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YURI BOZHKOV (*)

The specific Hermitian geometry of certain three-folds (**)

1 - Introduction

The *non-Kählerian Hermitian manifolds* appear to be quite challenging due to the lack of analytic tools. For, to study an Hermitian manifold it is always useful to pick a metric with some special properties. Especially, for description of the Hermitian deformations, one needs to impose an additional condition which can be a substitute of the Kähler one. It is natural to suppose in the first place that such conditions are to be found in the terms of the torsion. Here is a possible list:

I. Kähler metrics: $T = 0$

II. Semi-Kähler or balanced metrics: $\theta = 0$

III. Gauduchon metrics: $\partial\bar{\partial}F^{m-1} = 0$

IV. Metrics with vanishing conformal torsion: $\partial F + \frac{1}{m-1}\theta \wedge F = 0$

V. Metrics with holomorphic torsion: $\bar{\partial}T = 0$

VI. Metrics with holomorphic (2, 1) torsion: $\partial\bar{\partial}F = 0$

VII. Hermitian-Einstein Metrics.

(*) Dip. di Matem., Univ. Trieste, Piazzale Europa 1, 34127 Trieste, Italia.
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Here T is the $(2, 0)$ vector-valued torsion of the Chern connection of an Hermitian metric g on a complex manifold of complex dimension m, θ its torsion $(1, 0)$ form and F is the fundamental (or Kähler) form of g.

For notations and further elements of Hermitian non-Kählerian differential geometry see [5], [9], [12], [8], [1].

Our purpose is to examine which of the above conditions hold for a class of *compact, simply connected 3-folds M* with trivial canonical bundle and the following Hodge numbers:

$$h^{1,0} = h^{0,1} = 0 \qquad h^{2,0} = h^{0,2} = h^{1,1} = 0 \qquad h^{3,0} = h^{0,3} = 1\,.$$

Existence of such M is due to R. Friedman [4], who proved that M is diffeomorphic to the connected sum of n copies of $S^3 \times S^3$, where $n \geqslant 103$ [4]. Recently, P. Lu and G. Tian [11] showed that for any $n \geqslant 2$ the connected sum of n copies of $S^3 \times S^3$ possesses a complex structure with trivial canonical class. It can also be proved that $h^{2,1} = n - 1$ [2].

With the paper [2] we have initiated a search for different sorts of conditions in order to have a rigidity theorem (as on K3 surfaces), which could suggest the existence of a canonical metric on M. There we proved that the holomorphic tangent bundle $\mathcal{T}$ of M is stable with respect to any Gauduchon metric. Hence, by the theorem of Li and Yau [8], we concluded that there is an Hermitian-Einstein metric on M. Actually, it is shown in [2] that any Hermitian metric g determines a unique Hermitian metric h which is Hermitian-Einstein with respect to g. If we suppose that $g = h$, this metric could be a Calabi-Yau substitute for M. However, to investigate its deformations, supposing that such Hermitian-Einstein metric exists, we need to impose an additional condition, which replaces the Kähler one.

In Section 2 we discard the condition **IV**. The conditions **V**, **VI** and **VII** are considered in Section 3, where we prove that **VI** cannot happen on M and that there is no Hermitian-Einstein metric on M which satisfies **V**.

2 - Conditions on the torsion

The heuristic justification of the list of conditions **I-VII** presented in the Introduction is the classification of almost Hermitian manifolds by Gray and Hervella [7]. According to this classification there are sixteen classes of almost Hermitian manifolds. Further we have the following four classes of Hermitian manifolds of dimension $m \geqslant 3$:

i. Kähler manifolds: $dF = 0$

ii. Semi-Kähler manifolds: $\theta = 0$

iii. $W_4\colon \partial F + \dfrac{1}{m-1}\theta \wedge F = 0$

iv. H - Hermitian manifolds.

See [1]. The third condition is equivalent to $T_{\alpha}{}^{\lambda}{}_{\beta} = \dfrac{1}{m-1}(\theta_\beta \delta_\alpha^\lambda - \theta_\alpha \delta_\beta^\lambda)$, where $T_{\alpha}{}^{\lambda}{}_{\beta}$ are the components of the torsion T [1].

