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Product simple sets of polynomials in Clifford analysis (**)

1 - Notations and preliminaries

The theory of basic sets of polynomials in complex case has been already generalised to the Clifford setting (see [1], [2]). Clifford analysis is one of the possible generalizations of the theory of holomorphic functions in one complex variable to the Euclidean space of dimension larger than two. For details concerning this theory we refer the reader to [4].

The regular functions considered in the present work have values in a Clifford algebra and are null-solutions of a linear differential operator which linearizes the laplacian (see e.g. [9]).

Let $\mathcal{V}$ be an n-dimensional real (resp. complex) vector space with a bilinear form $(v|w)$, $v, w \in \mathcal{V}$ and an associated orthonormal basis $(e_1, e_2, \dots, e_n)$ such that

$$(e_i\,|\,e_j) = 0 \quad \text{if} \quad i \neq j \qquad (e_i\,|\,e_i) = -1 \qquad i, j = 1, \dots, n.$$

Consider the 2^n-dimensional real (resp. complex) vector space $\mathcal{A}$ (resp. $\mathcal{A}^C$) with the following basis $\{e_A = e_{h_1, \dots, h_r}\,|\,A = \{h_1, \dots, h_r\} \in P\{1, \dots, n\};\ 1 \leq h_1 < \dots < h_r \leq n\}$, $e_\emptyset$ being written as e_0 or 1. Then arbitrary element a of $\mathcal{A}$ (resp. $\mathcal{A}^C$) can be written as $a = \sum_A e_A a_A$, $a_A \in \boldsymbol{R}$ (resp. $a_A \in \boldsymbol{C}$).

One defines a product on $\mathcal{A}$ as follows $e_A e_B = (-1)^{\#(A \cap B)}(-1)^{P(A,B)} e_{A \triangle B}$,

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where $P(A, B) = \sum_{j \in B} P(A, j)$, $P(A, j) = \# \{i \in A,\ i > j\}$, the sets A, B and $A \triangle B$ being ordered in the prescribed way. It is easy to see that:

i e_0 is the identity element

ii $e_i e_j + e_j e_i = -2\delta_{ij}$

iii If $h_1 < h_2 < \ldots < h_r$ then $e_{h_1} \cdot e_{h_2} \ldots e_{h_r} = e_{h_1 \ldots h_r}$.

One proves also that under the above notations $\mathcal{A}$ (resp. $\mathcal{A}^C$) is a linear associative but non-commutative algebra, called the real (resp. complex) universal *Clifford algebra* over $\mathcal{V}$.

As $\mathcal{A}$ (resp. $\mathcal{A}^C$) is isomorphic to $\boldsymbol{R}^{2^n}$ (resp. $\boldsymbol{C}^{2^n}$) we may provide it with the Euclidean norm $|a| = (\sum_A |a_A|^2)^{\frac{1}{2}}$ and it is easy to show that $|a \cdot b| \leq 2^{\frac{n}{2}} |a| \cdot |b|$ $a, b \in \mathcal{A}$ (resp. $\mathcal{A}^C$).

Clifford analysis is developed within the following frame-work. On the one hand we have the Euclidean space $\boldsymbol{R}^{m+1}$, the points of which are denoted alternatively by $x = (x_0, x_1, \ldots, x_m) = (x_0, \vec{x})$; $\vec{x}$ laying in the hyperplane $x_0 = 0$ which is identified with $\boldsymbol{R}^m$. On the other hand we have the Clifford algebra $\mathcal{A}$, its space of 1-vector $\mathcal{A}_1 = \mathrm{sp}\{e_i\colon i = 1, \ldots, n\}$ having dimension n; it is assumed that $m \leq n$.

For $x \in \boldsymbol{R}^{m+1}$, $\vec{x} \in \boldsymbol{R}^m$ we put $x = \sum_{i=0}^{m} e_i x_i = x_0 + \vec{x}$, $\vec{x} = \sum_{j=1}^{m} e_j x_j$, and $\bar{x} = \sum_{i=0}^{m} \bar{e}_i x_i = x_0 - \vec{x}$.

