Riv. Mat. Univ. Parma (5) 4 (1995), 1-5

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The construction of the similar Ceva-triangle (**)

1 - Introduction

Let ABC be a given triangle. If A_1, B_1, C_1 are three points on the sides BC, AC, AB respectively, then $A_1B_1C_1$ is said to be an inscribed triangle in ABC. If furthermore, the lines AA_1, BB_1, CC_1 are concurrent, then $A_1B_1C_1$ is said to be a *Ceva-triangle* in ABC.

In [3], K. Seebach proved an interesting theorem. Let $A_0B_0C_0$ be a triangle. Then there is one and only one Ceva triangle $A_1B_1C_1$ inscribed in ABC such that $A_1B_1C_1$ is *similar* to $A_0B_0C_0$ with $\widehat{A}_1 = \widehat{A}_0$, $\widehat{B}_1 = \widehat{B}_0$, $\widehat{C}_1 = \widehat{C}_0$.

A very natural problem arises: *How to construct the unique Ceva triangle $A_1B_1C_1$ if ABC and $A_0B_0C_0$ are given?*

In this note, using the idea in [4], we prove that, *in general, the above mentioned problem is an impossible construction by ruler and compass.* We first deduce a cubic trigonometric polynomial, then give a counterexample to show that, for some given ABC and $A_0B_0C_0$, the uniquely determined Ceva-triangle cannot be constructed by ruler and compass.

2 - A cubic trigonometric equation

Let ABC and $A_0B_0C_0$ be the given triangles, and $A_1B_1C_1$ be the inscribed Ceva-triangle such that $\widehat{A}_1 = \widehat{A}_0$, $\widehat{B}_1 = \widehat{B}_0$, $\widehat{C}_1 = \widehat{C}_0$. All triangles occurring in the paper are assumed to be *oriented triangles.* For example, AB, BC, CA are re-

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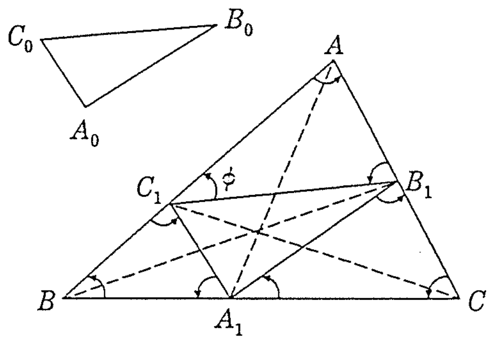
(**) Received August 30, 1993. AMS classification 51 M 15.

garded as positive segments. Correspondingly the angles BAC, CBA, ACB shall be regarded as positive angles.



Denote by ϕ the angle B_1C_1A. Then it is easily seen that

$$\text{angle } AB_1C_1 = \pi - \hat{A} - \phi$$

$$\text{angle } BC_1A_1 = \pi - \hat{C}_0 - \phi \qquad \text{angle } BA_1C_1 = \hat{B} - \hat{C}_0 - \phi$$

$$\text{angle } CA_1B_1 = \hat{B} + \hat{B}_0 - \phi \qquad \text{angle } CB_1A_1 = \hat{B}_0 - \hat{A}_0 - \phi\,.$$

Since $A_1B_1C_1$ is a Ceva-triangle, we have

(1) $$AC_1 \cdot BA_1 \cdot CB_1 = -AB_1 \cdot BC_1 \cdot CA_1\,.$$

It is easily seen that in the triangles A_1BC_1, A_1B_1C and AB_1C_1 we have

(2) $$\frac{BA_1}{\sin(\hat{C}_0 + \phi)} = -\frac{BC_1}{\sin(\hat{C}_0 - \hat{B} + \phi)}$$

(3) $$\frac{CB_1}{\sin(\hat{B} + \hat{B}_0 - \phi)} = -\frac{CA_1}{\sin(\hat{A} - \hat{B}_0 + \phi)}$$

(4) $$\frac{AC_1}{\sin(\hat{A} + \phi)} = -\frac{AB_1}{\sin\phi}\,.$$

Hence

$$\begin{aligned}(5)\qquad &\sin(\hat{A}+\phi)\sin(\hat{C}_0+\phi)\sin(\hat{B}+\hat{B}_0-\phi)\\ &=\sin\phi\ \sin(\hat{C}_0-\hat{B}+\phi)\sin(\hat{A}-\hat{B}_0+\phi).\end{aligned}$$

Changing the product to be difference, we have

$$\begin{aligned}(6)\qquad &[\cos(\hat{A}+\hat{C}_0+2\phi)-\cos(\hat{A}-\hat{C}_0)]\sin(\hat{B}+\hat{B}_0-\phi)\\ &=\sin\phi[\cos(\hat{C}_0+\hat{A}-\hat{B}-\hat{B}_0+2\phi)-\cos(\hat{C}_0+\hat{B}_0-\hat{A}-\hat{B})]\end{aligned}$$

$$\begin{aligned}(7)\qquad &\sin(\hat{A}+\hat{B}+\hat{B}_0+\hat{C}_0+\phi)-\sin(\hat{A}+\hat{C}_0-\hat{B}-\hat{B}_0+3\phi)\\ &\quad -2\cos(\hat{A}-\hat{C}_0)\sin(\hat{B}+\hat{B}_0-\phi)\\ &=\sin(\hat{C}_0+\hat{A}-\hat{B}-\hat{B}_0+3\phi)-\sin(\hat{C}_0+\hat{A}-\hat{B}-\hat{B}_0+\phi)\\ &\quad -2\cos(\hat{C}_0+\hat{B}_0-\hat{A}-\hat{B})\sin\phi\,.\end{aligned}$$

