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On the semi-Riemannian structure of the tangent bundle of a two-point homogeneous space (**)

1 - Introduction

The tangent bundle TM of a manifold M is an object used to study the manifold M. Moreover, as its own construction is closely related to the structure of the manifold M, it seems reasonable to wonder what type of geometric information goes through them each other.

The diagonal lift g^D, associated with a Riemannian metric g on M, is a Riemannian metric on TM called the *Sasaki metric of TM*. Such a metric on TM makes the projection $\pi: TM \rightarrow M$ a Riemannian submersion. This fact seems particularly interesting for the study of the geometric properties of both spaces. In [**6**] the curvature of the metric g^D was studied.

The complete lift g^C, associated with the Riemannian metric g, is a semi-Riemannian metric on TM of signature (n, n), $(n = \dim M)$. This metric was initially investigated by Yano and Kobayashi, showing that (TM, g^C) has vanishing scalar curvature, and moreover, that (TM, g^C) is an Einstein space if and only if M is Ricci-flat [**16**].

The aim of this paper is to investigate the *curvature of the metric* g^C. Since (TM, g^C) is a locally symmetric space, if and only if the base manifold (M, g) is so, we will concentrate on the study of TM, when M is a *two-point homogeneous space* (except the Cayley plane), showing how certain conditions on the curva-

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ture of (TM, g^C) turn out to be characteristic of the tangent bundle of a two-point homogeneous space.

The paper is organized as follows. In Section 2 we summarize some known results about the curvature tensor of g^C. We refer to [16] for a general treatment of TM and some basic facts concerning g^C. For the purpose of this paper, we will express the curvature tensor $\widetilde{R}$ of g^C in terms of vertical and horizontal lifts of vector fields on M. The expression of the Ricci tensor, Ric, will allow us to show that it is not diagonalizable with respect to a g^C-orthonormal basis unless it vanishes.

Using the results of Section 2, in Section 3 we prove that (TM, g^C) is locally conformally flat, if and only if (M, g) is a space of constant curvature.

Section 4 is devoted to the study of the tangent bundle of a Kähler manifold (M, g, J). We show that (TM, g^C, J^C) is an indefinite Kähler manifold, if and only if (M, g, J) is Kähler, and moreover the tangent bundle has vanishing Bochner tensor, if and only if the base manifold is a complex space form. As an application of the results of Section 2, in Section 4 we exhibit examples of locally symmetric indefinite Kähler manifolds with vanishing Bochner tensor, but neither of constant holomorphic sectional curvature nor locally isometric to a product of Kähler manifolds of constant holomorphic sectional curvature.

Finally, Section 5 is devoted to the study of the tangent bundle of a quaternionic space form.

2 - Preliminaries

Let (M, g) be a *connected n-dimensional Riemannian manifold, TM*, its tangent bundle and g^C *the complete lift of the metric g to TM*. Then g^C is a semi-Riemannian metric of signature (n, n), which coincides with the horizontal lift g^H of g when this is considered with respect to the Levi Civita connection ∇ associated to g [16].

Taking into account the decomposition of the tangent space to TM given by ∇ at every point $\xi \in TM$, for each vector field X on M we denote X^V (resp. X^H) the vertical (resp. horizontal) lift to TM. In terms of these lifts, the complete lift of the metric is characterized by

$$g^C(X^H, Y^H) = g^C(X^V, Y^V) = 0 \qquad g^C(X^H, Y^V) = g^C(X^V, Y^H) = g(X, Y)^V \tag{2.1}$$

where $g(X, Y)^V$ denotes the vertical lift of the scalar $g(X, Y)$.

Note that the horizontal and vertical lifts of the vector fields on M are null vectors for g^C. Moreover, if $\{X_1, \dots, X_n\}$ is a local orthonormal basis for the vector fields on M, then

$$\{\frac{1}{\sqrt{2}}(X_i^H + X_i^V), \frac{1}{\sqrt{2}}(X_i^H - X_i^V)\}_{i=1,\ldots,n} \tag{2.2}$$

is a local orthonormal basis for the vector fields on TM, being $\frac{1}{\sqrt{2}}(X_i^H + X_i^V)$ *spacelike unit vectors*, and $\frac{1}{\sqrt{2}}(X_i^H - X_i^V)$ *timelike unit vectors.*

Let $\widetilde{\nabla}$ denote the metric connection associated to g^C, then [16]:

$$\widetilde{\nabla}_{X^V} Y^V = 0 \quad \widetilde{\nabla}_{X^V} Y^H = 0 \quad \widetilde{\nabla}_{X^H} Y^V = (\nabla_X Y)^V \quad \widetilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H + \gamma R(X, Y) \tag{2.3}$$

for any X, Y vector fields on M, where $\gamma R(X, Y)$ is the vector field on TM given by

$$\gamma R(X, Y) = [X, Y]^H - [X^H, Y^H]$$

and R is the curvature tensor, $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, of the Levi Civita connection ∇ on M.