By Ω we denote an open set in $\boldsymbol{R}^{m+1}$. The functions under consideration are of the form $f\colon \Omega \to \mathcal{A}$, $x \to f(x) = \sum_A e_A f_A(x)$, $f_A\colon \Omega \to \boldsymbol{R}$.

Introducing the generalised Cauchy-Riemann operator $D = \sum_{i=0}^{m} e_i \partial_{x_i} = e_0 \partial_{x_0} + D_0$, $D_0 = \sum_{j=1}^{m} e_j \partial_{x_j}$ one has

Definition 1. Let $\Omega \subset \boldsymbol{R}^{m+1}$ be open, then an $\mathcal{A}$-valued function f is called *(left) monogenic* in Ω, iff $f \in \boldsymbol{C}_1(\Omega; \mathcal{A})$ and $Df = 0$ in Ω.

Definition 2. A polynomial $P(x)$ is *special monogenic*, iff $DP(x) = 0$ (so $P(x)$ is monogenic) and there exists $a_{ij} \in \mathcal{A}$ for which $P(x) = \sum_{i,j}^{\text{finite}} \bar{x}^i x^j a_{ij}$.

Definition 3. Let Ω be a connected open subset of $\boldsymbol{R}^{m+1}$ containing 0 and let f be monogenic in Ω, then f is called *special monogenic* in Ω, iff its Taylor series near zero (which is known to exist) has the form $f(x) = \sum_{n=0}^{\infty} P_n(x)$ for certain special monogenic polynomials $P_n(x)$.

The fundamental references for special monogenic functions are [4], [9].

Remark. A homogeneous special monogenic polynomial $P_n(x)$ of degree n is necessarily of the form $P_n(x) = p_n(x)\, c_n$. Hereby $c_n \in \mathcal{A}$ and $p_n(x)$ is given by the generating function $\dfrac{1-\overline{x}}{|1-x|^{m+1}}$ (see [1]).

2 – Product sets of special monogenic polynomials

From the begining we recall the notion of *basic set of special monogenic polynomials*, cf. [1]. Let $\{P_n(x)\}$, $n \in N$ be a set of special monogenic polynomials i.e.

$$P_n(x) = \sum_{j=0}^{\infty} p_j(x)\, P_{nj} \qquad P_{nj} \in \mathcal{A}\,. \tag{1}$$

We shall say that $\{P_n(x)\}$ is *$\mathcal{A}$-linearly independent* if $\sum_{j=0}^{l} P_k(x)\, a_k = 0$ implies $a_k = 0$, $k = 1, \dots, l$, for every finite sequence $(a_k)_{k=1}^{l}$ in $\mathcal{A}$.

The set $\{P_n(x)\}$ will be called *basic* iff it is a *basis* for the set of special monogenic polynomials; i.e. if every $p_n(x)$ can be expressed as a right $\mathcal{A}$-linear combination

$$p_n(x) = \sum_{k=0}^{\infty} P_k(x)\, \pi_{nk} \qquad \pi_{nk} \in \mathcal{A} \tag{2}$$

we shall call $\{P_n(x)\}$ a *simple set* if $\deg P_n(x) = n$ for every $n \in N$. If for a simple set $\{P_n(x)\}$ one has $P_{nn} = 1$ for all $n \in N$, then it will be called a *simple monic set.*

Generalising a result of Whittaker ([10], T. 34, p. 40) to the Clifford setting, we can see that *a set of special monogenic polynomials $\{P_n(x)\}$ is basic, iff its Clifford matrix of coefficients $P = (P_{nj})$ has a unique row-finite inverse* $\Pi = (\pi_{nk})$, which we call the *matrix of operators*; the proof is completely similar to the complex case. The inverse set $\{\overline{P}_n(x)\}$ was defined in [3] as the basic set whose Clifford matrix of coefficients is Π, and consequently its Clifford matrix of operators is P.

