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F. PAPALINI and S. VERCILLO (*)

Existence of solutions to a class of evolution equations (**)

1 - Introduction

The existence of solution to the Cauchy problem for evolution inclusions of the form

$$\dot{x} \in -\partial V(x) + f(t) \qquad x(0) = x_0 \qquad x_0 \in D(\partial V) \tag{1.1}$$

where ∂V is the sub-differential of a proper, convex and lower semicontinuous function V, defined on an Hilbert space, and f is a single valued perturbation, has been largely studied (cf. [4], [5], [6]). Later, some Authors (cf. [2] and [11]) have studied the problem (1.1) in the more general context that the perturbation is a multifunction.

In 1990, F. Ancona and G. Colombo [1] have studied the problem

$$\dot{x} \in F(x) + f(t, x) \qquad x(0) = x_0 \tag{1.2}$$

by assuming that $F: \boldsymbol{R}^n \to 2^{\boldsymbol{R}^n}$ is an upper semicontinuous and cyclically monotone multifunction with compact (not necessarily *convex*) values, while $f: \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a single valued function such that

β. for every $x \in \boldsymbol{R}^n$, $t \mapsto f(t, x)$ is measurable

ββ. for a.e. $t \in \boldsymbol{R}$, $x \mapsto f(t, x)$ is continuous on $\boldsymbol{R}^n$

βββ. $\exists m \in L^2(\boldsymbol{R}, \boldsymbol{R})$ such that $\|f(t, x)\| \leq m(t)$, for a.e. $t \in \boldsymbol{R}$ and for all $x \in \boldsymbol{R}^n$.

This result has been improved in [12]. We obtain the existence of solutions for

(*) Dip. di Matem., Univ. Perugia, Via Vanvitelli 1, Perugia, Italia.
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the problem (1.2) by substituting the condition ββ*β* (for us, it is enough that f is defined on $[0, b] \times \boldsymbol{R}^n$) with the weaker assumption

$\beta\beta\beta_w$. $\exists p \in]1, 2[$ and $\exists h \in L^p([0, b], \boldsymbol{R}) \cap L^2_{loc}([0, b], \boldsymbol{R})$, such that $\|f(t, x)\| \leq h(t)$, for a.e. $t \in [\in [0, b]$, for all $x \in \boldsymbol{R}^n$.

In 1991, A. Cellina and V. Staicu [9] obtained an existence result to the Cauchy problem of the form

$$(1.3) \qquad \dot{x} \in -\partial V(x) + F(x) \qquad x(0) = x_0 \qquad x_0 \in D(\partial V)$$

where V: $\boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function and F: $U(x_0) \to 2^{\boldsymbol{R}^n}$ is an upper semicontinuous and cyclically monotone multivalued operator, with compact values and defined on some neighbourhood of x_0.

V. Staicu in [13] has unified the results of [1] and of [9] by proving the existence of solutions for the problem

$$(1.4) \qquad \dot{x} \in -\partial V(x) + F(x) + f(t, x) \quad x(0) = x_0 \qquad x_0 \in D(\partial V)$$

where V and F are as in [9] and f is like in [1].

In this note we consider the Cauchy problem of the form (1.4), in the case that x_0 belongs to the interior of $D(\partial V)$ ($x_0 \in \operatorname{int} D(\partial V)$). We prove that (cf. Theorem) it has solutions by supposing that V: $\boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ and F: $U(x_0) \to 2^{\boldsymbol{R}^n}$ are, as for V. Staicu, respectively a proper, convex and lower semicontinuous function and an upper semicontinuous and cyclically monotone multifunction with compact values, while f: $[0, b] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ (not necessarily defined, as for V. Staicu, in $\boldsymbol{R} \times \boldsymbol{R}^n$) satisfies the weaker conditions β, ββ and $\beta\beta\beta_w$.

In the case that $x_0 \in \operatorname{int} D(\partial V)$, our theorem contains the theorem presented by V. Staicu in [13] (cf. Remark 3) and the one obtained by A. Cellina and V. Staicu [9]. When V is a constant function, our proposition reduces to the theorem of [12] and so, even in this particular case, our theorem contains the mentioned result of A. Ancona and G. Colombo [1].