Note that there are many Gauduchon metrics – one in each conformal class of any Hermitian manifold of complex dimension at least 2 (see [5]). Therefore **III** always holds.

Let us concentrate on M. We reject **i**, since M is not a Kähler manifold because $b_2(M) = 0$. Furthermore, we shall prove the following

Lemma 1. *M is not in the class W_4, that is, M does not admit metrics of vanishing conformal torsion.*

Proof. Suppose M has a metric g in W_4. Since W_4 is invariant under conformal changes of the metric [7], we can also suppose that g is a Gauduchon metric. Then applying the ∂-operator to the defining condition **iii** we obtain

$$\partial\theta \wedge F - \theta \wedge \partial F = 0\,.$$

Replace $\partial F = -\dfrac{1}{2}\theta \wedge F$ in the above equation. Therefore $\partial\theta \wedge F = 0$ since $\theta \wedge \theta \wedge F = 0$ because θ is an $(1, 0)$-form. Hence $\partial\theta = 0$ since the following lemma holds.

Lemma. *Let φ be a $(m-1, 0)$-form on a complex compact manifold N of complex dimension m such that $\varphi \wedge F = 0$, where F is the fundamental form of an Hermitian metric on N. Then $\varphi = 0$.*

This statement is Theorem 3.1 (c), p. 182, [12] with $p = m - 1$.

We continue with the proof of Lemma 1. From $\partial\theta = 0$ we deduce that $\theta \in H_{\bar\partial}^{1,0}(M) = H_{\bar\partial}^{0,1}(M)$. Thus, since $h^{0,1} = 0$, we have $\theta = \partial f$, where f is a function. Since g is a Gauduchon metric $\delta'\theta = 0$ (see [5]; δ' is the L_2 formally adjoint to ∂ operator). Hence and from the formula ([5])

$$\delta\varphi = \delta'\varphi = -D^{\lambda}\varphi_{\lambda} + T_{\bar{\mu}}^{\bar{\mu}\lambda}\varphi_{\lambda}$$

for $\varphi = \theta$, we obtain $D^{\lambda}\theta_{\lambda} = |\theta|^2$. But

$$D^{\lambda}\theta_{\lambda} = g^{\lambda\bar{\mu}} D_{\bar{\mu}}\theta_{\lambda} = g^{\lambda\bar{\mu}}\partial_{\bar{\mu}}\theta_{\lambda} = g^{\lambda\bar{\mu}}\partial_{\bar{\mu}}\partial_{\lambda} f.$$

Therefore $L(f) = |\theta|^2$, where $L = g^{\lambda\bar{\mu}} \dfrac{\partial^2}{\partial z^{\lambda}\partial \bar{z}^{\mu}}$.

From the maximum principle f is a constant and therefore $\theta = 0$. Since g is in the class W_4, this implies $T = 0$, which is impossible since M is not a Kähler manifold.

3 - Holomorphic torsion

As we showed in the previous section, the unique possible linear condition which involves only the first derivatives of the metric is $\theta = 0$. But if M is not semi-Kähler, M will be a *general* Hermitian manifold without any linear condition on the first derivatives of the metric. Thus, except the Hernitian-Einstein condition, we shall seek some other conditions in terms of second derivatives.

To begin with, note that the torsion T is a $(2, 0)$ vector-valued form, that is, $T \in \Lambda^{2,0} \otimes \mathcal{T}$, where $\mathcal{T}$ is the *holomorphic tangent bundle.* Now if T is holomorphic:

$$\partial_{\bar{\mu}} T_{\alpha}{}^{\sigma}{}_{\lambda} = 0\,, \tag{1}$$

that is
$$D_{\bar{\mu}} T_{\alpha}{}^{\sigma}{}_{\lambda} = T_{\alpha}{}^{\alpha}{}_{\lambda|\bar{\mu}} = 0\,.$$

where D is the Chern connection of g and $|$ denotes covariant differentiation with respect to D. Now, supposing that there is on M an Hermitian-Einstein metric (see [8]) we have the following

