Riv. Mat. Univ. Parma (5) 4 (1995), 41-47

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On graphs for which the point-distinguishing chromatic index equals the intersection number (**)

1 - Introduction

Let S be a set and $F = \{S_1, S_2, \ldots, S_p\}$ a non-empty family of distinct non-empty subsets of S whose union is S. The intersection graph of F has F as vertex set with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

Let $G = (V, E)$ be a finite graph, where $|E| = q$. The *intersection number* $\omega(G)$ of G is the minimum number of elements in a set S such that G is an intersection graph on S [1]. A point distinguishing (p.d.) h-coloring of G is a coloring of the edges of G using h colors, so that the color sets which correspond to distinct vertices are distinct. The *point distinguighing chromatic index* of G, denoted by $\chi_0(G)$, is the minimum number h of colors such that G has a p.d. h-coloring.

In [2] the problem of determining the graphs G for which

$$\omega(G) = \chi_0(G) \tag{1}$$

has been posed. In [3] we considered this problem and proved the following results

Proposition 1. *For every graph G we have* $\omega(G) \geq \chi_0(G)$.

Theorem 1. *A graph G satisfies the equality* $\omega(G) = \chi_0(G)$, *if and only if*

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(**) Received June 8, 1994. AMS classification 05 C 75. This research has been partially supported by MURST.

there exists a p.d. $\omega(G)$-coloring of G where non-adjacent vertices correspond to disjoint color sets.

In this paper we present some results concerning the graphs satisfying (1). In particular (Theorem 3) we prove that there are only three special graphs satisfying (1), having a cut-vertex not adjacent to any end-vertex.

Now, let us introduce some notation. Let $P(S)$ be the power set of a set S, where $|S| = \omega(G)$. A representation of G is an assignment of one member S_i of $P(S)$ to each vertex of G so that G is an intersection graph on S. A set assigned to a vertex v of a graph G in a suitable representation of G is denoted by $\alpha(v)$ and is said to correspond to v.

Throughout this paper we suppose G distinct from K_2.

2 - The point distinguishing chromatic index equals the number of edges

In [1] it is proved that, for $G \neq K_3$, $\omega(G) = q$ if and only if G has no triangles. Now we consider a similar problem concerning the parameter $\chi_0(G)$. In fact, in Theorem 2 we characterize the graphs G satisfying $\chi_0(G) = q$.

Theorem 2. *A connected graph G satisfies the equality $\chi_0(G) = q$, if and only if it is isomorphic to either $K_{1,n}$ or K_3 or P_3.*

Proof. We prove the non trivial implication. The condition $\chi_0(G) = q$ implies that in every p.d. q-coloring of G, every edge has a color different from the remaining ones.

Let v be a vertex of maximum degree. Since G is not K_2, v is adjacent to at least two vertices, say u_1, u_2.

If all other vertices are adjacent to v, then G is either isomorphic to the star graph or is obtained from it by adding some edges connecting vertices adjacent to v.

Suppose u_1, u_2 are adjacent. If G does not contain other vertices, then $G \simeq K_3$ and $\chi_0(K_3) = 3$.

If there is at least another vertex u_3 adjacent to v, then in a p.d. q-coloring the color assigned to the edge $u_1 u_2$ can be substituted by the color assigned to the edge vu_1. So we obtain new color sets corresponding to the vertices u_1 and u_2 denoted by $\alpha(u_1)$ and $\alpha(u_2)$.

Clearly, $\alpha(u_1) \neq \alpha(u_2)$; moreover they are distinct from $\alpha(v)$, since this set contains also the color assigned to the edge vu_3. So in the case when all vertices are adjacent to v, the graph G satisfying $\chi_0(G) = q$ is isomorphic to either $K_{1,n}$ or K_3.

Now, let w be a vertex non-adjacent to v. Whithout loss of generality, we may suppose that w is adjacent to u_1. If there are no other vertices or other adjacencies, then G is isomorphic to P_3.

Assume that there exists at least another edge e distinct from the preceding ones. We change its color to that assigned to vu_1 or wu_1 according to w has degree 1 or more than 1. It is easy to see that we can not obtain distinct vertices having the same color set; so again we obtain a contradiction.

