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Isocurved deformations of Riemannian homogeneous metrics (**)

1 - Introduction

It is very simple to construct deformations of a Riemannian metric on a manifold (M, g), but it is much more difficult to keep the curvature or some other Riemannian invariants of the deformed metric under control. Often the expected result is the *rigidity* of the special Riemannian structure which we are concerned with, namely the impossibility to construct non-trivial deformations remaining in the same special class. More generally, some *finiteness* of the space of the non-trivial deformations is often conjectured, and sometimes proved. We refer to [2] for a discussion of the case of Einstein metrics, and to [1], [5] for a conjecture of Gromov closely related to the subject developed in the present paper.

Here we construct and study deformations of Riemannian homogeneous metrics which preserve the Riemann curvature in a sense specified below. We were inspired by some examples constructed in [3], and motivated by the aim to complete the study started there.

In order to state our results in a precise form, it is convenient to consider the Riemannian curvature R as a map defined on the total space OM of the orthonormal frame bundle of (M, g) with values in the vector space $R(V)$ of alge-

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braic curvature tensor on $V = \boldsymbol{R}^n$. Such a map is defined by

$$R(u)(\xi_1, \xi_2, \xi_3, \xi_4) = (R_{\pi(u)})_{u\xi_1, u\xi_2, u\xi_3, u\xi_4} \qquad \xi_i \in \boldsymbol{R}^n = V$$

where the elements u of OM are just the isometries between the Euclidean vector space V endowed with the standard inner product, and the tangent space T_pM at $p = \pi(u)$, π being the projection of OM onto M. The orthogonal group $O(n)$ is acting on the right on OM, on the left on $R(V)$ and the map R is *equivariant* w.r.t. such actions. If (M, g) is *homogeneous*, $R(OM)$ is contained in one single orbit. The converse is not true, and the manifolds having this property are called *curvature homogeneous*. Thus, if (M, g) is curvature homogeneous, we have

$$R(OM) \subset O(n)K$$

where K is a fixed element of $R(V)$. If K is the curvature tensor of some homogeneous space $G/H = M_0$ endowed with a G-invariant metric g_0, we say that (M, g) has the *same curvature* as the *model space* (M_0, g_0).

The model spaces considered in the present paper are special solvable Lie groups endowed with left invariant metrics. As Riemannian manifolds, $M_0 = \boldsymbol{R}^p \times \boldsymbol{R}^q$ and $g_0 = g_C$ is a homogeneous metric depending on a linear map

$$C: \boldsymbol{R}^q \to \mathrm{Sym}(\boldsymbol{R}^p) \qquad x \mapsto C(x)$$

of $\boldsymbol{R}^q$ into the space of the *symmetric* $p \times p$ matrices, such that $C(x)$ and $C(x')$ commute for any x and x'. This metric appears as a special case (i.e. $A = 0$) of a metric $g_{A,C}$ depending also on a second linear map A. This time

$$A: \boldsymbol{R}^p \to \boldsymbol{so}(\boldsymbol{R}^q) \qquad w \mapsto A(w)$$

maps $\boldsymbol{R}^p$ into the space of the *skew-symmetric* $q \times q$ matrices in such a way that $A(w)$ and $A(w')$ commute for any w and w'. If the following *compatibility condition*

$$C(A(w)x)w' = C(A(w')x)w$$

is satisfied then all the metrics $g_{A,C}$ have the *same curvature* as g_C. In general, they are not locally homogeneous and therefore not isometric to the model metric. So, the family $g_{A,C}$ is a *non-trivial deformation* of the homogeneous metric g_C preserving the Riemann curvature.