Lemma 1. *At any point $\xi \in TM$, the curvature tensor $\widetilde{R}_\xi$ of $\widetilde{\nabla}$ is given by*

$$\widetilde{R}_\xi(X^V, Y^V) Z^V = \widetilde{R}_\xi(X^V, Y^V) Z^H = \widetilde{R}_\xi(X^H, Y^V) Z^V = 0$$

$$\widetilde{R}_\xi(X^H, Y^V) Z^H = \widetilde{R}_\xi(X^H, Y^H) Z^V = \{R_{\pi(\xi)}(X, Y) Z\}^V$$

$$\widetilde{R}_\xi(X^H, Y^H) Z^H = \{R_{\pi(\xi)}(X, Y) Z\}^H + \{(\nabla_\xi R)(X, Y) Z\}^V$$

for arbitrary vector fields X, Y, Z on M.

Proof. Direct computation. (See [16]).

An well-known result of Ambrose-Singer asserts that a Riemannian space (M, g) is *locally homogeneous*, if and only if there exists an homogeneous structure on it, that is, there exists a tensor field T on M of type $(1, 2)$ such that the connection $D = \nabla - T$ satisfies [15]

$$Dg = DR = DT = 0 . \tag{2.4}$$

This result has been generalized to semi-Riemannian spaces in [4], proving that the existence of an homogeneous structure T satisfying (2.4) is characteristic for reductive semi-Riemannian homogenous spaces [4].

Using the results in [15] and [4], one obtains

Theorem 1. *Let T be an homogeneous structure on a Riemannian space (M, g). Then the complete lift T^C determines an homogeneous semi-Riemannian structure on (TM, g^C). Moreover T^C is naturally reductive or locally symmetric if and only if T is so.*

Proof. The Levi Civita connection $\tilde{\nabla}$ of g^C and its curvature tensor $\tilde{R}$ coincide with the complete lifts ∇^C and R^C of the metric connection ∇ and of the curvature tensor R of g [16]. Moreover, since $\tilde{D} = \tilde{\nabla} - T^C = \nabla^C - T^C = (\nabla - T)^C$, it follows that $\tilde{D} = D^C$ and therefore

$$\tilde{D}g^C = (Dg)^C \qquad \tilde{D}\tilde{R} = D^C R^C = (DR)^C \qquad \tilde{D}T^C = D^C T^C = (DT)^C$$

which shows that T^C is an homogeneous structure on (TM, g^C), if an only if T is an homogeneous structure on M.

An homogeneous structure T is *naturally reductive* (resp. *locally symmetric*) if and only if $T_X X = 0$ (resp. $T = 0$) for all vector fields X. Hence the second assertion follows directly from the definition of the complete lift of a tensor fields of type $(1, 2)$ [16].

One of the interesting facts in the previous theorem is that (M, g) is a locally symmetric space if and only if (TM, g^C) is so, which suggests to study the relation between these two kinds of Riemannian and semi-Riemannian locally symmetric spaces. Recall that a Riemannian locally symmetric space is either irreducible or locally isometric to a product of locally symmetric Einstein spaces. This fact lies on the diagonalizability of the Ricci tensor, which is a specific feature of Riemannain metric. Since the Ricci tensor of a semi-Riemannian metric is not necessarily diagonalizable, an analogous decomposition should not be expected in semi-Riemannian geometry.

As a first stage to the study of the locally symmetric structure on (TM, g^C), we establish

Theorem 2. *The Ricci tensor,* Ric, *of (TM, g^C) is given by*

$$\text{(2.5)} \qquad \mathrm{Ric}(X^V, Y^V) = \mathrm{Ric}(X^V, Y^H) = 0 \qquad \mathrm{Ric}(X^H, Y^H) = 2\rho(X, Y)^V$$

where ρ denotes the Ricci tensor of (M, g). Moreover, the Ricci tensor Ric *is never diagonalizable with respect to any g^C-orthonormal basis, unless it vanishes.*

Proof. Let us consider the local orthonormal basis (2.2) of the vector fields on TM. Then

$$\mathrm{Ric}(\widetilde{X}, \widetilde{Y}) = \sum_{i=1}^{n} g^C(\widetilde{R}(\widetilde{X}, \frac{1}{\sqrt{2}}(X_i^H + X_i^V))\frac{1}{\sqrt{2}}(X_i^H + X_i^V), \widetilde{Y})$$

$$- \sum_{i=1}^{n} g^C(\widetilde{R}(\widetilde{X}, \frac{1}{\sqrt{2}}(X_i^H - X_i^V))\frac{1}{\sqrt{2}}(X_i^H - X_i^V), \widetilde{Y})$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(TM)$, and (2.5) follows easily from the expressions of $\widetilde{R}$ in Lemma 1.

In order to prove the non-diagonalizability of Ric, we proceed as follows. At an arbitrary point $\xi \in TM$, let us consider a symmetric bilinear form

$$\phi : T_\xi(TM) \times T_\xi(TM) \to \boldsymbol{R}$$

and the associated linear map $\widetilde{\phi} : x \in T_\xi(TM) \mapsto \widetilde{\phi}(x) \in T_\xi^*(TM)$ defined by $\widetilde{\phi}(x)y = \phi(x, y)$, for every $y \in T_\xi(TM)$.

For any basis $\{x_i\}$ of $T_\xi(TM)$, the matrix B associated to ϕ coincides with the matrix associated to ϕ with respect to the basis $\{x_i\}$ of $T_\xi(TM)$ and $\{x_i^*\}$ of $T_\xi^*(TM)$. If the bilinear form ϕ is *non-degenerated* (B is *invertible*), then there exists a linear map $\widetilde{\phi}^{-1}$ with associated matrix B^{-1}.

