Riv. Mat. Univ. Parma (5) 3 (1994), 295-299

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Direct product of weakly periodic subsets (**)

1 - Introduction

Let G be a finite abelian group written multiplicatively with identity element e. The product of subsets $A_1, \ldots, A_n$ of G is said to be *direct* if each element g of the product $A_1 \ldots A_n$ is uniquely expressible in the form

$$g = a_1 \ldots a_n \qquad a_j \in A_j . \tag{1}$$

Direct product of subsets is a generalization of the direct product of subgroups which is a commonly used construction. In case each element g of the product $A_1 \ldots A_n$ can be expressible precisely k ways in form (1), then we say that the product is a *k-fold product* of the subsets.

Three types of subsets will occur. A subset A of G is called *periodic* if there is an element $g \in G \backslash \{e\}$ with $Ag = A$. The element g is called a *period* of A. If Ag and A differ in at most one element, that is, Ag contains at most one element not in A, and A contains at most one element not in Ag, then A is called *weakly periodic*. The subset A is *cyclic* if it is of form $\{e, a, a^2, \ldots, a^{r-1}\}$. This cyclic subset is denoted by $[a, r]$.

G. Hajós [3] proved that if a finite abelian group is a direct product of cyclic subsets, then at least one of the factors is periodic. L. Fuchs [1] generalized Hajós' theorem by replacing the cyclic factors by weakly periodic subsets. In this paper we show that Fuchs' result holds for multiple products as well, provided that the multiplicity is relatively prime to the order of the group.

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(**) Received November 24, 1993. AMS classificazione 20 K 01.

As open problem 82 of [1] L. Fuchs proposed the following question: if a direct product of weakly periodic subsets of a finite abelian group is periodic, then at least one of the factors is in fact periodic?

O. Fraser and B. Gordon [2] showed that the answer is affirmative if the group is cyclic or a p-group or the number of the factors is less than four, but gave a negative answer in general by exhibiting a counterexample. In this example a periodic subset of order 16 of the non-cyclic group of order 18 is a direct product of subsets of order 2. Note that as 16 does not divide 18 the periodic subset cannot be a direct factor of the group.

In this paper we shall prove that Fuchs' question has a positive answer if the additional condition is imposed that the periodic subset is a direct factor of the group.

2 - Factoring a periodic subset

First we describe the periodic and weakly periodic subsets. Let g be an element of the finite abelian group G. The map $h \to hg$, $h \in G$ permutes the elements of G. This permutation is a union of disjoint cycles. The lengths of these cycles are the same, namely $|g|$, the order of g. If A is a periodic subset of G with period g, then the elements of A must fill complete cycles. Consequently, A has the form $C\langle g\rangle$. Here C is a subset of G, $\langle g\rangle$ is the subgroup generated by g and the product is direct. A similar argument gives that if A is a weakly periodic subset, then A has one the forms

$$C\langle g\rangle \qquad x[g, r] \qquad C\langle g\rangle \cup x[g, r] \tag{2}$$

where $C \subset G$, $x \in G$, $1 \leqslant r \leqslant |g| - 1$, the product is direct and the union is disjoint.

We work in the group ring $Z(G)$ whose elements are the formal sums $\sum_{g \in G} \lambda_g g$, where λ_g is an integer. (The addition and multiplication are defined like for polynomials.) A subset A of G corresponds to the element $\overline{A} = \sum_{a \in A} a$ of $Z(G)$. The fact that the subset A is a direct product of subsets $A_1, \ldots, A_n$ corresponds to the equation $\overline{A} = \overline{A}_1 \ldots \overline{A}_n$ in $Z(G)$.

Theorem 1. *If a periodic subset of a finite abelian group is a direct factor of the group and itself is a direct product of weakly periodic subsets, then at least one of the weakly periodic factors is in fact periodic.*