Finally in the Corollary, we obtain the existence of solutions to the problem (1.4) where the single valued perturbation f is replaced by a multifunction G: $[0, b] \times \boldsymbol{R}^n \to 2^{\boldsymbol{R}^n}$ such that

j. $G(t, x)$ is nonempty, closed and convex, $\forall (t, x) \in [0, b] \times \boldsymbol{R}^n$

jj. $\forall x \in \boldsymbol{R}^n$, $t \mapsto G(t, x)$ is measurable jjj. $\forall t \in [0, b]$, $x \mapsto G(t, x)$ is lower semicontinuous and has closed graph

jv. $\exists p \in]1, 2[$ and $\exists h \in L^p([0, b], \boldsymbol{R}) \cap L^2_{loc}([0, b], \boldsymbol{R})$, such that $\|y\| \leq h(t)$, $\forall y \in G(t, x)$, for a.e. $t \in [0, b]$ and for all $x \in \boldsymbol{R}^n$.

2 - Preliminaries

Let $[a, b]$ be an interval and μ the Lebesgue measure on it. For $x \in \boldsymbol{R}^n$ and $\varepsilon > 0$ we set $B(x, \varepsilon) = \{y \in \boldsymbol{R}^n: \|y - x\| < \varepsilon\}$, where $\|\cdot\|$ is the Euclidean norm in $\boldsymbol{R}^n$ endowed by the scalar product $\langle \cdot, \cdot \rangle$, and, given a subset A of $\boldsymbol{R}^n$, we put $B(A, \varepsilon) = \{x \in \boldsymbol{R}^n: \rho(x, A) < \varepsilon\}$, where $\rho(x, A) = \inf\{\|y - x\|: y \in A\}$. For a closed and convex subset A of $\boldsymbol{R}^n$, we denote with $m(A)$ the element of A such that $\|m(A)\| = \inf\{\|y\|: y \in A\}$.

Let be $1 \leq p < +\infty$, we put

$$L^p_{loc}([a, b], \boldsymbol{R}^n) = \{x: [a, b] \to \boldsymbol{R}^n: \ x \text{ is measurable in } [a, b]\}$$

$$\cap \{x: [a, b] \to \boldsymbol{R}^n: \int_c^d \|x(t)\|^p \, dt < +\infty, \quad \forall c, d \in]a, b[\}$$

$$W^{1,p}([a, b], \boldsymbol{R}^n) = \{x: [a, b] \to \boldsymbol{R}^n: \ x \text{ is absolutely continuous on } [a, b]\}$$

$$\cap \{x: [a, b] \to \boldsymbol{R}^n: \ \dot{x} \in L^p([a, b], \boldsymbol{R}^n)\}.$$

A function $V: \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ is said to be *proper* if $D(V) \neq \emptyset$, where $D(V) = \{x \in \boldsymbol{R}^n: V(x) < +\infty\}$. If V is proper, convex and lower semicontinuous, the multifunction $\partial V: \boldsymbol{R}^n \to 2^{\boldsymbol{R}^n}$, defined by

$$\partial V(x) = \{y \in \boldsymbol{R}^n: \ V(\xi) - V(x) \geq \langle y, \xi - x \rangle, \ \forall \xi \in \boldsymbol{R}^n\}, \quad \forall x \in \boldsymbol{R}^n$$

is called *sub-differential* of V. We put $D(\partial V) = \{x \in \boldsymbol{R}^n: \ \partial V(x) \neq \emptyset\}$.

A multifunction $F: \boldsymbol{R}^n \to 2^{\boldsymbol{R}^n}$ is called *lower semicontinuous* (*upper semicontinuous*) if $\forall x \in \boldsymbol{R}^n$ and $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$F(x) \subset B(F(y), \varepsilon) \qquad (F(y) \subset B(F(x), \varepsilon)) \qquad \forall y \in B(x, \delta).$$