A simple consequence of the above results is

Corollary 1. *A connected graph G satisfies $\omega(G) = \chi_0(G) = q$, if and only if G is isomorphic to $K_{1,n}$ or to K_3 or to P_3.*

3 - Graphs with a cut-vertex

The graphs R_1, R_2 and R_3 of Fig. 1 have a cut-vertex not adjacent to any end-vertex and satisfy (1). In this case we have $\omega(G) = \chi_0(G) = 4$ and a p.d. 4-coloring satisfying Theorem 1 is represented using the colors $1, 2, 3, 4$. We shall prove that they are the only graphs satisfying these conditions.

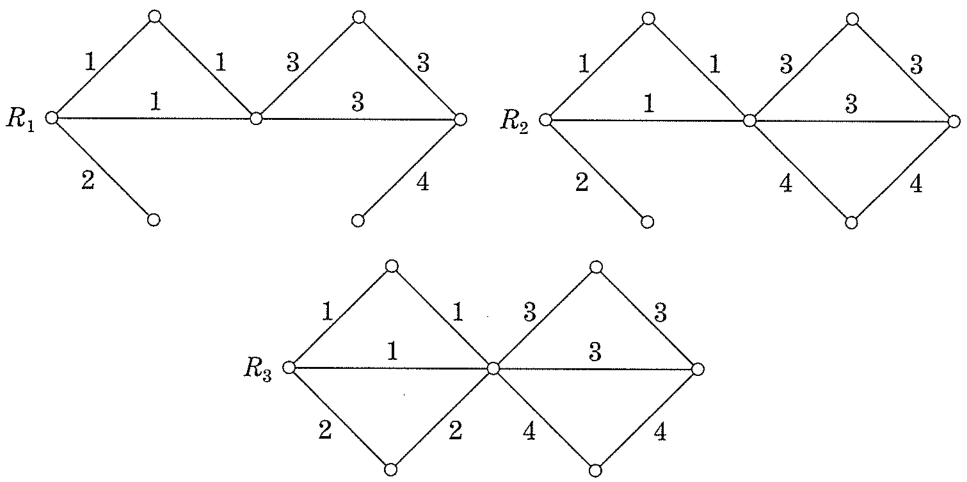

Fig. 1.

Theorem 3. *Let G be a graph with a cut-vertex not adjacent to any end-vertex. Then G satisfies $\omega(G) = \chi_0(G)$, if and only if it is isomorphic to one of R_1, R_2 or R_3.*

Proof. If G is isomorphic to one of R_1, R_2 or R_3, then clearly $\omega(G) = \chi_0(G)$. Thus let us assume that G satisfies $\omega(G) = \chi_0(G)$.

Let v be a cut-vertex of G not adjacent to any end-vertex and u_1, w_1 two vertices adjacent to v and belonging to different components in $G - v$. Denote by $[u_1]$ and $[w_1]$ the components of $G - v$ containing u_1, w_1 respectively, with the addition of v and the edges connecting v with the vertices of $[u_1]$, $[w_1]$.

Let ϕ be a p.d. χ_0-coloring of G satisfying the condition of Theorem 1 i.e. such that the color sets corresponding to non-adjacent vertices are disjoint. So we have that $\alpha(u_1) \cap \alpha(w_1) = \emptyset$.

By the condition on v, all components of G-v are colored with at least two colors.

With respect to the components $[u_1]$ and $[w_1]$ we have to distinguish two cases:

I the components are colored by two colors

II at least one of the components is colored by at least three colors.