Let ψ denote another symmetric bilinear form on $T_\xi(TM)$, and consider the composite $F = \widetilde{\phi}^{-1} \circ \widetilde{\psi}$. For all $X \in T_\xi(TM)$, $F(X) = (\widetilde{\phi}^{-1} \circ \widetilde{\psi})(X) = \widetilde{\phi}^{-1}(\psi(X, -))$ and hence

$$\phi(F(X), -) = \widetilde{\phi}(F(X)) = \psi(X, -). \tag{2.6}$$

Now, if there exists a basis $\{E_i\}$ simultaneously diagonalizing both bilinear forms ϕ and ψ, it follows that

$$F(E_i) = \sum_{k=1}^{2n} \alpha_{ik} E_k$$

where $\psi(E_i, E_j) = \phi(F(E_i), E_j) = \phi(\sum_{k=1}^{2n} \alpha_{ik} E_k, E_j) = \alpha_{ij}\phi(E_i, E_j) = 0$, for $i \neq j$.

Consequently, if ϕ is non-degenerated, then $\phi(E_j, E_j) \neq 0$ for any $j = 1, \ldots, 2n$, and hence $\alpha_{ij} = 0$, $\forall i \neq j$. This shows that $F(E_i) = \alpha_{ii} E_i$, and therefore $\{E_i\}$ diagonalizes F.

The non-diagonalization of the Ricci tensor on (TM, g^C) follows from the previous argument, taking $\widetilde{\phi} = g^C$, and $\widetilde{\psi} = \mathrm{Ric}$. Let us show that $F = (g^C)^{-1} \circ \mathrm{Ric}$ is never diagonalizable, and hence the non-existence of a g^C-othonormal basis diagonalizing Ric.

For that, let $\{X_1, \dots, X_n\}$ be an orthonormal basis on M diagonalizing the Ricci tensor ρ (i.e. $\rho(X_i, X_j) = \lambda_i \delta^i_j$). Considering on $T_\xi(TM)$ the basis $\{X_1^H, \dots, X_n^H, X_1^V, \dots, X_n^V\}$ of null vectors, it follows that the metric g has associated matrix

$$g^C = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

and the Ricci tensor is a diagonal matrix Ric = diag $[\lambda_1, \dots, \lambda_n, 0, \dots, 0]$. Hence the endonorphism F defined in (2.6) has associated matrix

$$F = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix},$$

where Q is the diagonal matrix $Q = \text{diag}[\lambda_1, \dots, \lambda_n]$.

Since the characteristic polynomial of F is given by $\det(F - xI) = x^{2n}$ and its minimal polynomial is x^2, it follows that Ric is not diagonalizable with respect to a g^C-orthonormal basis, unless Ric vanishes.

As a direct consequence of the expression of the Ricci tensor (2.5), we get the known result [**16**]

Corollary 1. *The tangent bundle TM of a Riemannian space (M, g), endowed with the semi-Riemannian metric g^C, has vanishing scalar curvature. Moreover, (TM, g^C) is an Einstein space, if and only if (M, g) is Ricci flat.*

Using the results of the previous theorem and corollary, for the locally symmetric semi-Riemannian structure of (TM, g^C), we obtain:

Corollary 2. *Let (M, g) be an Einstein locally symmetric space. Then (TM, g^C) is locally symmetric, but neither Einstein nor locally isometric to a product of Einstein spaces, unless it is locally flat.*

Proof. It is clear from the previous theorem that (TM, g^C) is Einstein, if and only if (M, g) is Ricci flat. Since a Riemannian Ricci flat locally symmetric space is locally flat [2], we obtain the result. Moreover, if (TM, g^C) is locally isometric to a product of Einstein spaces, then the Ricci tensor Ric of TM diagonalizes, and hence it must vanish, which shows that (M, g) is locally flat.

The simplest locally symmetric Riemannian spaces (M, g), are the two-point homogeneous spaces. The results of this section show that their tangent bundles

(TM, g^C) are locally symmetric spaces, and one may expect that their geometry generalizes that of the two-point homogeneous spaces.

In what follows we will study the tangent bundle of a space of constant sectional curvature, of a Kähler manifold of constant holomorphic sectional curvature, and of a quaternionic Kähler manifold of constant quaternionic sectional curvature. In all the cases we will show that the study of the Jacobi operator along null geodesics on (TM, g^C) allows a characterization of the two-point homogeneous spaces. In particular we will show that (M, g) is a space of constant curvature, if and only if (TM, g^C) is locally conformally flat, and (M, g, J) is a Kähler manifold of constant holomorphic sectional curvature, if and only if (TM, g^C, J^H) is an indefinite Kähler manifold with vanishing Bochner tensor.

3 - The tangent bundle of a space of constant curvature

Let $(M, \langle\,,\rangle)$ be a *(semi)-Riemannian manifold*. The sectional curvature is the function

$$K(\alpha) = \frac{\langle R(X, Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} \qquad \alpha = \{X, Y\}. \tag{3.1}$$

For a Riemannian manifold, K is a function defined on the whole Grassmannian $G_2(T_m M)$ at each $m \in M$, but for semi-Riemannian metrics (3.1) only makes sense for non degenerated planes (i.e. when $\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2 \neq 0$).

It is well-known that a semi-Riemannian manifold has constant sectional curvature c if and only if the curvature tensor takes the form

$$R(X, Y)Z = c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} \tag{3.2}$$

and hence, for any unit-speed geodesic γ, the Jacobi operator R_γ is a multiple of the identity $(R(-, \gamma')\gamma' = cI)$.