Moreover F is said to have *closed graph*, if the set

$$\operatorname{Gr} F = \{(x, y) \in \boldsymbol{R}^n \times \boldsymbol{R}^n: \ y \in F(x)\}$$

is closed in $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

Let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable subsets of $\boldsymbol{R}^n$. The multifunction F is called *measurable* if for any closed subset $C \subset \boldsymbol{R}^n$, we have

$$\{x \in \boldsymbol{R}^n : F(x) \cap C \neq \emptyset\} \in \mathcal{A} .$$

The multivalued operator F is said to be *cyclically monotone* if for every cyclical sequence $x_0, x_1, \ldots, x_N = x_0$ and for every sequence $y_1, \ldots, y_N$ such that $y_i \in F(x_i)$, $i = 1, \ldots, N$, we have

$$\sum_{i=1}^{N} \langle y_i, x_i - x_{i-1} \rangle \geqslant 0 .$$

Remark 1. We recall that (cf. [7], Theorem 2.5) F is cyclically monotone if and only if there exists a proper, convex, lower semicontinuous function $W: \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ such that

$$F(x) \subset \partial W(x) \qquad \forall x \in \boldsymbol{R}^n .$$

From Theorem 3.4 and Proposition 3.8 of [7] it is easy to deduce the following proposition that will be used in Sec. 3.

Lemma. *Let $V: \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. For every $x_0 \in \operatorname{int} D(\partial V)$ and $h \in L^1([0, b], \boldsymbol{R}^n)$, there exists a unique absolutely continuous function $x^h : [0, b] \to \boldsymbol{R}^n$ with the property*

$$\text{(2.1)} \qquad \dot{x}^h(t) \in -\partial V(x^h(t)) + h(t) \text{ a.e. } \textit{in } [0, b] \textit{ and } x^h(0) = x_0 .$$

Remark 2. Let $x^0 : [0, +\infty[\to \boldsymbol{R}^n$ be the unique absolutely continuous function such that

$$\dot{x}^0(t) \in -\partial V(x^0(t)) \text{ a.e. in } [0, +\infty[\text{ and } x^0(0) = x_0 .$$

From Theorem 3.2.1 of [3], using (26) of [7], if $x^h : [0, b] \to \boldsymbol{R}^n$ is the unique function that satisfies (2.1), it follows that

$$\text{(2.2)} \qquad \|x^h(t) - x_0\| \leqslant \int_0^t \|h(s)\| \, ds + t \|m(\partial V(x_0))\| \qquad \forall t \in [0, b] .$$

3 - Existence theorem

We consider the Cauchy problem

$$(3.1) \qquad \dot{x} \in -\partial V(x) + F(x) + f(t, x) \qquad\qquad x(0) = x_0 \in \boldsymbol{R}^n$$

where $V: \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$, $F: U(x_0) \to 2^{\boldsymbol{R}^n}$ ($U(x_0)$ is a neighbourhood of x_0) and $f: [0, b] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ verify respectively the properties:

i. V is a proper, convex, lower semicontinuous function

α. $F(x)$ is non empty and compact, $\forall x \in U(x_0)$

αα. F is upper semicontinuous

ααα. F is cyclically monotone

β. $\forall x \in \boldsymbol{R}^n$ the function $t \mapsto f(t, x)$ is measurable

ββ. for a.e. $t \in [0, b]$ the function $x \mapsto f(t, x)$ is continuous on $\boldsymbol{R}^n$

$\beta\beta\beta_w$. $\exists p \in]1, 2[$, $\exists h \in L^p([0, b], \boldsymbol{R}) \cap L^2_{loc}([0, b], \boldsymbol{R})$ such that $\|f(t, x)\| \leq h(t)$, for a.e. $t \in [0, b]$, for all $x \in \boldsymbol{R}^n$.

We observe that for every compact set K, containing x_0, there exists $x^* \in K$ such that $\inf\{V(x): x \in K\} = V(x^*)$. Since $\partial(V(x) - V(x^*)) = \partial V(x)$, we can assume $V \geq 0$.

An absolutely continuous function $x: [0, T] \to \boldsymbol{R}^n$ is called a *solution* of the Cauchy problem (3.1) if there exists $u \in L^2([0, T], \boldsymbol{R}^n)$, a selection of $F(x(\cdot))$ (i.e. $u(t) \in F(x(t))$ a.e. in $[0, T]$), such that $\dot{x}(t) \in -\partial V(x(t)) + u(t) + f(t, x(t))$ a.e. in $[0, T]$ and $x(0) = x_0$.