Since $\hat{A}+\hat{B}+\hat{B}_0+\hat{C}_0=2\pi-(\hat{C}+\hat{A}_0)$ one gets

$$\sin(\hat{A}+\hat{B}+\hat{B}_0+\hat{C}_0+\phi)=\sin(2\pi-(\hat{C}+\hat{A}_0-\phi))=-\sin(\hat{C}+\hat{A}_0-\phi)$$

and relation (7) becomes

$$\begin{aligned}(8)\qquad &-\sin(\hat{C}+\hat{A}_0-\phi)-2\cos(\hat{C}_0-\hat{A})\sin(\hat{B}+\hat{B}_0-\phi)-2\sin(\hat{P}+3\phi)\\ &=\sin(P+3\phi)-2\cos(C-A_0)\sin\phi\end{aligned}$$

where $\hat{P}=\hat{A}-\hat{B}+\hat{C}_0-\hat{B}_0$.

Now, using the formulas

$$\cos 3\phi=4\cos^3\phi-3\cos\phi \qquad \sin 3\phi=3\sin\phi-4\sin^3\phi$$

we obtain

$$(9)\qquad K\cos^3\phi+L\sin^3\phi+M\cos\phi+N\sin\phi=0$$

where

$$(10)\qquad K=8\sin\hat{P} \qquad L=-8\cos\hat{P}$$

$$(11)\qquad M=\sin(\hat{C}+\hat{A}_0)+2\cos(\hat{A}-\hat{C}_0)\sin(\hat{B}+\hat{B}_0)-7\sin\hat{P}$$

$$\begin{aligned}(12)\qquad N=&-\cos(\hat{C}+\hat{A}_0)-2\cos(\hat{A}-\hat{C}_0)\cos(\hat{B}+\hat{B}_0)\\ &+5\cos\hat{P}-2\cos(\hat{C}-\hat{A}_0).\end{aligned}$$

Remark now that, since we have $\cos^2\phi = (1+\tan\phi)^{-1}$, then relation (9) results to be equivalent to

$$K\cos^2\phi + L\tan\phi\,\sin^2\phi + M + N\tan\phi = 0\,. \tag{13}$$

Hence we have the trigonometric equation

$$(N+L)\tan^3\phi + M\tan^2\phi + N\tan\phi + (K+M) = 0\,. \tag{14}$$

3 - A counterexample

Suppose that, given the triangles ABC and $A_0B_0C_0$, the Ceva-triangle $A_1B_1C_1$ can be constructed by ruler and compass, then surely the angle ϕ can be constructed by ruler and compass too. Therefore the number $\tan\phi$ must be a constructible number (see [1] or [2]). But the following counterexample shows that for some triangles ABC and $A_0B_0C_0$, the number $\tan\phi$ is not constructible.

Let ABC be an equilateral triangle with unit side. Let $A_0B_0C_0$ be a triangle with $\hat{A}_0 = \frac{\pi}{6}$, $\hat{B}_0 = \frac{\pi}{3}$, $\hat{C}_0 = \frac{\pi}{2}$, and consequently $\hat{P} = \frac{\pi}{6}$. Then

$$\begin{aligned}
K &= 8\sin\frac{\pi}{6} = 4 \\
L &= -8\cos\frac{\pi}{6} = -4\sqrt{3} \\
M &= -\sin\frac{\pi}{2} + 2\cos\frac{\pi}{6}\sin\frac{2\pi}{3} - 7\sin\frac{\pi}{6} = -1 \\
N &= \cos\frac{\pi}{2} - 2\cos\frac{\pi}{6}\cos\frac{2\pi}{3} + 5\cos\frac{\pi}{6} - 2\cos\frac{\pi}{6} = 2\sqrt{3}\,.
\end{aligned}$$

Therefore $\tan\phi$ is the solution of the cubic equation.

$$2\sqrt{3}\,x^3 + x^2 - 2\sqrt{3}\,x - 3 = 0\,. \tag{15}$$

Let $x = \sqrt{3}\,y$. Then we have

$$6y^3 + y^2 - 2y - 1 = 0\,. \tag{16}$$

Now we check that the polynomial $6y^3 + y^2 - 2y - 1$ is *irreducible in rational numbers*. Suppose the contrary, $6y^3 + y^2 - 2y - 1 = (y-r)\,(ay^2+by+c)$, then r must be a rational solution of the equation (16). By a well known result ([2], p. 160), r must be one of ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{6}$. It is easily checked by synthetic division that none of them is a solution of equation (16). This shows that $6y^3 + y^2 - 2y - 1$ is an irreducible cubic polynomial. Therefore y is not a constructible number. Since $\tan \phi = \sqrt{3}\, y$, ϕ cannot be constructed by ruler and compass.

References

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Sommario

Un controesempio mostra che, assegnati due triangoli ABC e $A_0B_0C_0$, *non è possibile, in generale, costruire con riga e compasso un triangolo di Ceva* $A_1B_1C_1$, *inscritto in ABC e simile ad* $A_0B_0C_0$.

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