If f is a special monogenic function such that $f(x) = \sum_{n=0}^{\infty} p_n(x)\, c_n$ (near 0), then

$$f(x) = \sum_{k=0}^{\infty} P_k(x) \cdot \Big(\sum_{n=0}^{\infty} \pi_{nk}\, c_n\Big)\,. \tag{3}$$

A basic set $\{P_n(x)\}$ is said to be *effective* if for every special monogenic function f, defined in a closed neighbourhood of zero $\overline{B}(R)$ of the radius $R > 0$, the series (3) converges normally to f in $\overline{B}(R)$.

Set

$$\lambda(R) = \limsup_{n \to \infty} (\lambda_n(R))^{\frac{1}{n}} \tag{4}$$

where

$$\lambda_n(R) = \sum_{k=0}^{\infty} \sup_{|x|=R} |P_k(x)\pi_{nk}| = (\sum_k \|P_k(x)\pi_{nk}\|_R). \tag{5}$$

The last sum is called the *Cannon sum* cf. [1]. We showed in [1], Theorem 1, that *a simple set* $\{P_n(x)\}$ *is effective in* $\overline{B}(R)$, *iff* $\lambda(R) = R$.

Now we are ready to give the definition of the *product set* $\{P_n(x)\}$ *of special monogenic polynomials*; a similar definition for the complex case was given by Nassif [7].

Let $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$ be two basic sets of special monogenic polynomials whose respective matrices of Clifford coefficients are P_1 and P_2; the product $P = P_2 P_1$ is the matrix of the Clifford coefficients of a set $\{P_n(x)\}$ of special monogenic polynomials.

In fact, if Π_1 and Π_2 are the respective matrices of operators of the above sets, then by a similar way of Whittaker [10], T. 34, p. 40, we deduce that $\{P_n(x)\}$ is basic. This basic set $\{P_n(x)\}$ is defined as the *product set* of the sets $\{P_n^2(x)\}$ and $\{P_n^1(x)\}$ in this order. We shall write

$$\{P_n(x)\} = \{P_n^2(x)\}\{P_n^1(x)\}. \tag{6}$$

According to this definition we can define any positive or negative power of a given basic set and also the product of more than two sets.

It is noteworthy that each of the product sets $\{P_n(x)\}\{p_n(x)\}$ and $\{p_n(x)\}\{P_n(x)\}$ is the set $\{P_n(x)\}$. Also the inverse and in fact any positive or negative power of the set $\{p_n(x)\}$ is the same set $\{p_n(x)\}$. In this theory the set $\{p_n(x)\}$ plays the part of unity and may be accordingly called the *unit set.*

From (6) one deduces

$$P_n(x) = \sum_j P_j^2(x) P_{nj}^1. \tag{7}$$

In [10] Whittaker asked, when the product set of complex polynomials is effective? In this note we answer this question for the product set of special monogenic polynomials but only for the case of simple sets.

The following example shows that in general the product of two effective basic sets need not be effective.

Example. Set

$$P_n^1(x) = p_n(x) + \frac{p_{2n}(x)}{n^n} \quad n \text{ odd} \qquad P_n^1(x) = p_n(x) \quad n \text{ even}$$

and

$$P_n^2(x) = p_n(x) + n^n p_{n+1}(x) \qquad n \text{ odd}$$

$$P_n^2(x) = (n-1)^{2(n-1)} p_{n-1}(x) + p_n(x) \quad n > 0 \text{ even}, \qquad P_0^2(x) = 1 .$$

It is easy to see that

$$P_n(x) = p_n(x) + n^n p_{n+1}(x) + (2n-1)^{2(2n-1)} \frac{p_{2n-1}(x)}{n^n} + \frac{p_{2n}(x)}{n^n} \quad n \text{ odd}$$

$$P_n(x) = (n-1)^{2(n-1)} p_{n-1}(x) + p_n(x) \quad n > 0 \text{ even}, \qquad P_0(x) = 1 .$$

This shows that the set $\{P_n(x)\}$ cannot be effective in the closed ball $\overline{B}(R)$.