Our existence result is the following

Theorem. *Let V, F and f satisfy the conditions* i, α, αα, ααα, β, ββ, $\beta\beta\beta_w$ *and let* $x_0 \in \operatorname{int} D(\partial V)$. *Then there exist $T > 0$ and a solution $x: [0, T] \to \boldsymbol{R}^n$ of the Cauchy problem* (3.1).

We start by observing that from i, α, αα and from Theorem 0.7.2 of [3], it is possible to find two positive real number R and M with the properties:

$$(3.2) \qquad \|y\| < M \qquad \forall y \in F(x) \text{ and } \forall x \in \operatorname{cl} B(x_0, R)$$

$$(3.3) \qquad \|z\| < M \qquad \forall z \in \partial V(x) \text{ and } \forall x \in \operatorname{cl} B(x_0, R)$$

where $\operatorname{cl} A$ denotes the closure of the set A.

By $\beta\beta\beta_w$ there exists $T_1 > 0$ such that

$$\int_0^{T_1} (h(t) + M)\, dt < \frac{R}{2}. \tag{3.4}$$

Let T_2 be a positive number such that

$$T_2 < \frac{R}{2(R + \|m(\partial V(x_0))\|)}. \tag{3.5}$$

We shall consider a sequence of functions defined in $[0, T]$, $T = \min\{T_1, T_2\}$, and prove that a subsequence converges to a solution of the Cauchy problem (3.1).

For every $m \in N$ we set

$$I_{m,1} = [0, \frac{T}{m}] \qquad I_{m,i} =](i-1)\frac{T}{m}, i\frac{T}{m}] \qquad \forall i \in \{2, \ldots, m\}.$$

1^{st} *step* $i = 1$. Choose $y_{m,0} \in F(x_0)$ and define $f_{m,1}: I_{m,1} \to \boldsymbol{R}^n$ by $f_{m,1}(t) = y_{m,0} + f(t, x_0)$, $\forall t \in I_{m,1}$. Since $f_{m,1} \in L^1(I_{m,1}, \boldsymbol{R}^n)$, by the above Lemma there exists a unique absolutely continuous function $x_{m,1}: I_{m,1} \to \boldsymbol{R}^n$ such that

$$\dot{x}_{m,1}(t) \in -\partial V(x_{m,1}(t)) + f_{m,1}(t) \text{ a.e. in } I_{m,1} \text{ and } x_{m,1}(0) = x_0.$$

Therefore, by (2.2), (3.2), $\beta\beta\beta_w$, (3.4), (3.5) and (3.3), we obtain

$$\|x_{m,1}(t) - x_0\| < R \qquad \forall t \in I_{m,1} \quad \text{and} \quad \|\dot{x}_{m,1}(t)\| < 2M + h(t) \qquad \text{a.e. in } I_{m,1}.$$

2^{nd} *step* $i = 2$. Now we take $y_{m,1} \in F(x_{m,1}(Tm^{-1}))$ and define $f_{m,2}: [0, 2Tm^{-1}] \to \boldsymbol{R}^n$ by

$$f_{m,2}(t) = \begin{cases} f_{m,1}(t) & \forall t \in I_{m,1} \\ y_{m,1} + f(t, x_{m,1}(Tm^{-1})) & \forall t \in I_{m,2}. \end{cases}$$

We have that $f_{m,2} \in L^1([0, 2Tm^{-1}], \boldsymbol{R}^n)$, and so there exists a unique absolutely continuous function $x_{m,2}: [0, 2Tm^{-1}] \to \boldsymbol{R}^n$ such that

$$\dot{x}_{m,2}(t) \in -\partial V(x_{m,2}(t)) + f_{m,2}(t) \text{ a.e. in } [0, 2Tm^{-1}] \text{ and } x_{m,2}(0) = x_0.$$

Obviously $x_{m,2} = x_{m,1}$ on $I_{m,1}$ and

$$\|x_{m,2}(t) - x_0\| < R \quad \forall t \in [0, 2Tm^{-1}] \text{ and } \|\dot{x}_{m,2}(t)\| < 2M + h(t) \quad \text{a.e. in } [0, 2Tm^{-1}].$$

Analogously we proceed until the step $i = m$. We obtain a sequence $(x_m)_m$,

$x_m : [0, T] \to \boldsymbol{R}^n$, of absolutely continuous functions, defined by

$$x_m(t) = x_{m,m}(t) = \sum_{i=1}^{m} x_{m,i}(t)\chi_{I_{m,i}}(t) \qquad \forall t \in [0, T]$$

where $\chi_{I_{m,i}}$ is the characteristic function of the set $I_{m,i}$.

