial f_j}{\partial \xi} = \frac{\partial f_j}{\partial \eta} \qquad j = 1, \dots, N \tag{7}$$

$$G(\xi, \eta) = \frac{1}{2\pi i} \int_{\gamma} z_1 \frac{\mathrm{d}g}{g} = \sum_{j=1}^{N} \varepsilon_j f_j . \tag{8}$$

Proof.

i$\Rightarrow$ii: the existence of the functions f_j satisfying (7) comes from Lemma 2. Let $p_j^* = (f_j(\xi^*, \eta^*), \xi^* + \eta^* f_j(\xi^*, \eta^*))$, Δ_j be a small enough disc on supp S, centered at p_j^* and $\Gamma_j = b\Delta_j$, then $\gamma - \sum \varepsilon_j \Gamma_j = b(S - \sum \varepsilon_j \Delta_j)$ in $\boldsymbol{C}P^2 \setminus D(\xi^*, \eta^*) \cong \boldsymbol{C}^2$, and satisfies the moment condition. (8) follows from Lemma 3.

ii$\Rightarrow$i: from **ii** and Lemma 2, for (ξ, η) in a convenient neighborhood of (ξ^*, η^*), $p_j = (f_j(\xi, \eta), \xi + \eta f_j(\xi, \eta))$ generates a connected Riemann surface Σ_j embedded in an open set of $\boldsymbol{C}^2$ such that, for $j \neq k$, either $\Sigma_j = \Sigma_k$ or $\Sigma_j \cap \Sigma_k \neq \emptyset$. Let Δ_j be a disc of Σ_j centered at p_j^* and $\Gamma_j = b\Delta_j$; from Lemma 3 and condition (8), $\gamma - \sum \varepsilon_j \Gamma_j$ satisfies the moment condition in $\boldsymbol{C}P^2 \setminus D(\xi^*, \eta^*) \simeq \boldsymbol{C}^2$. Then, from Wermer's theorem, there exists a holomorphic 1-chain T of $\boldsymbol{C}^2$ such that $\gamma - \sum \varepsilon_j \Gamma_j = bT$; so $\gamma = b(T + \sum \varepsilon_j \Delta_j)$. From the structure theorem of Harvey-Shiffman [8], $S = T + \sum \varepsilon_j \Delta_j$ is a holomorphic 1-chain of $\boldsymbol{C}P^2 \setminus \gamma$.

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