Proposition 1. *The torsion of any Hermitian-Einstein metric on M is not holomorphic.*

Proof. Suppose that T is holomorphic. According to a lemma in [1], if the torsion of an Hermitian-Einstein metric on a compact manifold is holomorphic, then it is parallel. Thus $T_{\alpha}{}^{\sigma}{}_{\lambda|\mu} = 0$. Relation (1) means that

$$T \in H^{2,0}_{\bar\partial}(M, \mathcal{T}) = H^0(M, \Omega^2(\mathcal{T})) = H^0(M, \Theta \otimes \mathcal{T})$$

since the canonical bundle $K_M = \Lambda^3 \mathcal{T}^*$ is trivial and therefore $\Omega^2 = \Theta$, where

$$\Omega^2 = \{\text{holomorphic 2-forms}\} \qquad \Theta = \mathcal{O}(\mathcal{T}) = \{\text{holomorphic vector fields}\}.$$

In this way we see that T determines a map $t\colon \mathcal{T} \to \mathcal{T}^*$, and also

$$\det(t)\colon \Lambda^3 \mathcal{T} \to \Lambda^3 \mathcal{T}^* .$$

But the canonical bundle $K_M = \Lambda^3 \mathcal{T}^*$ is trivial. Hence, if the rank of $\det(t)$ is not maximal, the kernel of t would be a non-trivial holomorphic subbundle of $\mathcal{T}$ since T is parallel and therefore nowhere vanishing. However, according to the Corollary in Section 4 of [2], $\mathcal{T}$ does not have any non-trivial holomorphic subbundles. Therefore $\det(t)$ has maximal rank and in this case t must be an isomorphism. This means that the holonomy group of M is included in $\mathcal{O}(3, \boldsymbol{C})$. Hence, since M is an Hermitian manifold, its holonomy group is the maximal compact subgroup of $SU(3) \cap \mathcal{O}(3, \boldsymbol{C})$. Thus, the holonomy group is reduced to $SO(3)$ and the tangent bundle has the following decomposition

$$\mathcal{T} = E \otimes \boldsymbol{C} = E \otimes iE ,$$

where E is a real rank 3 bundle. By the defining propeties of the Chern classes we know that $c_i(E \otimes \boldsymbol{C}) = 0$ if i - odd. But the Euler characteristic of M is $-2(n-1) \leqslant -2 < 0$ since $h^{2,1} = n - 1$ [2]. Therefore $c_3(\mathcal{T}) = c_3(E \otimes \boldsymbol{C}) \neq 0$, which leads to contradiction.

So far, the conditions **V** and **VII** in the Introduction can not mutually hold.

Now let consider another possibility. For complex surfaces the Gauduchon condition is

$$\bar\partial \partial F = 0 . \tag{2}$$

Since in [2] we have not essentially used the Gauduchon condition, (2) seems to be a nice substitute in higher dimensions. From (2) we obtain

$$\bar{\partial} \boldsymbol{T} = 0 , \tag{3}$$

where $\partial F = \frac{i}{2}\boldsymbol{T}$ and the (2, 1) torsion $\boldsymbol{T}$ has components $T_{\alpha\bar{\gamma}\beta} = g_{\lambda\bar{\gamma}} T_{\alpha}{}^{\lambda}{}_{\beta}$.

In local coordinates (3) has the form

$$\partial_{\bar{\mu}} T_{\alpha\bar{\lambda}\beta} = \partial_{\bar{\lambda}} T_{\alpha\bar{\mu}\beta} . \tag{4}$$

On the other hand, using (4), it is easy to show that

$$2\,|\,\theta\,|^2 + 2\delta'\,\theta = \,|\,T\,|^2 . \tag{5}$$

Integrating (5) we have

$$2\|\theta\|_{L^2} = \|T\|_{L^2} . \tag{6}$$

Therefore among all metrics which satisfy (2) or (3) none is semi-Kählerian. Otherwise (6) will give $T = 0$, which is impossible on M as we have already seen. Also on non-Kähler manifolds there are no metrics in W_4 for which (2) holds. Indeed, if a metric is in W_4, it is easy to see that $\|\theta\|_{L^2} = \|T\|_{L^2}$, which together with (6) implies $T = \theta = 0$.

