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Positive bundles over foliations with complex leaves (**)

1 - Introduction

In this paper we are dealing with positive CR-vector bundles over a foliation X with complex leaves ([3]). We give the notion of *strictly q-pseudoconvexity* and we focus our attention on those foliations for which N_F, the transverse bundle to the leaves is *q-positive* (Sec. 2).

Let ϕ: $X \to \boldsymbol{R}$ be an exhaustion function for X which is q-strictly plurisubharmonic along the leaves of X, outside a compact set K and $\overline{X}_c = \{\phi \leq c\}$ (Sec. 2). Then we prove:

i. if X is real analytic, N_F is q-positive and $\sup_K \phi < c$, $\overline{X}_c = \{\phi \leq c\}$ has a fundamental system of neighbourhoods in $\widetilde{X}$ (the complexification of X) which are strictly q-pseudoconvex complex manifolds (Theorem 1).

Let $L \to X$ be a CR-bundle of rank one, L' its dual, $\mathcal{L}_{an}$ the sheaf of germs of real analytic CR-sections of L. Then we have the vanishing theorem

ii. under the hypothesis of **i**, if L' is positive, then the groups $H^r(X_c, \mathcal{L}_{an}{}^s)$ vanish for $r \geq q$ and $s \gg 0$ (Theorem 3).

2 - Preliminaires

Let X be a foliation with complex leaves of dimension n and real codimension d ([3]). Let $\{U_j\}$ be a distinguished open covering of X, $U_j = V_j \times B_j$, $V_j \subset \boldsymbol{C}^n$,

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$B_j \subset \boldsymbol{R}^d$ and

$$z_j = f_{jk}(z_k, t_k) \qquad t_j = h_{jk}(t_k) \tag{2.1}$$

be the local transformations of coordinates: $z_k \in V_k$, $t_j \in B_j$, f_{jk}, h_{jk} are smooth and f_{jk} is holomorphic with respect to z_k.

We denote by $\mathcal{D} = \mathcal{D}_X$ the sheaf of germs of smooth CR-functions on X and when X is real analytic, $\mathcal{O}' \subset \mathcal{D}$ denotes the subsheaf of those germs which are real analytic.

The matrices $\dfrac{\partial h_{jk}}{\partial t_k} = \dfrac{\partial(h_{jk}^1, \ldots, h_{jk}^d)}{\partial(t_k^1, \ldots, t_k^d)}$ determine a real vector bundle N_F, the *transverse bundle* to the leaves of X.

Let us suppose that X is real analytic. Then the complexification $\widetilde{X}$ of X in the sense of Whitney and Bruhat carries in a natural way a structure of holomorphic foliation such that the leaves of X are leaves of $\widetilde{X}$ ([3]).

To construct $\widetilde{X}$ we complexify $\boldsymbol{R}^d$ and $\{U_j\}$ in such a way to obtain $\{\widetilde{U}_j\}$, $\widetilde{U}_j \subset \boldsymbol{C}^n \times \boldsymbol{C}^d$. Next we patch together the $\widetilde{U}_j$'s by the local holomorphic transformations

$$z_j = \widetilde{f}_{jk}(z_k, \tau_k) \qquad \tau_j = \widetilde{h}_{jk}(z_k, \tau_k) \tag{2.2}$$

which are obtained complexifying f_{jk}, h_{jk} with respect to t_k.

In particular the transverse bundle to the leaves of $\widetilde{X}$, $\widetilde{N}_F$ is determined by $\dfrac{\partial \widetilde{h}_{jk}}{\partial \tau_k} = \dfrac{\partial(\widetilde{h}_{jk}^1, \ldots, \widetilde{h}_{jk}^d)}{\partial(\tau_k^1, \ldots, \tau_k^d)}$.