A particular feature of semi-Riemannian metrics is the existence of *null geodesics* γ, that is geodesics γ such that $\langle\gamma', \gamma'\rangle = 0$. The study of the Jacobi operator along these geodesics presents some significant differences coming from the fact that $R_\gamma X = R(X, \gamma')\gamma'$ may have a non null component, tangential to γ.

In order to unify the study of the Jacobi operator for semi-Riemannian metrics, in [7] it is considered the nondegenerate normal bundle

$$\bar{\gamma}^\perp = \gamma'^\perp/\langle\gamma'\rangle$$

where $\langle\gamma'\rangle$ is the space spanned by γ', and the Jacobi operator $\bar{R}_\gamma$, $\bar{R}_\gamma(\bar{X}) = \pi(R(X, \gamma')\gamma')$, being $\pi: \langle\gamma'\rangle^\perp \to \bar{\gamma}^\perp$ the projection. (Note that $\bar{\gamma}^\perp$ coincides with $\langle\gamma'\rangle^\perp$ if the geodesic γ is non null).

A geodesic γ is called *isotropic* [7], if $\overline{R}_\gamma = \overline{r}(\gamma')\overline{I}$ along γ, for some real valued function $\overline{r}$ defined along γ. Moreover, (M, g) is a space of constant curvature if and only if any non null geodesic is isotropic, or equivalently, every null geodesic γ is isotropic with $\overline{r}(\gamma') = 0$.

Hence, an interesting condition on a semi-Riemannian manifold generalizing the spaces of constant curvature is to assume all null geodesics to be isotropic. Such condition for every null geodesic γ is shown to be equivalent to the vanishing of the Weyl tensor [7], and hence to locally conformally flatness for $\dim M \geqslant 4$.

The following theorem shows that locally conformally flatness of the semi-Riemannian metric g^C on the tangent bundle TM of a Riemannian manifold (M, g) characterizes the spaces of constant curvature.

Theorem 3. *Let (M, g) be a Riemannian manifold. Then (TM, g^C) is locally conformally flat if and anly if (M, g) is a space of constant curvature.*

Proof. First of all, note that a semi-Riemannian manifold $(M, \langle\,,\rangle)$ of metric signature ν, $2 \leqslant \nu \leqslant \dim M - 2$, is null isotropic if and only if

$$R(u, v, v, u) = 0 \qquad \forall u, v \text{ orthogonal null vectors}. \tag{3.3}$$

In fact, it is clear that any null isotropic semi-Riemannian space satisfies (3.3). Conversely, for any null vector u in $(M, \langle\,,\rangle)$, consider the nondegenerate normal $\overline{u}^{\perp}$, and define the symmetric bilinear form $f(X, Y) = \overline{R}(u, X, u, Y)$ for each $X, Y \in \overline{u}^{\perp}$. Since $\dim \overline{u}^{\perp} = \dim M - 2$, and the projected metric tensor has signature $\nu - 1$, we obtain from [10] that (3.3) implies null isotropy.

Hence (TM, g^C) is locally conformally flat if and only if, at each point $\xi \in TM$, $\widetilde{R}(u, v, v, u) = 0$ for all $u, v \in T_\xi(TM)$ orthogonal null vectors.

Now, let $\xi \in TM$ and cosider u, v orthogonal null vectors tangent to TM at ξ. Since the tangent space to TM splits into vertical and horizontal parts due to the metric connection ∇ on M, there exist $X, Y, A, B \in T_{\pi(\xi)}M$ such that $u = X_\xi^H + Y_\xi^V$, $v = A_\xi^H + B_\xi^V$, and moreover, since u, v are orthogonal null vectors, it follows that:

$$g(X, Y) = 0 = g(A, B) \qquad g(X, B) = -g(Y, A). \tag{3.4}$$

From Lemma 1, it follows that

$$\begin{aligned} \widetilde{R}_\xi(u, v, v, u) &= g^C(\widetilde{R}_\xi(u, v)v, u) \\ &= g((\nabla_\xi R)(X, A)A, X)^V + 2g(R(A, X)Y, A)^V + 2g(R(X, A)B, X)^V. \end{aligned} \tag{3.5}$$

Now, if (M, g) is a space of constant curvature c, then its curvature tensor takes the form (3.2) and hence, (3.5) together with (3.4) shows that (TM, g^C) is null isotropic. Since $\dim M > 2$, and because of the signature of g^C, it follows that TM is locally conformally flat.

Conversely, let (TM, g^C) be a null isotropic manifold, and consider orthogonal null vectors $u = X^H$, $v = Y^H$; then, from the expressions above, it follows that

$$\tilde{R}_\xi(u, v, v, u) = g^C(\tilde{R}_\xi(X^H, Y^H)Y^H, X^H) = g((\nabla_\xi R)(X, Y)Y, X)^V = 0$$

and hence (M, g) is locally symmetric.

In the case of $\dim M = 2$, the local symmetry implies the constancy of the sectional curvature of M. Now if $\dim M \geqslant 3$, let us take X, Y, Z arbitrary orthogonal vectors at $m \in M$, and consider the orthogonal null vectors $u = X_\xi^H + Y_\xi^V$, $v = Z_\xi^H + X_\xi^V$ at $\xi \in TM$, $(\pi(\xi) = m)$. From (3.5), and using the fact that (M, g) is locally symmetric, we get $R(Z, X, Y, Z) = 0$. Therefore the sectional curvature of M is constant provided that $\dim M \geqslant 3$.