Suppose further that the factor sets $\{P_n^2(x)\}$ and $\{P_n^1(x)\}$ are not effective in $\overline{B}(R)$; is the product set $\{P_n(x)\}$ not effective there either?

The answer is negative since we can take $\{P_n^1(x)\}$ as the inverse set of $\{P_n^2(x)\}$ to yield, for the product set, the unit set $\{p_n(x)\}$ which is everywhere effective.

We conclude this section with some examples.

Examples.

i. Let the basic sets $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$ be given by

$$P_n^1(x) = 1 + p_n(x) \quad \text{for } n \geq 1 \qquad P_0^1(x) = 1$$

$$P_n^2(x) = p_n(x) \qquad \text{for all } n \geq 0 .$$

The product set $\{P_n(x)\}$ which coincides with $\{P_n^1(x)\}$, is effective in $\overline{B}(R)$, for $R \geq 1$, where both the sets $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$ are effective.

ii. Now let $\{P_n^2(x)\}$ be given by

$$P_n^2(x) = \frac{1}{3^n}\, p_n(x) \qquad \text{for} \quad n \geqslant 0$$

while $\{P_n^1(x)\}$ is the same as before. It is easy to see that the product set $\{P_n(x)\}$ in this case is not effective in $\overline{B}(R)$ for $1 \leqslant R < 3$, where both the sets $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$ are effective.

We showed that to obtain the effectiveness of the product of two sets it is necessary to impose some additional conditions on $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$. We start with the following special case.

3 - Effectiveness of simple monic sets

In this section we give the key lemmas to obtain the main results of Section 4.

Lemma 1. *Let $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$ be simple monic sets of special monogenic polynomials both effective in $\overline{B}(R)$. Then the product set $\{P_n(x)\} = \{P_n^2(x)\}\{P_n^1(x)\}$ is effective in $\overline{B}(R)$.*

Proof. Since $\{P_n^\alpha(x)\}$ $(\alpha = 1, 2)$ is effective in $\overline{B}(R)$, then by Cannon's theorem (see [1]), if ρ is any finite number greater than R, then $\lambda^\alpha(R) < \rho$. Hence

$$\lambda_n^\alpha(R) < K\rho^n \qquad n \geqslant 0\,. \tag{8}$$

Since $\lambda_n^\alpha(R) \geqslant \|P_k^\alpha(x)\,\pi_{nk}^2\|_R$ and $\{P_n^\alpha(x)\}$ is effective, then for every $\rho > R$, there exists K such that

$$\|P_k^\alpha(x)\,\pi_{nk}^\alpha\|_R \leqslant K\rho^n \qquad 0 \leqslant k \leqslant n\,. \tag{9}$$

Since the set $\{P_n^\alpha(x)\}$ is monic, then for all n we get $P_{nn}^\alpha = 1$ and $\pi_{nn}^\alpha = 1$.

Using Cauchy's inequality one obtains

$$1 \leqslant \sqrt{\frac{n!}{(m)_n}}\, \frac{\|P_n^\alpha(x)\|_R}{R^n} < \frac{\lambda_n^\alpha(R)}{R^n} \qquad (m)_n = m(m+1)\ldots(m+n-1)\,. \tag{10}$$

Applying again Cauchy's inequality (to the polynomial $P_k^1(x)$) together with

(7) we have

$$\|P_k(x)\|_R \leqslant 2^{\frac{m}{2}} \sum_{j=0}^{k} \|P_j^2(x)\|_R \sqrt{\frac{k!}{(m)_k}} \frac{\|P_k^1(x)\|_R}{R^j}. \tag{11}$$