Now we set:

$$\delta_m, \gamma_m : [0, T] \to [0, T] \quad \text{and} \quad f_m, g_m : [0, T] \to \boldsymbol{R}^n$$

where

$$\delta_m(t) = \sum_{i=1}^{m} (i-1)\frac{T}{m}\chi_{I_{m,i}}(t) \qquad \gamma_m(t) = \sum_{i=1}^{m} i\frac{T}{m}\chi_{I_{m,i}}(t) \qquad \forall t \in [0, T]$$

$$f_m(t) = f_{m,m}(t) = \sum_{i=1}^{m} f_{m,i}(t)\chi_{I_{m,i}}(t) \qquad \forall t \in [0, T]$$

$$g_m(t) = \sum_{i=1}^{m} y_{m,i-1}\chi_{I_{m,i}}(t) \qquad \forall t \in [0, T].$$

Moreover, by construction, we have

$$\delta_m(t) \to t \text{ and } \gamma_m(t) \to t \text{ uniformly in } [0, T] \tag{3.6}$$

$$g_m(t) \in F(x_m(\delta_m(t))) \qquad \forall t \in [0,T] \quad \forall m \in \boldsymbol{N} \tag{3.7}$$

$$f_m(t) = g_m(t) + f(t, x_m(\delta_m(t))) \qquad \forall t \in [0,T] \quad \forall m \in \boldsymbol{N} \tag{3.8}$$

$$\dot{x}_m(t) \in -\partial V(x_m(t)) + f_m(t) \qquad \text{a.e. in } [0,T] \quad \forall m \in \boldsymbol{N} \tag{3.9}$$

$$\|x_m(t) - x_0\| < R \qquad \forall t \in [0,T] \quad \forall m \in \boldsymbol{N} \tag{3.10}$$

$$\|\dot{x}_m(t)\| < 2M + h(t) \qquad \text{a.e. in } [0,T] \quad \forall m \in \boldsymbol{N} \tag{3.11}$$

and, by (3.7), (3.10) and (3.2), it is trivial to prove that

$$\|g_m(t)\| < M \qquad \forall t \in [0, T] \quad \forall m \in \boldsymbol{N}. \tag{3.12}$$

By (3.11) and βββ$_w$ we have that $(\dot{x}_m)_m$ is bounded in $L^p([0, T], \boldsymbol{R}^n)$. Hence, by taking Arzela-Ascoli Theorem and Theorem III.27 of [8] into account, it follows that there exist a subsequence of $(x_m)_m$, still denoted by $(x_m)_m$, and an absolutely continuous function $x: [0, T] \to \boldsymbol{R}^n$ such that:

(3.13) $(x_m)_m$ converges uniformly to x

(3.14) $(\dot{x}_m)_m$ converges weakly in $L^p([0, T], \boldsymbol{R}^n)$ to $\dot{x}$.

Moreover, by (3.12) and Theorem III.27 of [8], we can assume that

(3.15) $(g_m)_m$ converges weakly in $L^2([0, T], \boldsymbol{R}^n)$ to g.

On the other hand, from (3.7), (3.6) and (3.13) we obtain

$$\lim_{m \to +\infty} \rho((x_m(t), g_m(t)), \operatorname{Gr} F) \leqslant \lim_{m \to +\infty} \|x_m(t) - x_m(\delta_m(t))\| = 0 \tag{3.16}$$

a.e. in $[0, T]$.

From $\alpha\alpha$, (3.13), (3.15), (3.16) and from the convergence Theorem 1.4.1. of [3], there exists (Remark 1) a proper, convex and lower semicontinuous function $W: \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ such that

$$g(t) \in \partial W(x(t)) \qquad \text{a.e. in } [0, T]. \tag{3.17}$$

Now, fix a closed interval $J = [c, d] \subset]0, T[$. By using (3.11) and $\beta\beta\beta_w$ it follows that $(\dot{x}_m)_m$ is bounded in $L^2(J, \boldsymbol{R}^n)$, therefore, by (3.14) and Theorem 2 of [**10**], p. 222, we have that

(3.18) $(\dot{x}_m)_m$ converges weakly in $L^2(J, \boldsymbol{R}^n)$ to $\dot{x}$

and so $x \in W^{1,2}(J, \boldsymbol{R}^n)$.