Remark 1. Locally symmetric Riemannian spaces with vanishing Weyl tensor are spaces of constant curvature, or locally isometric to a product of two spaces of constant curvature c and $-c$, or locally isometric to a product of a space of constant curvature and the real line. The tangent bundle (TM, g^C) of a non flat Riemannian space of constant sectional curvature is a semi-Riemannian locally symmetric space with vanishing Weyl tensor, but never belongs to the classes listed before.

Corollary 3. *(TM, g^C) has constant sectional curvature if and only if (M, g) is flat.*

Proof. If (TM, g^C) has constant sectional curvature, then it is null isotropic with $\bar{r}(\gamma') = 0$. Since the function $\bar{r}$ is completely determined by Ricci tensor, from [7], we have

$$\bar{r}(\gamma') = \frac{1}{2(n-1)} \operatorname{Ric}(\gamma', \gamma')$$

and hence equation (2.5) shows that $\bar{r}(\gamma') = 0$ if and only if (M, g) has zero sectional curvature.

At each point m of a Riemmannian manifold, the values of the sectional curvature K at m are bounded, since K is defined on the Grassmannian $G_2(T_m M)$, which is compact. However, for semi-Riemannian metrics this assertion is no longer valid. In fact, Kulkarni showed that the sectional curvature function is bounded from above or from below if and only if it is constant. Moreover, in [**10**], [**11**] it is shown that bounds from above and from below on planes of signature $(+,+)$ or $(-,-)$ is equivalent to constant sectional curvature.

We close this section with examples of semi-Riemannian manifolds with sectional curvature bounded from below (or from above) on planes of signature $(+,+)$ (or $(-,-)$) but not constant.

Theorem 4. *Let (M^n, g) be a Riemannian space of constant curvature $c > 0$ (resp. $c < 0$). Then the sectional curvature K of (TM, g^C) is non negative (resp. non positive) on planes of signature $(+,+)$, and non positive (resp. non negative) on planes of signature $(-,-)$.*

Proof. According to Theorem 3, (TM, g^C) is locally conformally flat. Since the scalar curvature of (TM, g^C) vanishes, the sectional curvature is completely determined by the Ricci tensor through

$$K(\alpha) = K(\widetilde{X} \wedge \widetilde{Y}) = \frac{1}{2(n-1)} \left\{ \frac{\mathrm{Ric}(\widetilde{X}, \widetilde{X})}{g^C(\widetilde{X}, \widetilde{X})} + \frac{\mathrm{Ric}(\widetilde{Y}, \widetilde{Y})}{g^C(\widetilde{Y}, \widetilde{Y})} \right\}$$

where $\{\widetilde{X}, \widetilde{Y}\}$ is an orthogonal basis of the non degenerate plane α.

For each vector field $\widetilde{Z} \in \mathfrak{X}(TM)$, we decompose $\widetilde{Z}_\xi = A_\xi^H + B_\xi^V$. Using (2.5) we have

$$\mathrm{Ric}(\widetilde{Z}, \widetilde{Z}) = 2\rho(A, A)^V = 2c(n-1)(\|A\|^2)^V$$

where $\|A\|^2 = g(A, A)$.

Hence the sectional curvature K is non negative (resp. non positive) on planes of signature $(+,+)$ (resp. $(-,-)$) for $c > 0$.

4 - An indefinite Kähler structure on the tangent bundle of a Kähler manifold

Since the constancy of the sectional curvature for Kähler manifolds leads to locally flat spaces, we consider the holomorphic sectional curvature, which is defined as the restriction of the sectional curvature function to holomorphic planes, (J-invariant planes). Explicitly

$$H(\alpha) = \frac{R(X, JX, JX, X)}{\langle X, X \rangle^2} \qquad \alpha = \{X, JX\}. \tag{4.1}$$

Nomizu [**9**] gives an equivalent condition on a Kähler manifold to have constant holomorphic sectional curvature in terms of the Jacobi operator, condition which has been extended by Barros and Romero [**1**] to indefinite Kähler manifolds.

Due to the existence of null geodesics, it seems interesting to study further these conditions, and in [8], it is shown the following

Theorem 5. *Let (M, g, J) be an indefinite Kähler manifold. Then the holomorphic sectional curvature H of M is constant if and only if one of the following equivalent conditions holds*

$$R(X, JX)JX \sim X, \qquad \textit{for all unit } X$$

$$R(u, Ju)Ju = 0, \qquad \textit{for all null } u.$$

where $\sim$ means «is proportional to».

In view of this theorem, we will say that an indefinite Kähler manifold is *null holomorphically flat* [3], [8] if

$$R(u, Ju)Ju = \lambda_u u \qquad \text{for all null } u \tag{4.2}$$

where λ_u is a certain real valued function.

Condition (4.2) is equivalent to being zero the restriction of the curvature tensor to holomorphic degenerate planes:

$$R(u, Ju, Ju, u) = 0$$

and moreover, it defines a class of indefinite Kähler manifolds which contains the complex space forms (cf. [3]).