Let z_j, τ_j denote the complex coordinates on $\widetilde{U}_j$ and let $\theta_j = \operatorname{Re} \tau_j$. Then on $\widetilde{U}_j \cap \widetilde{U}_k$ we have $\theta_j^s = \operatorname{Im} h_{jk}^s(\tau_k)$, $1 \leq s \leq d$ and consequently, since $\operatorname{Im} \widetilde{h}_{jk}^s = 0$, $1 \leq s \leq d$, $\theta_j^r = \sum_{s=1}^{d} \psi_{jk}^{rs} \theta_k^s$, where $\psi_{jk} = (\psi_{jk}^{rs})$ is a $d \times d$ invertible matrix whose entries are real analytic functions on $\widetilde{U}_j \cap \widetilde{U}_k$. Moreover, since $\widetilde{h}_{jk}$ is holomorphic and $\widetilde{h}_{jk|X} = h_{jk}$, we also have $\psi_{jk|X} = \dfrac{\partial h_{jk}}{\partial t_k}$.

In what follows we prove a vanishing theorem for *positive CR-vector bundles*. One of the main points in the proof is the local existence of a fundamental system of strongly q-pseudoconvex neighbourhoods of X in $\widetilde{X}$ and this is related to the *positivity* of N_F.

3 – q-pseudoconvex foliations

We recall that a function $\phi = \phi(z)$ is said to be *strictly q-plurisubharmonic* (strictly q-p.s.h) if its Levi form

$$\sum \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \xi^j \bar{\xi}^k$$

has at least $n - q + 1$ positive eigenvalues.

A foliation X is said to be *strictly q-pseudoconvex* (strictly q-p.c.) if X carries a smooth exhaustion function ϕ: $X \to \boldsymbol{R}^+$ which is strictly q-p.c. along the leaves outside a compact $K \subset X$, $1 \leqslant q \leqslant n + 1$ and $\sup \phi = +\infty$. In particular, if $K = \emptyset$, X is said to be a *q-complete foliation* ([3]).

Consider a metric $\{\lambda_j^0\}$ on the fibres of N_F. For every j, $\{\lambda_j^0\}$ is a smooth map from U_j into the space of positive definite symmetric $d \times d$ matrices such that

$$\lambda_k^0 = \frac{{}^t\partial h_{jk}}{\partial t_k} \lambda_j^0 \frac{\partial h_{jk}}{\partial t_k} .$$

If ∂ and $\bar{\partial}$ denote the complex differentiations along the leaves of X then the local tangential forms

$$\omega_j = \bar{\partial}\partial \log \lambda_j^0 - \bar{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0$$

actually give a global tangential form ω. N_F is said to be *q-positive* (along the leaves of X) if the functions λ_j^0 can be chosen in such a way that the hermitian form associated to ω has at least $n - q + 1$ positive eigenvalues.

Remark 1. If l is a leaf of X and $N_{F|l}$ denotes the restriction of N_F to l then $\{\lambda_{j|l}^0\}$ gives a *hermitian metric* on the fibres of $N_{F|l} \oplus \boldsymbol{C}$ and

$$\omega_j + \bar{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0$$

is the *curvature form* of this metric.

Let X be a real analytic strictly q-p.c. (with exhaustion function ϕ) and let $\overline{X}_c = \{\phi \leqslant c\}$, $\sup_K \phi \leqslant c$. We have

Theorem 1. *If N_F is q-positive, $\overline{X}_c$ has a fundamental system of neighbourhoods in $\widetilde{X}$ which are strictly q-p.c. complex manifolds.*

Proof. Let N_F be q-positive and let $\{\lambda_j^0\}$ be a smooth metric on the fibres of N_F. Let E be the vector bundle on $\tilde{X}$ defined by the cocycle ψ_{jk} (Sec. 2) and let $\{\mu_j\}$ be a metric on the fibres of E. On $\tilde{U}_j \cap \tilde{U}_k$ we have

$$\mu_k = {}^t\psi_{jk}\, \mu_j \psi_{jk}$$

and consequently, if $\mu_j^0 = \mu_{j|X}^0$ we have on X

$$\mu_k^0 \lambda_k^{0^{-1}} = \frac{\partial h_{jk}}{\partial t_k} \mu_j^0 \lambda_j^{0^{-1}} \frac{{}^t\partial h_{jk}}{\partial t_k}^{-1} .$$

Thus $\sigma^0 = \{\mu_j^0 \lambda_j^{0^{-1}}\}$ is a smooth section of $N_F' \otimes N_F'$, N_F' the dual of N_F. Extend σ^0 by a smooth section $\sigma = \{\sigma_j\}$ of $E \otimes E$. Then $\{\sigma_j^{0^{-1}} \mu_j^0\}$ is a new metric on the fibres of E whose restriction to X is $\{\lambda_j^0\}$. In particular E is q-positive on a neighbourhood of X provided N_F is.