Proof. Let $G = AB$ and let $A = A_1 \dots A_n$, where each A_i is a weakly periodic subset of G and the products are direct. We prove the result by induction. If $n = 1$, then $A = A_1$ and so A_1 is periodic. The direct product $A = A_1 \dots A_n$ corresponds to the equation $\overline{A} = \overline{A}_1 \dots \overline{A}_n$ in $Z(G)$. Let h be a period of A and multiply by $e - h$ to get $0 = \overline{A}_1 \dots \overline{A}_n(e - h)$. If A_n is periodic, then the proof ends. So we may assume that A_n is not periodic. This means that A_n is of form $(2)_2$ or $(2)_3$. Multiplying by $e - g$ we have $0 = \overline{A}_1 \dots \overline{A}_{n-1}(e - h)x(e - g^r)$. Cancel x from this equation. By Theorem 2 [5] there are $K, L \subset G$ such that $A_1 \dots A_{n-1} = K\langle h \rangle \cup L\langle g^r \rangle$, where the products are direct and the union is disjoint. If $L = \emptyset$, then $A_1 \dots A_{n-1}$ is periodic with period h and it is a direct factor of G. The induction assumption now gives that one of the factors is periodic. Thus we may suppose that $L \neq \emptyset$. Consider first the case when A_n is of form $(2)_3$. Then $C \neq \emptyset$. Therefore $\langle g \rangle \subset (c^{-1}A_n)$, where $c \in C$. Further $g^r \in l^{-1}A_1 \dots A_{n-1}$, where $l \in L$. This contradicts the directness of the product $(l^{-1}c^{-1}A) = (l^{-1}A_1 \dots A_{n-1})(c^{-1}A_n)$. Thus A_n is a cyclic subset.

In the remaining part of the proof we may assume that each A_i factor is cyclic. Using the identity $[a, rs] = [a, r][a^r, s]$ we can replace each cyclic subset by a direct product of cyclic subsets of prime orders. Further, if one of the new cyclic subsets is periodic, so is one of the originals. Therefore A is a direct product of cyclic subsets $A_1, \dots, A_m$ of prime order. The case $m = 1$ is trivial and so we start a new induction on m. Multiplying by $e - h$ we get an equation $0 = \overline{A}_1 \dots \overline{A}_m(e - h)$ in $Z(G)$ as we already seen in the first part of the proof. The factor $e - h$ cannot be cancelled from this equation without violating the equation, since $A_1 \dots A_m$ is equal to A. If a term $\overline{A}_i$ can be omitted, then we can conclude by the inductive assumption. Thus we may assume that no term may be omitted from the equation. Now Hajós' zero divisor theorem (see Lemma 84.8 [1]) applies and gives that the number of the (not necessarily distinct) prime divisors of $|\langle A_1, \dots, A_m, h \rangle|$ does not exceed m. Let $M = \langle A_1, \dots, A_m \rangle$. Then $G = AB$ and $A \subset M$ implies $M = A(B \cap M)$. From this it follows that $|A|$ divides $|M|$. From $|A| = |A_1| \dots |A_m|$ we deduce that $A = M$. Now Hajós' theorem completes the proof.

3 - Multiple factoring

Rédei [4] developed a method using characters of G to study direct products of subsets of G. If A is a subset and χ is a character of G, then we put $\chi(A) = \sum_{a \in A} \chi(a)$. Rédei observed that if $\chi(A) = \chi(B)$ for each character χ of G, then the subsets A and B must be equal.

If A and A' are subsets of the finite abelian group G such that for every subset B of G, if the direct product AB gives G, then $A'B$ is also a direct product giving G, then we shall say that A is *replaceable* by A'. Consequently if $|A| = |A'|$ and if from $\chi(A) = 0$ it follows that $\chi(A') = 0$, then A can be replaced by A'. This criterion extends to multiple products as well and this is what we will use.

Theorem 2. *If the product of weakly periodic subsets is a multiple product of a finite abelian group and the multiplicity is relatively prime to the order of the group, then at least one of the factors is periodic.*

Proof. Let G be a finite abelian group and let $A_1, \dots, A_n$ be weakly periodic subsets of G. The equation $k\overline{G} = \overline{A}_1 \dots \overline{A}_n$ in $Z(G)$ corresponds to the fact that the product $A_1 \dots A_n$ is a k-fold product of G.

The weakly periodic subset A_i is of one of the forms $(2)_1$, $(2)_2$, $(2)_3$. Using the observation that if $k\overline{G} = \overline{A}_1 \dots \overline{A}_n$, then $k\overline{G} = (a_1^{-1}\overline{A}_1)\dots(a_n^{-1}\overline{A}_n)$, where $a_1, \dots, a_n \in G$, we may assume that $x = e$ in the previous representations.