The metrics $g_{A,C}$ can be deformed once more. This leads to a class of metrics

$g_{A,C,h}$ depending also on a *diffeomorphism* h of $\boldsymbol{R}^p$ whose Jacobian matrix, at any point, is commuting with $C(x)$, for all $x \in \boldsymbol{R}^q$. Again, all these metrics have the same curvature as the model metric g_C. So we get non-trivial deformations of a homogeneous metric preserving the Riemann curvature and depending in an essential way on some arbitrary functions. In fact, we shall prove that, if some genericity assumptions are satisfied, the *isometry classes* of these metrics depend on a certain number of arbitrary functions. Namely, their *moduli space* is not *finite-dimensional.*

The metrics $g_{A,C}$ and $g_{A,C,h}$ are already defined as deformation of a *flat right invariant metric* on a Lie group. In particular, we prove that they are *curvature homogeneous* if A and C satisfy the compatibility condition stated above and we show that a *generic* metric $g_{A,C,h}$ is not homogeneous (see Section 3). We find general conditions which guarantees the irreducibility and the completeness of the metrics $g_{A,C,h}$ (the metrics $g_{A,C}$ are always complete) and the isometries between two metric $g_{A,C,h}$ and $g_{A',C,h'}$ (with the same C, and therefore the same curvature) are studied and completely characterized under the assumption of *weakly generic Ricci curvature.*

It is worthwhile to observe here that our method works because the model space $(M_0, g_0 = g_C)$ is *reducible.* This is equivalent to the requirement that the kernel of the map $x \mapsto C(x)$ is not trivial. Otherwise the compatibility condition forces A to be zero. In such a case, all metric $g_{0,A,h}$ are isometric to g_C, and our deformations are *trivial.*

It would be interesting, in case there are any, to construct non-trivial isocurved deformations of an *irreducible* homogeneous Riemannian metric.

2 - The metric $g_{A,C}$, $g_{A,C,h}$ and the model space

A Lie group admits a flat right (or left) invariant Riemannian metric if and only if it is the semidirect product of two Abelian Lie subgroups whose Lie algebras are mutually orthogonal and one factor is acting on the other by isometries ([4] p. 298). Such a group can be realized as the semidirect product $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$, where $\boldsymbol{R}^p$ is acting on $\boldsymbol{R}^q$ as follows

$$(2.1) \qquad (w, x)(w_0, x_0) = (w + w_0, e^{-A(w)} x_0 + x).$$

In this formula, $e^{-A(w)}$ denotes the exponential of the operator $A(w)$, and $A: \boldsymbol{R}^p \to \boldsymbol{so}(q)$ is a linear map into the Lie algebra $\boldsymbol{so}(q)$ of the skew-symmetric operators on $\boldsymbol{R}^q$ such that

$$[A(w), A(w')] = 0 \tag{2.2}$$

for each $w, w' \in \boldsymbol{R}^p$.

It is easily checked that the Maurer-Cartan form on $G = \boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ is given by (dw, θ) where

$$\theta = dx + A(dw)\,x\,. \tag{2.3}$$

We adopt here an index-free matrix notation; so dw and dx denote the vector valued one-forms

$$dw = \begin{pmatrix} dw^1 \\ \vdots \\ dw^p \end{pmatrix} \qquad dx = \begin{pmatrix} dx^1 \\ \vdots \\ dx^p \end{pmatrix} \tag{2.4}$$

where $(w^1, \ldots, w^p)$ and $(x^1, \ldots, x^q)$ are, respectively, the coordinate functions of the vector spaces $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$. dw is biinvariant and θ is right invariant. Therefore,

$$g_0 = {}^t dw \otimes dw + {}^t\theta \otimes \theta \tag{2.5}$$

defines a right invariant Riemannian metric on $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$, which turns out to be flat.

The subgroup $\boldsymbol{R}^p$ is acting on $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ on the left by

$$w_0(w, x) = (w_0 + w,\, e^{-A(w_0)}x)\,. \tag{2.6}$$

The orbit space of this action can be identified with $\boldsymbol{R}^q$. With this identification, the projection π of $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ on $\boldsymbol{R}^q$ is given by

$$\pi(w, x) = e^{A(w)}x\,. \tag{2.7}$$

On the other hand, $\boldsymbol{R}^p$ can be identified with the orbit of (w_0, x_0) via the immersion

$$i_{(w_0, x_0)} : \boldsymbol{R}^p \to \boldsymbol{R}^p \ltimes \boldsymbol{R}^q \qquad w \mapsto (w_0 + w,\, e^{-A(w)}x_0)\,. \tag{2.8}$$

Then, we have

Proposition 1. *The metric g_0 is the unique Riemannian metric on $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ such that the induced metric on the orbits is the Euclidean metric ${}^t dw \otimes dw$, and π is a Riemannian submersion on the Euclidean space $(\boldsymbol{R}^q, {}^t dx \otimes dx)$.*