Now consider on $\tilde{X}$ the smooth function ρ locally defined by $\theta_j^t \lambda_j \theta_j$; ρ is non negative and positive outside X. Drop the subscripts and compute the Levi form $\mathcal{L}(\rho)$ of $\theta^t \lambda \theta = \sum \lambda_{\alpha\beta} \theta^\alpha \theta^\beta$ with respect to $z_1, \ldots, z_n, \tau_1, \ldots, \tau_d$. Denoting by $\xi^1, \ldots, \xi^n, \eta^1, \ldots \eta^d$ the variables of $\mathcal{L}(\rho)$ we have

$$\mathcal{L}(\rho) = \sum A_{jk} \xi^j \bar{\xi}_k + \sum B'_{\alpha\beta} \eta^\alpha \bar{\eta}_\beta + 2 \operatorname{Re} \sum C'_{j\alpha} \xi^j \bar{\eta}_\alpha$$

where

$$A_{jk} = \sum \frac{\partial^2 \lambda_{\alpha\beta}}{\partial z_j \partial \bar{z}_k} \theta^\alpha \theta^\beta$$

$$B'_{\alpha\beta} = \frac{1}{2} \lambda_{\alpha\beta} + O(|\theta|) = B_{\alpha\beta} + O(|\theta|)$$

$$C'_{j\alpha} = i(\sum \frac{\partial \lambda_{\beta\alpha}}{\partial z_j} \theta^\beta) + O(|\theta|^2)) = i(C_{j\alpha} + O(|\theta|^2)).$$

It follows that if $|\theta|$ is small enough and the $(n+d) \times (n+d)$ matrix

$$H(\rho) = \begin{pmatrix} A & B \\ {}^tC & D \end{pmatrix}$$

where $A = (A_{jk})$, $B = (B_{\alpha\beta})$, $C = (C_{j\alpha})$ is positive definite, then $\mathcal{L}(\rho)$ is also strictly positive definite.

Since these conditions are homogeneous in θ, we deduce that the positivity of $H(\rho)$ will follow from that of the matrix $H^0(\rho)$ obtained from $H(\rho)$ replacing A_{jk},

$B_{\alpha\beta}$, $C_{j\alpha}$ respectively by

$$A^0_{jk} = {}^t\theta \frac{\partial \lambda^0_{\alpha\beta}}{\partial z_j \partial z_k} \theta \qquad B^0_{\alpha\beta} = \frac{1}{2} \lambda^0_{\alpha\beta} \qquad C^0_{j\alpha} = i \sum \frac{\partial \lambda^0_{\beta\alpha}}{\partial z_j} \theta^\beta .$$

Now it is a simple matter to check that the hermitian form associated to $H^0(\rho)$ can be written

$$ {}^t\theta({}^t\bar{\xi}\Omega\xi)\,\theta + |2i(\lambda^0)^{-\frac{1}{2}}(\partial\lambda^0 \cdot \xi)\,\theta + (\lambda^0)^{\frac{1}{2}}\eta|^2$$

where $\partial\lambda^0 \cdot \xi$ is the matrix $\sum \frac{\partial \lambda}{\partial z_j} \xi^j$ and $\Omega = \lambda\omega$.

This shows that $H^0(\rho)$ has at least $n - q + 1$ positive eigenvalues in a neighbourhood of X and consequently that if ε is small the domains $\{\rho \leqslant \varepsilon\}$ are strictly q-p.c.

To produce a fundamental system of strictly q-p.c. neighbourhoods of $\overline{X}_c$, $\sup_K \phi < c$ we use ϕ as in [3] to construct a function ψ, which is strictly q-p.c. in a neighbourhood of $\overline{X}_c \setminus K$ in $\widetilde{X}$ and such that $\bigcap_{\varepsilon > 0} [\{\psi \leqslant \varepsilon\}] = \overline{X}_c$. Then the domains $\{\sup(\rho, \psi) < \varepsilon\}$ have the desired properties.

Remark 2. In particular, if X is a Stein foliation every $\overline{X}_c$ has a fundamental system of Stein neighbourhoods in $\widetilde{X}$ ([3]).