Now, since $\{P_k^\alpha(x)\}$ is monic, then

$$P_k^\alpha(x)\pi_{nk} = p_k(x)\pi_{nk} + p_{k-1}(x)P_{k,k-1}^\alpha \pi_{nk} + \dots + p_0(x)P_{k,0}^\alpha \pi_{nk}.$$

Consequently,

$$|\pi_{nk}^\alpha| \leqslant \sqrt{\frac{k!}{(m)_k}} \frac{\|P_k^\alpha(x)\pi_{nk}^\alpha\|_R}{R^k} \qquad \alpha = 1, 2. \tag{12}$$

From the definiton of the *Cannon sum* (5) and by (8), (9), (10), (11) and (12) we get

$$\begin{aligned}
\lambda_n(R) &\leqslant 2^{\frac{m}{2}} \sum_{k=0}^{n} \|P_k(x)\|_R |\pi_{nk}| \\
&< \sum_{k=0}^{n} \sqrt{\frac{k!}{(m)_k}} 2^m \sum_{j=0}^{k} \|P_j^2(x)\|_R \frac{\|P_k^1(x)\|_R}{R^j} \sum_{i=0}^{n} |\pi_{ik}^1| \, |\pi_{ni}^2| \\
&< 2^m \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{\lambda_j^2(R)}{R^j} \lambda_k^1(R) \sum_i \sqrt{\frac{k!}{(m)_k}} \frac{\|P_k^1(x)\pi_{ik}^1\|_R}{R^k} \sqrt{\frac{i!}{(m)_i}} \frac{\|P_i^2(x)\pi_{ni}^2\|_R}{R^i} \\
&< 2^m \sum_{k=0}^{n} \sum_{j=0}^{k} K\left(\frac{\rho}{R}\right)^j K \frac{\rho^k}{R^k} \sum_i \frac{\lambda_i^1(R)}{R^i} \|P_i^2(x)\pi_{ni}^2\|_R \\
&< 2^m \sum_{k=0}^{n} K\left(\frac{\rho}{R}\right)^k \sum_{j=0} K\left(\frac{\rho}{R}\right)^j \sum_i K\left(\frac{\rho}{R}\right)^i \|P_i^2(x)\pi_{ni}^2\|_R \\
&< K2^m(n+1)^3\left(\frac{\rho}{R}\right)^{3n} \lambda_n^2(R).
\end{aligned}$$

Making use of (4) we obtain $\lambda(R) \leqslant R(\frac{\rho}{R})^3$. As ρ can be chosen arbitrarily close to R, we get $\lambda(R) = R$. By Cannon's theorem of [1] the product set $\{P_n(x)\}$ is effective in $\overline{B}(R)$ as required.

Corollary 1. *If $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$ are such that $\{P_n^1(x)\}$ is simple and $\{P_n^2(x)\}$ is simple monic and both are effective in $\overline{B}(R)$ then $\{P_n(x)\} = \{P_n^2(x)\}\{P_n^1(x)\}$ is effective in $\overline{B}(R)$.*

Proof. Take $P_n^1(x) = M_n^1(x) P_{nn}^1$ then $M_n^1(x) = P_n^1(x)(P_{nn}^1)^{-1}$ is still special monogenic in the sense that $\{M_n^1(x)\}$ and $\{P_n^1(x)\}$ have the same region of effectiveness in $\overline{B}(R)$. Now, due to (6) we have

$$P_n(x) = M_n(x) P_{nn}^1$$

where $\{M_n(x)\} = \{P_n^2(x)\}\{M_n^1(x)\}$. Since $\{P_n^2(x)\}$ and $\{M_n^1(x)\}$ are monic sets and effective in $\overline{B}(R)$, then applying Lemma 1, $\{M_n(x)\}$ is effective in $\overline{B}(R)$. This means that $\{P_n(x)\}$ is effective in $\overline{B}(R)$.