By Lemma 3.3 of [7] (cf. (3.17)) it follows that

$$W(x(d)) - W(x(c)) = \int_c^d \langle g(s), \dot{x}(s) \rangle \, ds . \tag{3.19}$$

On the other hand, by (3.7) and by the definition of ∂W, we have

$$W(x_m(i\frac{T}{m})) - W(x_m((i-1)\frac{T}{m})) \geqslant \langle y_{m,i-1}, \int_{(i-1)Tm^{-1}}^{iTm^{-1}} \dot{x}_m(s) \, ds \rangle$$

$$= \int_{(i-1)Tm^{-1}}^{iTm^{-1}} \langle g_m(s), \dot{x}_m(s) \rangle \, ds \qquad \forall i \in \{\gamma_m(c)\frac{m}{T} + 1, \ldots, \delta_m(d)\frac{m}{T}\}, \qquad \forall m \in \boldsymbol{N}$$

and by adding for $i = \gamma_m(c)\dfrac{m}{T} + 1, \ldots, \delta_m(d)\dfrac{m}{T}$, we obtain

$$W(x_m(\delta_m(d))) - W(x_m(\gamma_m(c))) \geq \int_{\gamma_m(c)}^{\delta_m(d)} \langle g_m(s), \dot{x}_m(s)\rangle \, ds \qquad \forall m \in \boldsymbol{N}.$$

Hence, by taking (3.13), (3.6), Proposition 2.12 of [7] and (3.19) into account, we have

$$\limsup_{m\to+\infty} \int_c^d \langle g_m(s), \dot{x}_m(s)\rangle \, ds \leq \int_c^d \langle g(s), \dot{x}(s)\rangle \, ds. \tag{3.20}$$

Now, by using Lemma 3.3 of [7] (cf. (3.18), (3.15), (3.8), βββ_w and (3.9)), it follows that

$$\begin{aligned}\int_c^d \|\dot{x}_m(s)\|^2 \, ds &= V(x_m(c)) - V(x_m(d)) + \int_c^d \langle g_m(s), \dot{x}_m(s)\rangle \, ds \\ &\quad + \int_c^d \langle f(s, x_m(\delta_m(s))), \dot{x}_m(s)\rangle \, ds \qquad \forall m \in \boldsymbol{N}.\end{aligned} \tag{3.21}$$

Analogously, by taking theorem 1.4.1. of [3] into account (cf. (3.15), (3.18), (3.13) and (3.9)), from Lemma 3.3 of [7], we obtain

$$\int_c^d \|\dot{x}(s)\|^2 \, ds = V(x(c)) - V(x(d)) + \int_c^d \langle g(s), \dot{x}(s)\rangle \, ds + \int_c^d \langle f(s, x(s)), \dot{x}(s)\rangle \, ds. \tag{3.22}$$

Moreover, since (cf. (3.6), ββ, βββ_w and (3.18))

$$\lim_{m\to+\infty} \int_c^d \langle f(s, x_m(\delta_m(s))), \dot{x}_m(s)\rangle \, ds = \int_c^d \langle f(s, x(s)), \dot{x}(s)\rangle \, ds$$

by (3.21), (3.20), (3.22) and the lower semicontinuity of V, we obtain

$$\limsup_{m\to+\infty} \|\dot{x}_m\|_{L^2(J)} \leq \|\dot{x}\|_{L^2(J)}.$$

Therefore (cf. (3.18) and Proposition III.30 of [8], p. 52) $(\dot{x}_m)_m$ *converges strongly* in $L^2(J, \boldsymbol{R}^n)$ to $\dot{x}$. Hence (cf. [8], Theorem IV.9, p. 58), there exist a subsequence of $(\dot{x}_m)_m$, still denoted $(\dot{x}_m)_m$, which converges pointwise a.e. in J to $\dot{x}$ and $\lambda \in L^2(J, \boldsymbol{R})$ such that $\|\dot{x}_m(t)\| \leq \lambda(t)$ a.e. in J, $\forall m \in \boldsymbol{N}$.