Note also that (2) is equivalent to the vanishing of the invariant K_1 [6] and also that (2) appeared in [3] as a technical condition.

The equation (3) means that the (2, 1) torsion $\boldsymbol{T}$ is holomorphic and therefore (2) is actually the condition **VI** in the Introduction. Hence, the torsion determines a class in $H^{2,1}_{\bar{\partial}}(M)$, which could play the role of the cohomology class of the Kähler form on K3 surfaces. Unfortunately, on M it is trivial as the following argument shows.

First, note that due to Friedman-Lu-Tian's construction [4], [11], the $\partial\bar{\partial}$-lemma holds on M. This lemma says: if a form is ∂-exact and $\bar{\partial}$-closed, then it is $\partial\bar{\partial}$-exact. We shall use it to investigate the solutions of (2).

Let $\varphi = \partial F$. φ is ∂-exact. By (2) φ is $\bar{\partial}$-closed. Therefore the $\partial\bar{\partial}$-lemma implies $\varphi = \partial\bar{\partial}\psi$, where ψ is a (1, 0) form. Thus $\partial F = \partial\bar{\partial}\psi$ or $\partial(F - \bar{\partial}\psi) = 0$. Therefore the (1, 1) form $F - \bar{\partial}\psi$ belongs to $H^{1,1}_{\bar{\partial}} = H^{2,2}_{\bar{\partial}} = 0$. Hence we have

$F = \bar{\partial}\psi + \partial\eta = \bar{\partial}\bar{\eta} + \partial\bar{\psi}$ since F is real. Further $\partial F = \partial\bar{\partial}\bar{\eta} = \partial\bar{\partial}\psi$ and $\bar{\partial}(\partial(\bar{\eta} - \psi)) = 0$. It follows that $\partial(\bar{\eta} - \psi) \in H_{\bar{\partial}}^{2,0}(M) = 0$, that is, $\partial(\bar{\eta} - \psi) = 0$. This means that $\bar{\eta} - \psi \in H_{\partial}^{1,0} = H_{\bar{\partial}}^{2,3} = 0$ and therefore $\bar{\eta} - \psi = \partial\bar{f}$ or $\eta = \bar{\psi} + \bar{\partial}f$. Thus

$$F = \bar{\partial}\psi + \partial\bar{\psi} + \partial\bar{\partial}f \tag{7}$$

where the function f satisfies $\bar{f} = -f$.

It is obvious that any F determined by (7) is a solution of (2). Summarizing, we have obtained

Lemma 2. *On M any solution* (2) *is given by* (7).

However, if F is as in (7), the torsion $\boldsymbol{T}$ does not determine a non-trivial cohomology class in $H_{\bar{\partial}}^{2,1}(M)$ since

$$\partial F = \frac{i}{2}\boldsymbol{T} = \partial\bar{\partial}\psi\,. \tag{8}$$

Moreover, actually (2) does not hold on M. Indeed,

$$\int_M \boldsymbol{T} \wedge \bar{\boldsymbol{T}} = 4\int_M \partial F \wedge \bar{\partial} F = 4\int_M \partial\bar{\partial}\psi \wedge \bar{\partial}\partial\bar{\psi} = 0\,,$$

by Lemma 2, (8) and the Green's formula. Hence $\boldsymbol{T} = 0$. Therefore there are no Hermitian metrics on M, which satisfy (2).

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Sommario

Si propone una lista di metriche hermitiane naturali, non-kähleriane, determinate da condizioni sulla loro torsione. Si esamina quali di queste condizioni sono valide nel caso particolare di varietà complesse compatte di dimensione complessa 3, semplicemente connesse con $c_1 = 0$ *e aventi i seguenti numeri di Hodge:* $h^{1,0} = h^{0,1} = 0$, $h^{2,0} = h^{0,2} = h^{1,1} = 0$, $h^{3,0} = h^{0,3} = 1$.

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