The following result of [3], Theorem 1, will be used in the sequel.

Lemma 2. *If $\{P_n(x)\}$ is a simple monic set of special monogenic polynomials, effective in $\overline{B}(R)$, then the inverse set $\{\overline{P}_n(x)\}$ is effective in $\overline{B}(R)$.*

4 – Effectiveness of non-monic simple sets

In this section we prove two theorems.

Theorem 1. *If $\{P_n^1(x)\}$ is a simple set effective in $\overline{B}(R_1)$ and $\{P_n^2(x)\}$ is effective in $\overline{B}(R_2)$ and such that*

$$\text{(A)} \qquad \lim_{n \to \infty} |P_{nn}^2|^{\frac{1}{n}} = H \qquad 0 < H < \infty$$

then the product set $\{P_n(x)\} = \{P_n^2(x)\}\{P_n^1(x)\}$ is effective in $\overline{B}(R)$, $R = \max(\frac{R_1}{H}, R_2)$.

Theorem 2. *If $\{P_n(x)\}$ is a simple set effective in $\overline{B}(R)$ and such that*

$$\text{(B)} \qquad \lim_{n \to \infty} |P_{nn}|^{\frac{1}{n}} = H \qquad 0 < H < \infty$$

then the inverse set $\{\overline{P}_n(x)\}$ is effective in $\overline{B}(HR)$.

In order to prove these theorems we shall need also the following

Lemma 3. If *$\{P_n^1(x)\}$ is a simple set of special monogenic polynomials effective in $\overline{B}(R)$, and $\{P_n^2(x)\}$ is the diagonal set $\{p_n(x) P_{nn}^2\}$, where P_{nn}^2 satisfies (A), then the product set $\{P_n(x)\} = \{P_n^2(x)\}\{P_n^1(x)\}$ is effective in $B(RH^{-1})$.*

It is clear that this Lemma 3, is a particular case of Theorem 1, where the set

$\{P_n^2(x)\}$ of Theorem 1, is now everywhere effective. But we shall deduce Theorem 1 from Corollary 1 and Lemma 3.

Proof of Lemma 3. Keeping the same notation we used in the proof of Lemma 1, we have

$$\lambda_n(RH^{-1}) \leqslant 2^{\frac{m}{2}} \sum_k \|P_k(xH^{-1})\|_R \, |\pi_{nk}|$$

$$< 2^m \sum_k \| \sum_{j=0}^{k} P_j^2(xH^{-1}) P_{kj}^1 \|_R \, |\pi_{nk}^1| \, |\pi_{nn}^2|$$

$$= 2^m \sum_{k=0}^{n} \| \sum_{j=0}^{k} p_j(xH^{-1}) P_{jj}^2 P_{kj}^1 \|_R \, \frac{|\pi_{nk}^1|}{|P_{nn}^2|} .$$

From (A), given any $H' > H$, there exists a constant K such that $P_{nn}^2 < KH'^n$, $n \geqslant 0$. Hence

$$\lambda_n(RH^{-1}) \leqslant 2^m KH'^n \sum_{k=0}^{n} \|P_k^1(xH^{-1})\|_R \, \frac{|\pi_{nk}^1|}{|P_{nn}^2|} .$$

Using Cauchy-Kowalewski extension theorem (see e.g. [4], [9]) it is easy to prove that $p_n(xH^{-1}) = \frac{1}{H^n} p_n(x)$.

Then, by (8), (9), (10), (11) and (12) we get

$$\lambda_n(RH^{-1}) \leqslant 2^m K(n+1)(H'H^{-1})^n \sum_{k=0}^{n} \|P_k^1(x)\|_R \sqrt{\frac{k!}{(m)_k}} \, \frac{\|P_k^1(x)\pi_{nk}^1\|_R}{R^k} \, \frac{1}{|P_{nn}^2|}$$

$$< 2^m K(n+1)(H'H^{-1})^n \sum_k (\frac{\rho}{R})^k \|P_k^1(x)\pi_{nk}^1\|_R \, \frac{1}{|P_{nn}^2|}$$

$$< 2^m K(n+1)^2 (H'H^{-1})^n \, \frac{\lambda_n^1(R)}{|P_{nn}^2|} .$$

Since the set $\{P_n^1(x)\}$ is effective in $\overline{B}(R)$, then $\lambda^1(R) = R$.