Now, set $H\colon J \to 2^{\boldsymbol{R}^n}$, $\sigma\colon J \to \boldsymbol{R}$ and η_m, $\eta\colon [0, T] \to \boldsymbol{R}^n$, to be:

$$H(t) = F(x(t)) + f(t, x(t)) - \dot{x}(t) \qquad \sigma(t) = M + h(t) + \lambda(t)$$

$$\eta_m(t) = f_m(t) - \dot{x}_m(t) \qquad \eta(t) = g(t) + f(t, x(t)) - \dot{x}(t).$$

By construction, $\eta_m(t) \in F(x_m(\delta_m(t))) + f(t, x_m(\delta_m(t))) - \dot{x}_m(t)$, a.e. in J. Hence $\|\eta_m(t)\| \leq \sigma(t)$, a.e. in J, $\forall m \in N$, and

$$\rho(\eta_m(t), H(t)) \leq \|\dot{x}_m(t) - \dot{x}(t)\| + \|f(t, x_m(\delta_m(t))) - f(t, x(t))\|$$

$$+ \sup\{\rho(z, F(x(t))) \colon z \in F(x_m(\delta_m(t)))\} \qquad \text{a.e. in } J, \ \forall m \in \boldsymbol{N}.$$

Then, taking (3.13), (3.6), ββ and $\alpha\alpha$ into account, we have

$$\lim_{m \to +\infty} \rho(\eta_m(t), H(t)) = 0 \qquad \text{a.e. in } J.$$

Therefore, by Lemma 3.2 of [9], it follows that the multifunction $\psi\colon J \to 2^{R^n}$, defined by $\psi(t) = \bigcap_{m \in N} \mathrm{cl}(\bigcup_{i \geq m} \{\eta_i(t)\})$, $\forall t \in J$ is such that $\psi(t)$ is nonempty and compact, $\forall t \in J$, ψ is measurable in J and

(3.23) $$\psi(t) \subset F(x(t)) + f(t, x(t)) - \dot{x}(t) \qquad \text{a.e. in } J.$$

Consider now, the multifunction H^* defined by $H^*(t) = \partial V(x(t)) \cap \mathrm{cl}\,B(0, \sigma(t))$. Since $\eta_m(t) \in \partial V(x_m(t)) \cap \mathrm{cl}\,B(0, \sigma(t))$, a.e. in J, and $x \mapsto \partial V(x) \cap \mathrm{cl}\,B(0, \sigma(t))$ is upper semicontinuous in $\mathrm{cl}\,B(x_0, R)$, $\forall t \in J$, we have

$$\lim_{m \to +\infty} \rho(\eta_m(t), H^*(t)) = 0 \qquad \text{a.e. in } J.$$

Hence, by using Lemma 3.2 of [9], we get

(3.24) $$\psi(t) \subset \partial V(x(t)) \cap \mathrm{cl}\,B(0, \sigma(t)) \qquad \text{a.e. in } J.$$

Let $v_J\colon J \to \boldsymbol{R}^n$ be a measurable selection of ψ, and set $u_J\colon J \to \boldsymbol{R}^n$, $u_J(t) = v_J(t) + \dot{x}(t) - f(t, x(t))$. By (3.23) and (3.24), we have that $u_J \in L^2(J, \boldsymbol{R}^n)$, $u_J(t) \in F(x(t))$ and $\dot{x}(t) \in -\partial V(x(t)) + u_J(t) + f(t, x(t))$, a.e. in J.

Since J is arbitrary, it follows that $\forall s \in \boldsymbol{N}$, $\exists$ a closed interval $J_s \subset]0, T[$, $\mu([0, T] \setminus J_s) < \frac{1}{s}$, and $\exists$ a function $u_s \in L^2(J_s, \boldsymbol{R}^n)$ such that $u_s(t) \in F(x(t))$, a.e. in J_s, and $\dot{x}(t) \in -\partial V(x(t)) + u_s(t) + f(t, x(t))$, a.e. in J_s.