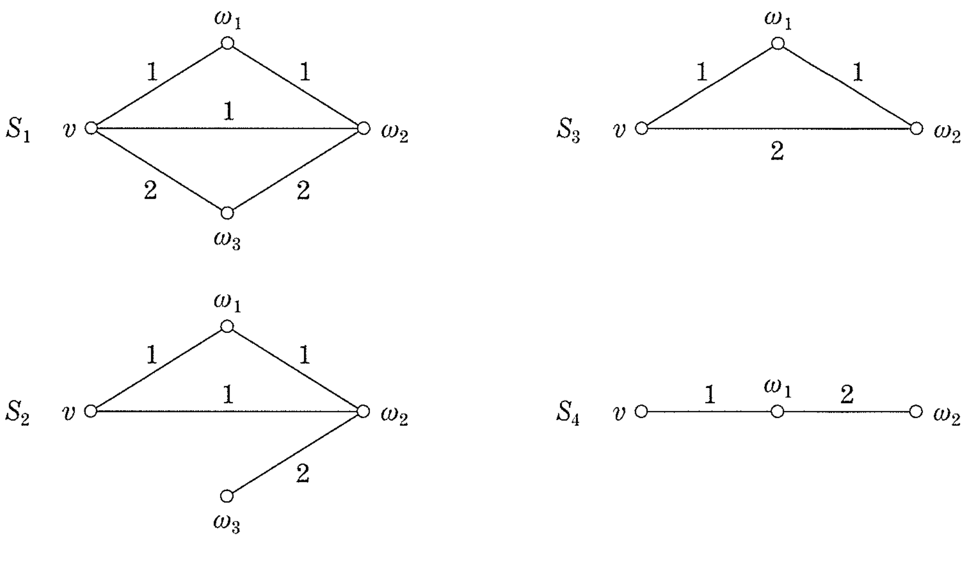

Fig. 2.

Case **I.** It is easy to see that $[u_1]$ and $[w_1]$ are isomorphic to one of the following graphs S_i, $1 \leqslant i \leqslant 4$ (Fig. 2), where the colors are denoted by 1 and 2 and $\alpha(v)$ is supposed to contain at least a color corresponding to the other component.

So G is obtained by identifying the vertex v of two subgraphs S_i, S_j, $1 \leqslant i, j \leqslant 4$ and is denoted by $S_i . S_j$.

It is clear that, for $i \neq j$, $S_i . S_j$ is isomorphic to $S_j . S_i$. We note that $S_1 . S_1$, $S_2 . S_2$ and $S_1 . S_2$ are the graphs R_3, R_1, R_2 of Fig. 1.

The remaining graphs $S_3 . S_3$, $S_4 . S_4$ and $S_i . S_j$, $i > j$, have a p.d. 3-coloring, not satisfying the condition of Theorem 1 (Fig. 3).

Case **II.** Assume that $[w_1]$ is colored by at least three colors b_1, b_2, b_3 and denote by a_1, a_2 two colors of $[u_1]$, where $a_1, b_1 \in \alpha(v)$.

Denote by H the subgraph induced by the vertices whose color set contains a_1. Let us distinguish the subcases in which H is isomorphic to K_2, K_3 or K_n, $n \geqslant 4$.

Assume that $H \simeq K_2$. Then we change a_1 in b_1. In the case when we obtain $\alpha(v) = \{b_1\}$ and there is a vertex of $[w_1]$, say w_1, whose color set is $\{b_1\}$, we no-

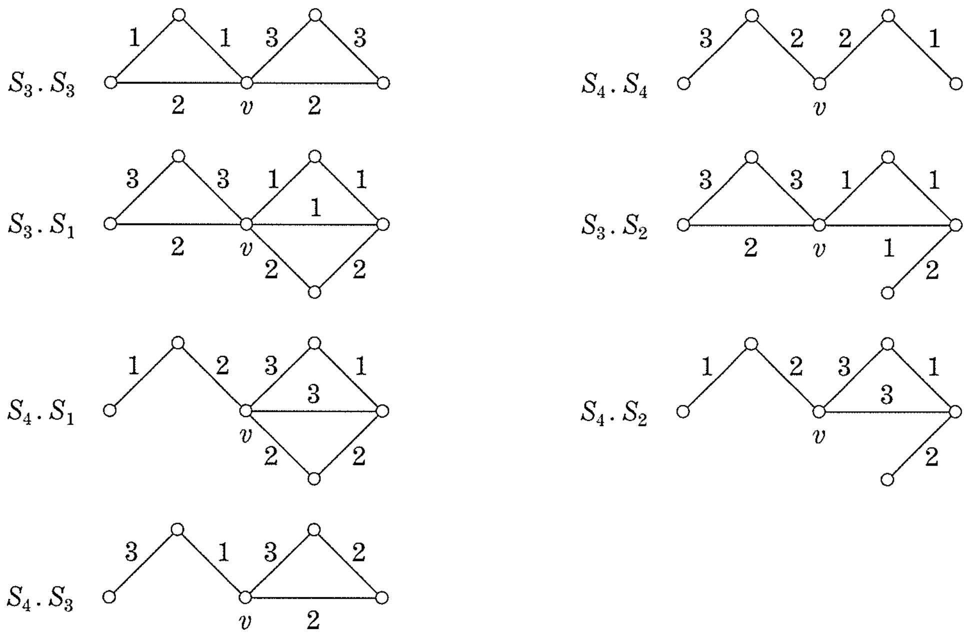

Fig. 3.