As H' is arbitrarily chosen greater than H, then $\lambda(RH^{-1}) \leqslant RH^{-1}$. Since $\lambda(RH^{-1}) \geqslant R$, for all R, $\lambda(RH^{-1}) = RH^{-1}$. This proves Lemma 3.

We now deduce Theorem 1. Let $\{\widetilde{P}_n^2(x)\}$ be the set obtained from $\{P_n^2(x)\}$ by dividing each polynomial $P_n^2(x)$ by P_{nn}^2. The sets $\{\widetilde{P}_n^2(x)\}$ and $\{P_n^2(x)\}$ are effectively the same. Thus $\{\widetilde{P}_n^2(x)\}$ is effective in $\overline{B}(R_2)$.

Writing $\{P_n(x)\} = \{\widetilde{P}_n^2(x)\}\{p_n(x)P_{nn}^2\}\{P_n^1(x)\} = \{\widetilde{P}_n^2(x)\}\{V_n(x)\}$, by Lemma 3 and (A), the set $\{V_n(x)\}$ is effective in $\overline{B}(R_1 H^{-1})$. Then, by Corollary 1 the product set $\{\widetilde{P}_n^2(x)\}\{V_n(x)\}$ is effective in $\overline{B}(R)$, $R = \max(R_1 H^{-1}, R_2)$. This is the result of Theorem 1.

We proceed to deduce Theorem 2. Since the set $\{\widetilde{P}_n(x)\}$ is monic, its inverse set $\{\overline{\widetilde{P}}_n(x)\}$ is (by Lemma 2) effective in $\overline{B}(R)$. From $\{P_n(x)\} = \{\widetilde{P}_n(x)\}\{p_n(x)P_{nn}\}$ we have $\{\overline{P}_n(x)\} = \{p_n(x)P_{nn}^{-1}\}\{\overline{\widetilde{P}}_n(x)\}$. Applying Lemma 3 and (B), we get Theorem 2.

Theorem 1 and Theorem 2 can be combined together to yield

Corollary 2. *Let $\{P_n^1(x)\}$ and $\{P_n^2(x)\}$ be simple sets of special monogenic polynomials and suppose that the set $\{P_n^2(x)\}$ is effective in $\overline{B}(R)$ and satisfying condition (A). Then the product set $\{P_n(x)\}$ is effective in $\overline{B}(R)$, iff the set $\{P_n^1(x)\}$ is effective in $\overline{B}(HR)$.*

Proof. To prove the *only if* we first observe from (A) that $\lim_{n\to\infty} |\overline{P}_{nn}^2|^{\frac{1}{n}} = \frac{1}{H}$, and by Theorem 2, the inverse set $\{\overline{P}_n^2(x)\}$ is effective in $\overline{B}(HR)$. Suppose now that the product set $\{P_n(x)\}$ is effective in $\overline{B}(R)$. As we can put

$$\{P_n^1(x)\} = \{\overline{P}_n^2(x)\}\{P_n(x)\}$$

the conditions of Theorem 1 are satisfied for the sets $\{P_n(x)\}$ and $\{\overline{P}_n^2(x)\}$ in $\overline{B}(HR)$ with $\frac{1}{H}$ instead of H. Hence the product set $\{P_n^1(x)\}$ is effective in $\overline{B}(HR)$.

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Sommario

Un teorema di Whittaker sull'effettività degli insiemi prodotto e degli insiemi inversi di polinomi nel caso complesso viene qui considerato nelle algebre di Clifford.

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