For $q = 1$ the above definition of positivity is too strong. Indeed suppose for simplicity $d = 1$. Then due to the fact that h_{jk} does not depend on z, $\omega = \{\bar{\partial}\partial \log \lambda^0_j\}$ and $\eta = \{\partial \log \lambda^0_j\}$ are global tangential forms on X and $\omega = \mathrm{d}\eta$. If N_F is 1-positive, according to the above definition ω is a d-exact Kähler form on each leaf of X and from this it follows that X has no positive dimension compact submanifold. In particular compact leaves cannot be present. On the other hand, in view of Theorem 1 for $c \geqslant \sup_K \phi$, $\overline{X}_c$ has a fundamental system of strictly 1-p.c. manifolds U. Since all these manifolds are obtained as point modifications of Stein spaces ([5]), from the above discussion it follows that U is in fact a Stein manifold. Conclusion: X is an increasing union of Stein foliations.

Similarly we see that for no compact foliation N_F can be 1-positive (according to the above definition). Thus for $q = 1$ we say that N_F is *1-positive* if for every $c > \sup_K \phi$, $\overline{X}_c$ has in $\widetilde{X}$ a fundamental system of neighbourhoods which are strictly 1-pseudoconvex.

Example 1. Consider in C^2 the subset $X' = \{|z_1| = |z_2|\}$. X' is a *singular foliation* with complex leaves, smooth for $z \neq 0$. Let π: $\tilde{C}^2 \to C^2$ be the blowing-up of C^2 at 0 and X be the proper transformation of X'. X is smooth and foliated by complex curves. Moreover, a small neighborhood U of X in $\tilde{C}^2$ is a complexification of X and there X is locally defined by $\theta_j = 0$ and $\theta_j = -\theta_k$ if $j \neq k$. It follows that $\rho = \{\theta_j^2\}$ is smooth and p.s.h. on U. Let $\phi(w) = |\pi(w)|^2$, $w \in U$ and $\overline{X}_c = \{\phi \leq c\} \cap X$. Then for c large $\psi = \phi - c + \rho$ is strictly p.s.h. in U and $\{\psi \leq \varepsilon\} \cap U$, $\varepsilon > 0$ gives a fundamental system of strictly 1-pseudoconvex neighbourhoods of $\overline{X}_c$. In particular N_F is 1-positive.

4 - Positive bundles

We consider for simplicity CR-bundles $L \to X$ of rank one. All results hold for arbitrary rank.

We suppose that L is real analytic i.e. the cocycle $\{g_{jk}\}$ is given by real analytic CR-functions ([4]). Then the total space of L is a foliation whose local transformations are

$$z_j = f_{jk}(z_k, t_k) \qquad \zeta_j = g_{jk}(z_k, t_k)\,\zeta_k \qquad t_j = h_{jk}(t_k).$$

We denote by $\mathcal{L}$ and $\mathcal{L}_{an}$ respectively the sheaves of germs of smooth and real analytic CR-sections of L.

Let $\tilde{L}$ be the complexification of L. In [4] we proved that if X is a Stein foliation then $H^r(X, \mathcal{L}) = 0$ for $r \geq 1$. Here we discuss the case when X is strictly q-p.c., in particular X compact.

$\tilde{L}$ is a holomorphic line bundle over X. L is said to be *positive* (along the leaves) in the sense of Kodaira, if there exists a smooth metric $\{\lambda_j\}$ on the fibres of L with *positive curvature* i.e. such that the hermitian form

$$\sum_{r,s=1}^{n} \frac{\partial^2 \log \lambda_j}{\partial z_r \partial \bar{z}_s}\, \xi^r \bar{\xi}^s$$

is strictly positive definite.

Then it is easily seen that $\{\lambda_j\}$ extends to a metric on $\tilde{L}$ in such a way that $\tilde{L}$ is positive on a neighbourhood of X ([4]). We have

Theorem 2. *Suppose that X is strictly q-pseudoconvex, N_F is q-positive and L is positive. Then, for every $c > \sup_K \phi$, $\overline{X}_c$ has in $\tilde{L}$ a fundamental system of neighbourhoods which are strictly q-pseudoconvex complex manifolds.*

Proof. In view of Theorem 1, $\overline{X}_c$ has in $\widetilde{X}$ a fundamental system of neighbourhoods which are strictly q-pseudoconvex manifolds and $\widetilde{L}$ is positive on a neighbourhood U of $\overline{X}_c$.