The main purpose of this section is to study this property on the tangent bundle of a Kähler manifold (M, g, J), giving a characterization of complex space forms. Also, some bounds for the holomorphic sectional curvature of TM are investigated.

Let (M, g, J) be a *Kähler manifold*, ∇ the Levi-Civita connection associated to g, and consider on TM the complete lifts g^C and J^C of the Riemannian metric and the complex structure, respectively. Since (M, g, J) is Kähler, it is easy to prove that the action of the complete lift J^C of the complex structure J over the horizontal and vertical lifts is as follows

$$J^C(X^H) = \{J(X)\}^H \qquad J^C(X^V) = \{J(X)\}^V \tag{4.3}$$

where $X^H, X^V \in T_\xi(TM)$.

Theorem 6. *(M, g, J) is a Kähler manifold of constant holomorphic sectional curvature if and only if (TM, g^C, J^C) is a null holomorphically flat indefinite Kähler manifold.*

Proof. Firstly note that the tangent bundle (TM, g^C, J^C) is an indefinite Kähler manifold if and only if (M, g, J) is Kähler.

Now, let ξ be a point of TM and u a tangent null vector at ξ. As in the proof of Theorem 3, we write $u = X^H_\xi + Y^H_\xi$ for some orthogonal $X, Y \in T_{\pi(\xi)} M$. Using the expressions obtained in Section 2 for the curvature tensor $\widetilde{R}$, together with the identities of the curvature tensor of a Kähler manifold, we get

$$\widetilde{R}(u, J^C u, J^C u, u) = \{g((\nabla_\xi R)(X, JX)JX, X) + 2g(R(X, JX)JX, Y)\}^V. \tag{4.4}$$

Let us suppose that (TM, g^C, J^C) is null holomorphically flat. Since X^H is a null vector for any $X \in \mathfrak{X}(M)$, we get $\widetilde{R}(X^H, J^C X^H, J^C X^H, X^H) = 0$. Hence, from (4.4), $g((\nabla_\xi R)(X, JX)JX, X) = 0$.

Moreover, if X and Y are orthogonal tangent vectors on M, then $u = X^H + Y^V$ is a null tangent vector to TM and again by (4.4) we get $g(R(X, JX)JX, Y) = 0$.

Now, the constancy of the holomorphic sectional curvature of M follows from the criteria of [9].

Conversely, since any complex space form is locally symmetric, it follows immediately from (4.4) that the tangent bundle is null holomorphically flat.

Note that previous theorem allows us to exhibit a family of null holomorphically flat indefinite Kähler manifolds, but not of constant holomorphic sectional curvature, in addition to the examples constructed in [3].

Moreover, the tangent bundle of a complex space form satisfies

Theorem 7. *Let (M, g, J) be a Kähler manifold. Then (TM, g^C, J^C) has vanishing Bochner tensor if and only if M has constant holomorphic sectional curvature.*

Proof. Since the scalar curvature of the tangent bundle (TM, g^C) is identically zero, the expression of the Bochner curvature tensor B [14], [17], reduces to

$$\begin{aligned}B(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) &= \widetilde{R}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W})\\ &-\frac{1}{4(n+1)}\{g^C(\widetilde{Y}, \widetilde{Z})\operatorname{Ric}(\widetilde{X}, \widetilde{W}) - g^C(\widetilde{X}, \widetilde{Z})\operatorname{Ric}(\widetilde{Y}, \widetilde{W})\\ &+g^C(J^C\widetilde{Y}, \widetilde{Z})\operatorname{Ric}(J^C\widetilde{X}, \widetilde{W}) - g^C(J^C\widetilde{X}, \widetilde{Z})\operatorname{Ric}(J^C\widetilde{Y}, \widetilde{W})\\ &-2g^C(J^C\widetilde{X}, \widetilde{Y})\operatorname{Ric}(J^C\widetilde{Z}, \widetilde{W}) - 2\operatorname{Ric}(J^C\widetilde{X}, \widetilde{Y})g^C(J^C\widetilde{Z}, \widetilde{W})\\ &+\operatorname{Ric}(\widetilde{Y}, \widetilde{Z})g^C(\widetilde{X}, \widetilde{W}) - \operatorname{Ric}(\widetilde{X}, \widetilde{Z})g^C(\widetilde{Y}, \widetilde{W})\\ &+\operatorname{Ric}(J^C\widetilde{Y}, \widetilde{Z})g^C(J^C\widetilde{X}, \widetilde{W}) - \operatorname{Ric}(J^C\widetilde{X}, \widetilde{Z})g^C(J^C\widetilde{Y}, \widetilde{W})\}\end{aligned} \tag{4.5}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}$ are arbitrary vector fields on TM.

If the Bochner tensor of TM vanishes, for each X, JX, Y orthogonal tangent vectors to M, $B(X^H, (JX)^H, X^H, Y^V) = 0$. Considering the expression (2.6) of the Ricci tensor we have $R(X, JX, X, Y) = 0$. Hence the holomorphic sectional curvature of M is constant [9].

Reciprocally, if (M, g, J) has constant holomorphic sectional curvature c, the Ricci tensor is Einstein with $\rho(X, Y) = \frac{n+1}{2} cg(X, Y)$, $\forall X, Y \in \mathfrak{X}(M)$. Moreover, since the curvature tensor of a Kähler manifold of constant holomorphic sectional curvature c satisfies

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ\}$$

we obtain the vanishing of the Bochner tensor of (TM, G^C, J^C).