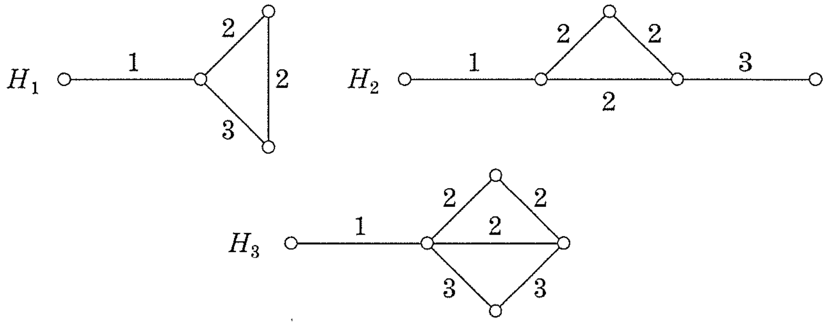

Fig. 4.

te first that w_1 is adjacent to w_2 distinct from v and that vw_2 is colored b_1. Then we assign to the edges vu_1, vw_1 the colors b_2, a_2 respectively.

If $H \simeq K_3$ and also $[u_1] \simeq K_3$, whose vertices are denoted v, u_1, u_2, then we assign to vu_1, vu_2 and u_1u_2 the colors a_2, b_2 and a_2 respectively. Otherwise, if $[u_1] \neq K_3$ and u_2 is assumed to be adjacent to at least another vertex, we assign to vu_1, vu_2 and u_1u_2 the colors a_2, b_3 and b_2 respectively.

Finally, assume that $H \simeq K_n$, $n \geq 4$. Let u_1, u_2 vertices whose color sets contain at least one color a_2, a_3 distinct from a_1. Assume that u_1 is the possible vertex whose color set is $\{a_1\}$. Then we change a_1 in b_1 and we assign to the edges vu_1, vu_2, u_1u_2 and u_1u_3 the colors b_2, a_2, b_1 and a_3.

In each subcase all the color sets are distinct and we obtain a p.d. $(\chi_0 - 1)$-coloring of G, a contradiction.

It is easy to see that, for $n \leq 6$, the only graphs G with a cut-vertex adjacent to end-vertices, satisfying (1) are: $K_{1,n}$, $2 \leq n \leq 5$, P_3 and the graphs H_1, H_2, H_3 depicted in Fig. 4.

We note that for these graphs the p.d. 3-coloring indicated in Fig. 4 satisfies Theorem 1.

4 - Graphs satisfying (1)

In [2], it is proved that for $2^{k-1} < n \leq 2^k$, $\chi_0(K_n) = k + 1$.

For $n = 2^k$, a p.d. $(k + 1)$-coloring of K_n is obtained by assigning to the vertices of K_n all the 2^k subsets of $A_k = \{1, 2, \ldots k\}$ with the addition of the $(k + 1)$-th color.

Denote by Y_n the intersection graph, whose vertex set is the union of all the subsets of A_k with the addition of the color $k + 1$ and of i, $0 \leq i \leq 2^k - 1$, subsets of A_k without the $(k+1)$-th color and distinct from the empty set. Then the order of Y_n satisfies $2^k \leqslant n \leqslant 2^{k+1}-1$. Clearly, $\omega(Y_n)=k+1$ and also $\chi_0(Y_n)=k+1$.

In fact, if $\chi_0(Y_n)<k+1$, since Y_n has a subgraph K_m, for $m=2^k$, we have a p.d. coloring of K_m with less than $k+1$ colors, a contradiction. Hence $\chi_0(Y_n)\geqslant k+1$. By Proposition 1, it follows that $\chi_0(Y_n)=k+1$.

References

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[2] F. HARARY and M. PLANTHOLT, *The point-distinguishing chromatic index*, Graphs and Applications, Wiley-Interscience, New York 1985, 147-162.

[3] N. ZAGAGLIA SALVI, *On the intersection number and point-distinguishing chromatic index of a graph*, Vishwa Internat. J. Graph Theory 1 (1992), 103-109.

Sommario

Siano $\omega(G)$ e $\chi_0(G)$, rispettivamente, il numero di intersezione e l'indice cromatico con distinzione vertici di un grafo G. Si stabiliscono alcuni risultati sui grafi soddisfacenti la condizione $\omega(G)=\chi_0(G)$. In particolare si dimostra che, tra questi grafi, solo tre hanno vertice di rottura non adiacente ad alcun vertice terminale.

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