Consider now a linear map

$$C: \boldsymbol{R}^q \to \mathrm{Sym}(\boldsymbol{R}^p) \qquad x \mapsto C(x)$$

of $\boldsymbol{R}^q$ into the space $\mathrm{Sym}(\boldsymbol{R}^p)$ of the *symmetric operators of* $\boldsymbol{R}^p$ such that

$$[C(x),\ C(x')] = 0 \tag{2.9}$$

for each x, x' in $\boldsymbol{R}^q$. Deform the metric g_0 along the $\boldsymbol{R}^p$-orbits of $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ by putting

$$g_{A,C} = {}^t\omega \otimes \omega + {}^t\theta \otimes \theta \tag{2.10}$$

where

$$w = e^{C(x)}\, dw\,. \tag{2.11}$$

Then, we obtain

Theorem 1. *If the maps A and C satisfy the following compatibility condition*

$$C(A(w)x)\,w' = C(A(w')x)\,w \tag{2.12}$$

then all the metrics $g_{A,C}$ are curvature homogeneous with Riemann curvature depending only on C.

It is expected that the isometry classes of these metrics depend on the maps A and C and therefore, on a finite number of parameters (i.e. the components of the tensors A and C). If we want to introduce other degrees of freedom in such a way that these classes depend also on some arbitrary functions, we shall modify again the one-form ω as follows. Consider the subgroup $\mathcal{H}$ of the diffeomorphisms of $\boldsymbol{R}^p$ whose differentials commute with all operators $C(x)$, i.e.

$$\mathcal{H} = \{h \in \mathrm{Diff}(\boldsymbol{R}^p) \mid [dh|_w,\ C(x)] = 0,\ \ \forall x \in \boldsymbol{R}^q,\ w \in \boldsymbol{R}^p\}\,. \tag{2.13}$$

Let h be an element of $\mathcal{H}$. Extend h to a diffeomorphism of $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$, denoted by the same symbol, by means of the formula

$$h(w,\ x) = (h(w),\ x) \tag{2.14}$$

and define the metric $g_{A,C,h}$ as follows

$$g_{A,C,h} = {}^t h^* \omega \otimes h^* \omega + {}^t\theta \otimes \theta . \tag{2.15}$$

We recover $g_{A,C}$ by choosing h equal to the identity of $\boldsymbol{R}^p$. In general, we have

Theorem 2. *If the compatibility condition* (2.12) *is satisfied, then the metrics* $g_{A,C,h}$ *are curvature homogeneous with the same curvature as* $g_{A,C}$.

Remark 1. All the metrics $g_{A,0,h}$ are *flat*, and all the metrics $g_{0,C,h}$ are *isometric* to

$$g_C = {}^t\omega_0 \otimes \omega_0 + {}^t dx \otimes dx \tag{2.16}$$

where $\omega_0 = e^{C(x)} dw$. In fact, if $A = 0$, then $\theta = dx$, and

$$h^* g_C = g_{0,C,h} . \tag{2.17}$$

For any choice of the map C the metrics g_C defined by (2.16) are *homogeneous*. In fact, g_C is a *left invariant metric* on the goup $G = \boldsymbol{R}^p \rtimes \boldsymbol{R}^q$ with the product defined by

$$(w, x)(w_0, x_0) = (w_0 + e^{-c(x_0)}(w), x + x_0) \tag{2.18}$$

since dx is biinvariant and $\omega = e^{C(x)}(dw)$ is left invariant with respect to this product. All the metrics $g_{A,C,h}$ have the same Riemann curvature as the *model space* (G, g_C). Therefore, by keeping C fixed, we have constructed a wide class of deformations of a homogeneous Riemannian metric which preserve the Riemann curvature. These metrics depend on a tensor A and on a diffeomorphism h of $\boldsymbol{R}^p$. If such *parameters* are chosen *generically*, then $g_{A,C,h}$ is not homogeneous, even locally.

3 - Main results and explicit example

We prove that, for a *generic choice* of the tensor A, the norm of the covariant derivative of the Ricci tensor is not constant. This fact implies that the metrics $g_{A,C,h}$ are not homogeneous, generically.