Remark 2. As a direct application of the results in Section 2, the tangent bundle of a Kähler manifold of constant holomorphic sectional curvature c is locally symmetric, and further has vanishing Bochner tensor, However (TM, g^C, J^C) is neither of constant holomorphic sectional curvature nor locally isometric to a product of Kähler manifolds of constant holomorphic sectional curvature unless it is flat. (Compare with the results in [14] for the positive definite case).

The holomorphic sectional curvature of a positive definite almost Hermitian manifold is bounded at each point, because it is a function defined on the spherical tangent bundle. This does not happen for indefinite almost Hermitian manifolds, and it has been shown in [3] that if the holomorphic sectional curvature is bounded from above and from below, then it is constant. Necessity of both bounds is proved in [1], where the authors exhibite an example of an indefinite Kähler manifold with holomorphic sectional curvature bounded from below but not constant.

In what follows, we study the holomorphic sectional curvature of (TM, g^C, J^C), obtainig some bounds for the holomorphic sectional curvature on spacelike and timelike holomorphic planes.

Theorem 8. *Let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature c. If $c < 0$ (resp. $c > 0$), then the holomorphic sectional curvature of TM is non positive (resp. non negative) on spacelike holomorphic planes, and non negative (resp. non positive) on timelike holomorphic planes.*

Proof. Since the tangent bundle of a complex space form has vanishing Bochner tensor, the holomorphic sectional curvature is completely determined by the Ricci tensor, and we get

$$\tilde{R}(\tilde{X}, J^C\tilde{X}, J^C\tilde{X}, \tilde{X}) = \frac{1}{4(n+1)}\, g^C(\tilde{X}, \tilde{X})(4\,\mathrm{Ric}(\tilde{X}, \tilde{X}) + 4\,\mathrm{Ric}(J^C\tilde{X}, J^C\tilde{X}))$$

$$= \frac{2}{n+1}\, g^C(\tilde{X}, \tilde{X})\,\mathrm{Ric}(\tilde{X}, \tilde{X})$$

for each vector field $\tilde{X}$ on TM.

Now, if $\tilde{X}$ is a tangent vector to TM at the point ξ, we write $\tilde{X} = X^H_\xi + Y^V_\xi$, and hence, from (2.5), it follows that

$$\tilde{R}(\tilde{X}, J^C\tilde{X}, J^C\tilde{X}, \tilde{X}) = c(\|X\|^2)^V g^C(\tilde{X}, \tilde{X})\,.$$

Consequently, the holomorphic sectional curvature H of (TM, g^C, J^C) is given by

$$H(\tilde{X}) = c\,\frac{(\|X\|^2)^V}{g^C(\tilde{X}, \tilde{X})}$$

from where we get the desired result.

Moreover, as a direct application of the previous expression, we get

Theorem 9. *(TM, g^C, J^C) has constant holomorphic sectional curvature if and only if (M, g, J) is locally flat.*

Proof. It is clear that if (TM, g^C, J^C) has constant holomorphic sectional curvature, then it is null holomorphically flat. Hence the curvature tensor at each point $\xi \in TM$ is given by the expression above. Now, if it is constant, it must be $c = 0$, which shows that (M, g, J) has constant zero holomorphic sectional curvature.

5 – The tangent bundle of a quaternionic space form

For a quaternionic Kähler manifold, the analogous of the holomorphic sectional curvature of Kähler manifolds is the quaternionic sectional curvature, which is defined as follows. Let $\{I, J, K\}$ be a local basis of the bundle of almost complex structures over M, for each vector X consider the 4-plane

$Q(X) = \{X, IX, JX, KX\}$. If the sectional curvature K is constant for every 2-plane in $Q(X)$, this common value is called the *quaternionic sectional curvature of* X. We refer to [13], [12], [17], for some basic facts concerning quaternionic Kähler manifolds with positive definite or indefinite metrics. (See also [2]). It is well-known that a quaternionic Kähler manifold has constant quaternionic sectional curvature c if and only if its curvature tensor is given by

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y \\ &+g(IY, Z)IX - g(IX, Z)IY - 2g(IX, Y)IZ \\ &+g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &+g(KY, Z)KX - g(KX, Z)KY - 2g(KX, Y)KZ\}. \end{aligned} \tag{5.1}$$

This formula is still valid in the indefinite case, and moreover, any quaternionic space form is well known to be locally symmetric.

As in the previous sections, we can impose on an almost quaternionic manifold a weaker condition than to be of constant quaternionic-sectional curvature, namely

$$R(u, \phi u, \phi u, u) = 0 \qquad \forall u \text{ null vector}, \quad \phi = I, J, K. \tag{5.2}$$

Note that this condition is automatically satisfied by any almost quaternionic manifold of constant quaternionic-sectional curvature, and moreover, that it defines a conformal invariant class of indefinite almost quaternionic manifolds.

In this section, we will construct an indefinite almost quaternionic structure on the tangent bundle of an almost quaternionic manifold, and study the effect of (5.2) on its curvature tensor.

For that, if $\xi \in TM$ and $m = \pi(\xi)$, consider an open subset U containing m and let $\{I, J, K\}$ be a local basis of the bundle of almost complex structures V over M. Cosider on $\pi^{-1}(U)$ the almost complex structures $\{I^H, J^H, K^H\}$. They define a bundle V^H of almost complex structures over TM, and moreover, (TM, g^H, V^H) is an indefinite almost quaternionic manifold.