Set $D = \bigcup_{s \in N} J_s$ and $u\colon [0, T] \to \boldsymbol{R}^n$ defined by

$$u(t) = \begin{cases} u_1(t) & t \in J_1 \\ \cdots & \\ u_i(t) & t \in J_i \setminus \bigcup_{j=1}^{i-1} J_j\,, \\ \cdots & \\ 0 & t \in [0, T] \setminus D\,. \end{cases}$$

It is easy to see that $u(t) \in F(x(t))$, a.e. in $[0, T]$, and so (cf. (3.2) and (3.10)) $u \in L^2([0, T], \boldsymbol{R})$. Since $\mu([0, T] \setminus D) = 0$, we have that

$$\dot{x}(t) \in -\partial V(x(t)) + u(t) + f(t, x(t)) \quad \text{a.e. in } [0, T].$$

Since $x_m(0) = x_0$, $\forall m \in \boldsymbol{N}$, it follows that x is a solution of the Cauchy problem (3.1).

Remark 3. In the case $x_0 \in \operatorname{int} D(\partial V)$, our proposition improves the existence theorem of V. Staicu [13]. In fact it is obvious that if $f\colon \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a function satisfying condition H_3 of [13], then f satisfies our assumptions β, $\beta\beta$, $\beta\beta\beta_w$ (cf. [10], Theorem 6, p. 101). On the other hand, the function $f\colon [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$, defined by

$$f(t, x) = \begin{cases} \dfrac{1}{\sqrt{t}} & (t, x) \in]0, 1] \times \boldsymbol{R} \\ 0 & (t, x) \in \{0\} \times \boldsymbol{R}. \end{cases}$$

satisfies the conditions β, $\beta\beta$, $\beta\beta\beta_w$ but does not satisfy the hypothesis H_3 of [13].

Remark 4. This existence result contains a proposition of [12] (cf. Theorem at p. 198). It is sufficient to assume $V(x) = 0$, $\forall x \in \boldsymbol{R}^n$.

Finally we observe that, by using a proposition of [12], p. 203, we obtain

Corollary. *If* $V\colon \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ *and* $F\colon U(x_0) \to 2^{\boldsymbol{R}^n}$ *satisfy respectively the conditions* i, α, $\alpha\alpha$, $\alpha\alpha\alpha$ *and* $G\colon [0, b] \times \boldsymbol{R}^n \to 2^{\boldsymbol{R}^n}$ *is a multifunction with the properties:*

j. $G(t, x)$ *is nonempty, closed and convex,* $\forall (t, x) \in [0, b] \times \boldsymbol{R}^n$

jj. $\forall x \in \boldsymbol{R}^n$, $t \mapsto G(t, x)$ *is measurable*

jjj. $\forall t \in [0, b]$, $x \mapsto G(t, x)$ *is lower semicontinuous and has closed graph*

jv. $\exists p \in]1, 2[$ *and* $\exists h \in L^p([0, b], \boldsymbol{R}) \cap L^2_{\mathrm{loc}}([0, b], \boldsymbol{R})$ *such that* $\|y\| \leq h(t)$, $\forall y \in G(t, x)$, *for* a.e. $t \in [0, b]$ *and for all* $x \in \boldsymbol{R}^n$, *then there exist a number* $T > 0$ *and an absolutely continuous function* $x\colon [0, T] \to \boldsymbol{R}^n$ *that is a solution of the Cauchy problem*

$$\dot{x} \in -\partial V(x) + F(x) + G(t, x) \qquad x(0) = x_0 \qquad x_0 \in \operatorname{int} D(\partial V).$$

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Sommario

In questo lavoro otteniamo un teorema di esistenza per problemi di Cauchy della forma $\dot{x} \in -\partial V(x) + F(x) + f(t, x)$, $x(0) = x_0$, *dove* F *è un operatore multivoco di* $\boldsymbol{R}^n$, ∂V *è il sottodiferenziale di una funzione reale* V *definita in* $\boldsymbol{R}^n$ *e* f *è una perturbazione monodroma. Questo teorema migliora i teoremi di esistenza conseguiti in* [1] *e in* [12], *e, nel caso* $x_0 \in \operatorname{int} D(\partial V)$, *contiene i teoremi di* [9] *e di* [13].

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