In order to study the irreducibility of the metrics $g_{A,C,h}$ it is convenient to suppose that $C(x)$ is given in diagonal form (the canonical form is not necessary here). Then we have

$$C(x)\, v_i = \langle c_i, x\rangle v_i \qquad 1 \leq i \leq p \tag{3.1}$$

where x is an elements of $\boldsymbol{R}^q$, $\{v_i,\ 1 \leq i \leq p\}$ the natural basis of $\boldsymbol{R}^p$ and c_i, $1 \leq i \leq p$, are suitable vectors in $\boldsymbol{R}^q$. We have

Theorem 3. *Let $\mathcal{C}$ be the set of $\{c_1, \ldots, c_p\}$ and $J(w)$ be the Jacobian matrix of h. If*

i. $c_i \neq 0,\ 1 \leq i \leq p$

ii. $\boldsymbol{R}^q = \mathcal{A} = \operatorname{Span}\{A(v_i)^m c_i,\ 1 \leq i \leq p,\ m \geq 0\}$

iii. *there is no partition of $\mathcal{C}$ in mutually orthogonal subsets*

iv. $\langle J(w)^{-1} v_i, v_i\rangle \neq 0$ *for all i, $1 \leq i \leq p$,*

then $g_{A,C,h}$ is irreducible.

About the completeness of the metrics $g_{A,C,h}$ we may prove

Theorem 4. *Suppose $C(x)$ and $J(w)$ are in diagonal form. Denote by $a^i(w_i)$ the (i, i)-entry of $J(w)$ (it depends only on w^i). If there exist two positive constants a^i and b^i such that $a^i \leq a^i(w^i) \leq b^i$, then the metric $g_{A,C,h}$ is complete.*

We study the isometry classes of the metrics $g_{A,C,h}$ when their Ricci curvature r is *generic* in the sense we are going to specify.

Indeed it is easy to see that we can put r in diagonal form just by performing an orthonormal change of basis in the space $\operatorname{Span}\{(c_1, \ldots, c_p)\}$. After this change, we note that the diagonal entries of r are given by

$$\lambda_i = \left\langle c_i, \sum_m c_m \right\rangle, \tag{3.2}$$

λ_α, $1 \leq \alpha \leq k$, and 0 with multiplicity $q - k$. This means that the *Ricci principal curvatures* are the scalars $\lambda_i (1 \leq i \leq p)$, $\lambda_\alpha (1 \leq \alpha \leq k)$, and $\lambda_0 = 0$ with multiplicity (at least) equal to $q - k$. We remark also that $q - k$ is just the *index of nullity* of the metric.

We say that the tensor r is (*weakly*) *generic* if the eigenvalues λ_i are non-zero, different from λ_α, and distinct, i.e. if

$$\lambda_i \neq 0 \qquad \lambda_i \neq \lambda_\alpha \qquad \lambda_i \neq \lambda_j$$

when $i \neq j$. Of course, some of the eigenvalues λ_α could be zero or coincide.

We have

Theorem 5. *Let f be an isometry between $g' = g_{A', C, h'}$ and $g = g_{A, C, h}$ with weakly generic Ricci curvature. Suppose that the genericity assumptions of Theorem 3 guaranteeing the irreducibility of both metrics, are satisfied.*

Then the metrics $g' = g_{A', C, h'}$ and $g = g_{A, c, h}$ are isometric if and only if there exist

i. *a $p \times p$ diagonal orthogonal matrix L*

ii. *a $q \times q$ orthogonal matrix P such that $C(Px) = C(x)$*

iii. *a non-singular $p \times p$ matrix Q which verifies*

$$A'(w) = P(A(Q^{-1}w))\,{}^tP \tag{3.3}$$

$$h'(w) = (\tilde{G} \circ h \circ G^{-1})(w)\,. \tag{3.4}$$

Moreover, the diffeomorphism $\tilde{G}$ is uniquely determined and

$$G(w) = w_0 + Q(w + \tilde{H}(w))\,, \tag{3.5}$$

where $\tilde{H}$ is a $(\mathrm{Ker}\, A)$*-valued function.*

We give now an explicit example of such a metric $g_{A, C, h}$.