Theorem 10. *Let (M, g, V) be a quaternionic Kähler manifold of real dimension $n \geqslant 8$. Then M has constant quaternionic sectional curvature if and only if the indefinite almost quaternionic structure (g^H, V^H) on TM satisfies*

$$\widetilde{R}(u, \phi^H u, \phi^H u, u) = 0 \qquad \forall u \textit{ null vector on } TM. \tag{5.3}$$

Proof. If $u \in T_\xi(TM)$ is a null vector, write $u = X_\xi^H + Y_\xi^V$ for orthogonal $X, Y \in T_{\pi(\xi)}(M)$; then

$$\widetilde{R}(u, \phi^H u, \phi^H u, u) = \{g((\nabla_\xi R)(X, \phi X)\phi X, X)\}^V$$

$$+\{g(R(X, \phi X)\phi X)\phi X, Y) - g(R(X, \phi X)X, \phi Y)\}^V \qquad \phi = I, J, K.$$

Since $\dim M > 4$, M is an Einstein space. Using the expressions in [**17**], it is easy to check that

$$R(X, \phi X, \phi X, Y) = -R(X, \phi X, X, \phi Y)$$

and hence, for $\phi = I, J, K$, we have

$$\widetilde{R}(u, \phi^H u, \phi^H u, u) = \{g((\nabla_\xi R)(X, \phi X)\phi X, X) + 2g(R(X, \phi X)\phi X, Y)\}^V. \tag{5.4}$$

Now, if the quaternionic sectional curvature is a constant c, then it follows from (5.1) that

$$R(X, \phi X)\phi X = cg(X, X)X \qquad \forall X \in \mathfrak{X}(M) \qquad \phi = I, J, K$$

and hence $g(R(X, \phi X)\phi X, Y) = 0$ for all orthognal X, Y. Moreover, since any quaternionic space form is locally symmetric, it follows that

$$\widetilde{R}(u, \phi^H u, \phi^H u, u) = 0 \qquad \forall u \text{ null vector on } TM.$$

In [**12**] it is shown that the quaternionic-sectional curvature of a quaternionic Kähler manifold of $\dim M \geqslant 8$ is constant if and only if

$$g(R(X, \phi X)\phi X, Y) = 0 \qquad \phi = I, J, K \tag{5.5}$$

for all X, Y spanning totally real planes ($Q(X) \perp Q(Y)$).

Now, if condition (5.3) holds for (TM, g^H, V^H), considering the null vector $u = X^H$, it follows from (5.4) that $g((\nabla_\xi R)(X, \phi X)\phi X, X) = 0$. Therefore $g(R(X, \phi X)\phi X, Y) = 0$, for any pair X, Y of orthogonal vectors, which shows that (5.5) holds, and hence, the constancy of the quaternionic sectional curvature on M.

Proposition 1. *(TM, g^H, V^H) is an indefinite quaternionic Kähler manifold if and only if (M, g, V) is a Ricci-flat quaternionic Kähler manifold,* ($\dim M \geqslant 4$). *Moreover, in such a case, both spaces are hyperkähler.*

Proof. Considering the covariant derivative of the local section ϕ^H of the bundle of almost complex structures on TM, it follows that

$$(\widetilde{\nabla}_{X^V}\phi^H)Y^V = 0 \qquad g^H((\widetilde{\nabla}_{X^H}\phi^H)Y^V, Z^V) = 0$$

$$g^H((\widetilde{\nabla}_{X^H}\phi^H)Y^V, Z^H) = g((\nabla_X\phi)Y, Z)^V$$

$$g^H((\widetilde{\nabla}_{X^H}\phi^H)Y^H, Z^H)_\xi = R(\phi Y, Z, X, \xi)^V + R(Y, \phi Z, X, \xi)^V .$$

Since dim $TM \geq 8$, if it is quaternionic Kähler, it must be Einstein, and that is possible only for a Ricci flat M, according to Corollary 2. Moreover, if TM is quaternionic Kähler, then M is so also as a consequence of the previous expressions.

Conversely, if (M, g, V) is a Ricci flat quaternionic Kähler manifold, then [17]

$$R(X, \xi, \phi Y, Z) + R(X, \xi, Y, \phi Z) = 0$$

and the Kähler condition for TM follows from previous expressions. Finally, note that a quaternionic Kähler manifold is hyperkähler if and only if it is Ricci flat.

Remark 3. Note that the tangent bundle of a quaternionic space has constant quaternionic sectional curvature if and only if it is locally flat, which also provides examples of indefinite almost quaternionic manifolds satisfying

$$\widetilde{R}(u, \phi^H u, \phi^H u, u) = 0 \qquad \forall u \text{ null vector on } TM$$

but which are of non-constant quaternionic-sectional curvature.

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Sommario

Il fibrato tangente TM di una varietà riemanniana (M, g)*, dotato del sollevamento completo* g^C *di g, è una varietà semi-riemanniana. Lo studio dell'operatore di Jacobi lungo le geodetiche nulle su TM dà luogo ad alcune caratterizzazioni degli spazi omogenei due-punti (ad eccezione del piano di Cayley). Sono anche indicati alcuni nuovi esempi di spazi semi-riemanniani e di varietà di Kähler indefinite con tensore di Bochner ovunque nullo.*

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