In such conditions, $\widetilde{L}|_U$ is a strongly q-pseudoconvex complex manifold ([2]) and from the proof given there, it follows that $\overline{X}_c$ has in $\widetilde{L}$ a fundamental system of neighbourhoods, which are strictly q-pseudoconvex complex manifolds.

Let $\mathcal{O}'$ be the sheaf of germs of real analytic CR-functions on L. Then

Corollary 1. *Under the above conditions every $\overline{X}_c$ has a fundamental system $\{U_m\}$ of neighbourhoods in L such that $H^r(\overline{U}_m, \mathcal{O}')$ is finite dimensional for $r \geqslant q$ and its dimension is independent on m.*

Proof. Let $\{\lambda_j\}$ be a smooth metric as above and let $\|v\|^2 = \lambda_j \zeta_j \overline{\zeta}_j$, $v \in \widetilde{L}$ and $W_c = \{v \in \widetilde{L}\colon \|v\|^2 \leqslant c\}$, $U_m = W_{\frac{1}{m}} \cap L$.

Since every germ of $\mathcal{O}'$ extends by a germ of $\mathcal{O}$ (the structure sheaf of $\widetilde{L}$) we have that $H^r(\overline{U}_m, \mathcal{O}')$ is the direct limit of the groups $H^r(\overline{W}_{\frac{1}{m}}, \mathcal{O})$. Thus our statement easily follows from the isomorphism theorem of Andreotti and Grauert ([1]).

Remark 3. The same statement holds when $\mathcal{O}'$ is replaced by a power of $\mathcal{M}$, the sheaf of the ideals of a finite set of points of X.

Now let $U = W \cap L$ be a neighbourhood of $\overline{X}_c$ where W is a relatively compact neighbourhood of $\overline{X}_c$ in $\widetilde{L}$, $W = \rho < 0$, ρ smooth and strictly p.s.h. outside a compact of W. Then in view of the above corollary the groups $H^r(\overline{U}_m, \mathcal{O}')$ are finite dimensional for $r \geqslant q$ and their dimension is independent on U.

Let $\{K_m\}$ be a covering of X such that $L|_{K_m}$ is trivial and K_m is isomorphic to $\{|z| \leqslant 1\} \times \{|t| \leqslant 1\}$; let $\overline{U}_m = \pi^{-1}(K_m) \cap \overline{U}$ and $\mathcal{U}$ be the covering $\{U_m\}$ of $\overline{U}$. Since every $\overline{U}_m$ has a fundamental system of neighbourhoods which are Stein manifolds Leray's theorem implies that $H^r(\mathcal{U}, \mathcal{O}') \simeq H^r(\overline{U}, \mathcal{O}')$ for $r \geqslant 1$. Furthermore the groups $H^r(\mathcal{U}, \mathcal{O}')$ have a natural filtration such that the graded group associated to this filtration is isomorphic to $\bigoplus_{s \geqslant 0} H^r(X, \mathcal{L}'^s_{an})$, $\mathcal{L}'_{an}$ being the sheaf of germs of real analytic CR-sections of the dual bundle L'. Thus from the above discussion we deduce

Theorem 3. *Under the hypothesis of Theorem 2, if L' is positive then the groups $H^r(X_c, \mathcal{L}^s_{an})$ vanish for $r \geqslant q$ and $s \gg 0$ for every $c > \sup_K \phi$.*

References

[1] A. ANDREOTTI and H. GRAUERT, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193-259.

[2] A. ANDREOTTI and G. TOMASSINI, *A remark on the vanishing of certain cohomology groups*, Compositio Math. **21** (1969), 417-430.

[3] G. GIGANTE and G. TOMASSINI, *Foliations with complex leaves*, Differential Geom. Appl., to appear.

[4] G. GIGANTE and G. TOMASSINI, *$G_{m,h}$-bundles over foliations with complex leaves*, to appear.

[5] H. GRAUERT, *On Levi's problem and the embedding of real-analytic manifolds*, Ann. of Math. **68** (1958), 460-472.

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