If A_i is of form $(2)_2$, then it is cyclic. Using Rédei's character test we show that if A_i is of form $(2)_3$, then it is replaceable by $[g, s]$, where $s = |A_i|$.

Clearly $\chi([g, s]) = s$ if $\chi(g) = 1$. If $\chi(g) \neq 1$, then we have

$$\chi([g, s]) = \frac{1 - \chi(g^s)}{1 - \chi(g)} .$$

Thus $\chi([g, s]) = 0$ if and only if $\chi(g) \neq 1$ and $\chi(g^s) = 1$. We are going to show that from $\chi(A_i) = 0$ it follows that $\chi(g) \neq 1$ and $\chi(g^s) = 1$.

To do so, suppose that $A_i = \{a_1, \dots, a_s\}$ and $a_1 \notin A_i g$ and $a_j g \notin A_i$ for some j, $1 \leqslant j \leqslant s$. Let χ be a character of G for which $\chi(A_i) = 0$. Multiplying by $\chi(g)$ we get $\chi(A_i g) = \chi(A_i)\chi(g) = 0$. Comparing $\chi(A_i) = 0$ and $\chi(A_i g) = 0$ we see that $\chi(a_1) = \chi(a_j g)$. Consequently $\chi(a_1), \chi(a_2), \dots, \chi(a_s)$ is a rearrangement of $\chi(a_1 g), \chi(a_2 g), \dots, \chi(a_s g)$. Therefore their products are equal and so, after cancelling, we get $\chi(g^s) = 1$. If $\chi(g) = 1$, then $0 = \chi(A_i) = \chi(C)t + r$, where $t = |g|$. Hence $\chi(C) = -\dfrac{r}{t}$. The left hand side is an algebraic integer of a cyclotomic field over the rationals, the right hand side is a rational number therefore it must be an integer. But this is not possible since $-1 < -\dfrac{r}{t} < 0$. Consequently A_i can be replaced by $[g, s]$ in any multiple product. If $[g, s]$ is periodic, then $g^s = e$ and so $|g|$ divides s. As $s = |C|\,|g| + r$, $|g|$ divides r and so A_i is periodic.

If A_i in $k\overline{G} = \overline{A}_1 \dots \overline{A}_n$ is of form $(2)_1$, then A_i is periodic and so there is nothing to prove. Thus we many assume that each A_i is of form $(2)_2$ or $(2)_3$. Replace each A_i of form $(2)_3$ by a cyclic subset. In the resulting k-fold product each factor is cyclic. Each cyclic subset is a direct product of cyclic subsets of prime order as we have already seen in the previous proof. Let $k\overline{G} = \overline{A}_1 \dots \overline{A}_m$ be the equation in $Z(G)$, which corresponds to the resulting k-fold product. Let χ be a nonprincipal character of G. As $\chi(G) = 0$, there must be a factor, say A_i, for which $\chi(A_i) = 0$. In case $A_i = [a_i, r_i]$, this is equivalent to $\chi(a_i) \neq 1$ and $\chi(a_i)^{r_i} = 1$. Since r_i is a prime and $\chi(a_i)$ is a $|G|$-th root of unity it follows that r_i is a divisor of $|G|$. Thus for each nonprincipal character χ of G there is a factor A_i such that $\chi(A_i) = 0$ and $|A_i|$ is a divisior of $|G|$.

Using the fact that the multiplicity k is relatively prime to $|G|$ and each $|A_i|$ is a prime and $k|G| = |A_1| \dots |A_m|$, we have that $|G|$ is equal to the product of all the $|A_i|$'s for which $|A_i|$ divides $|G|$.

Let D be the product of the factors A_i whose order divide $|G|$. We show that $\chi(D) = \chi(G)$ for each character χ of G. We have already seen this for the nonprincipal characters. For the principal character this reduces to $|D| = |G|$ what we have checked again. Consequently $D = G$, that is D is a direct product giving G. So by Hajós' theorem one of the occurring cyclic subsets is periodic and finally one of the original factors must be periodic. This completes the proof.

References

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Sommario

Si indicano condizioni perché in un prodotto diretto, in un prodotto multiplo, di sottoinsiemi debolmente periodici uno dei fattori sia periodico.

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