Suppose $p = r + 1$ and $q = 2r + 1$. Let $\{u_1, \ldots, u_q\}$ be the natural basis of $\boldsymbol{R}^q$ and c_i be the vectors of $\boldsymbol{R}^q$ given by

$$c_i = \gamma_i\, u_{2i-1} + u_{2r+1} \qquad i \leq r \tag{3.6}$$

$$c_{r+1} = u_{2r+1}\,. \tag{3.7}$$

Choose r vectors a_i of $\boldsymbol{R}^p$, $1 \leq i \leq r$, as follows

$$a_i = \alpha_i v_i \tag{3.8}$$

for some constant α_i. Let $A(w)$ and $C(x)$ be the operators associated to the matrices

$$\begin{pmatrix} 0 & \langle a_1, w\rangle & & & & & & \\ -\langle a_1, w\rangle & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \langle a_s, w\rangle & & & \\ & & & -\langle a_s, w\rangle & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \tag{3.9}$$

$$\begin{pmatrix} \langle c_1, x\rangle & & \\ & \ddots & \\ & & \langle c_p, x\rangle \end{pmatrix}. \tag{3.10}$$

Then, the compatibility condition is satisfied. It follows that all the metrics

$$g_{\alpha, \gamma, h} = g_{A, C, h} \tag{3.11}$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\gamma = (\gamma_1, \ldots, \gamma_r)$, are curvature homogeneous as soon as the Jacobian matrix $J(w)$ of the diffeomorphism h of $\boldsymbol{R}^p$ commutes with the operators $C(x)$.

Since $C(x)$ is diagonal, from (3.10) and (3.6) we can derive at once that $[J(w), C(x)] = 0$ if and only if

$$\gamma_i J^i_j(w) = 0 \tag{3.12}$$

for each $i \leqslant r = p - 1$ and for any j.

Therefore, if all the coefficients γ_i are *non-zero*, the matrix $J(w)$ must be *diagonal*. In such a case h is given by

$$h(w) = (h^1(w^1), \ldots, h^p(w^p)), \tag{3.13}$$

where each function h^i depends only on the variable w^i and is *invertible*.

Hence, the metrics $g_{\alpha,\gamma,h} = g_{A,C,h}$ depend on $2r$ real parameters and $p = r + 1$ functions $h^1(w^1), \ldots, h^p(w^p)$. Moreover, all these metrics are *curvature homogeneous* with curvature depending only on r real parameters $\gamma = (\gamma_1, \ldots, \gamma_r)$.

If all the parameters α_i are zero, then $g_{0,\gamma,h}$ is homogeneous and isometric to the *model space* $g_C = g_{0,\gamma,\mathrm{Id}}$. But, in general, these metrics are *not locally homogeneous*. In this case we prove that the *isometry classes* of the metrics $g_{\alpha,\gamma,h}$ as h is varying in $\mathrm{Diff}(\boldsymbol{R}^p)$, depend at least on $(p-1)$ arbitrary functions $h'^1(w^1), \ldots, h'^{p-1}(w^{p-1})$, modulo some finite-dimensional parameter space. In other words, the *moduli space* of the metrics $g_{\alpha,\gamma,h}$, $h \in \mathrm{Diff}(\boldsymbol{R}^p)$, is *not finite-dimensional.*

References

[1] M. Berger, *L'oeuvre de André Lichnerowicz en géométrie riemannienne*, Physique Quantique et Géométrie, ed. D. Bernard and Y. Choquet-Bruhat, Colloq. Géom. et Phys. 1986 à l'honneur d'André Lichnerowicz, Hermann, Paris 1988, 11-24.

[2] A. L. Besse, *Einstein manifolds*, Springer, Berlin 1987.

[3] O. Kowalski, F. Tricerri and L. Vanhecke, *Curvature homogeneous spaces with a solvable Lie group as homogeneous model*, J. Math. Soc. Japan 44 (1992), 461-484.

[4] J. Milnor, *Curvature of left invariant metrics on Lie group*, Adv. in Math. **21** (1976), 293-329.

[5] F. Tricerri and L. Vanhecke, *Curvature homogeneous Riemannian manifolds*, Ann. Sci. École Norm, Sup. 22 (1989), 535-554.

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