Riv. Mat. Univ. Parma (5) 4 (1995), 79-99

CARLES ORLANDO SARRICO (*)

Distributional products with invariance for the action of unimodular groups (**)

1 - Introduction

Suppose to each real x we associate a real or complex value $g(x)$, which results from an *average* concerning some physical system; e.g. for each x we can think of $g(x)$ as the *mean-value* of an unknown real or complex function $f(t)$ of a real variable t.

Under appropriate probabilistic hypotheses of markovian type it is possible to interpret $f(t)$ as a *random variable* with probability density $\alpha(t-x)$, the traslation by x of some function $\alpha(t) \geqslant 0$ with $\int_{\mathbf{R}} \alpha = 1$. By putting $\check{\alpha}(t) = \alpha(-t)$, $t \in \mathbf{R}$, we will have

$$g(x) = \int f(t)\,\alpha(t-x)\,\mathrm{d}t = \int_{\mathbf{R}} \check{\alpha}(x-t) f(t)\,\mathrm{d}t = (\check{\alpha} * f)(x).$$

So, we have obtained the mean value operator $f \to s_\alpha(f) = \check{\alpha} * f = g$ which is *linear* and verifies

$$(1) \qquad s_\alpha(Df) = D(s_\alpha f)$$

$$(2) \qquad \int_{\mathbf{R}} s_\alpha f = \int_{\mathbf{R}} f$$

(*) Centro de Matemática e Applicações Fundamentais, Av. Prof. Gama Pinto 2, 1699 Lisboa Codex, Portugal.

(**) Received October 18, 1994. AMS classification 46 F 10. The present research has been performed during the author's visit to the University of Bologna, where he had helpful discussions with Prof. Vaz Ferreira.

where D is the derivative operator, but *it does not verify*

$$s_a(f_1 f_2) = s_a(f_1)\, s_a(f_2) \tag{3}$$

Thus, if the mathematical description of the physical system involves formally the operator of product of functions to be averaged, we will not be able to operate simply using the usual product of functions.

This simple remark is at the origin of the present paper, as a matter of fact we will be able to multiply distributions in $\mathcal{D}'$ coherently with the average operations. To reach our goal it will be necessary to define a space E (a reminiscence of operator representations that we hidden in physical theories) endowed with the operations of sum, multiplication by a complex number, derivative, change of variable and product of two elements of E. We also define an integral on E and a mean-value operator, which is linear and verifies the properties (1), (2) and (3). Of course, E will be endowed with a whole class of products that satisfy (3) but, suprisingly enough, all these products (which are naturally non-commutative) are ruled by a simple system of axioms which is given at the beginning of Sec. **1**.

Then we prove existence of an isomorphism betweem E and $\mathcal{D}'$ preserving as much structure as possible; this enable us to define products on $\mathcal{D}'$ with physical significance (Sec. 4).

Next (Sec. 5) we present simple examples and consider the relationships between these products and the products we have defined in our preceding papers [2], [3].

A simplified scattering problem is studied (Sec. 6) in the form of a linear Cauchy problem and the existence of certains shock waves solutions of Burger's equation is reanalised (Sec. 7).

Finally (Sec. 8) we define new convolutive products of slow grouth distributions, which are closely related to the Fourier transform and to multiplicative products.

1 - The axioms of the product

We denote by $\mathcal{D}$ the algebra of indefinitely differentiable complex functions with bounded support defined on $\boldsymbol{R}^N$. $\mathcal{D}'$ means the space of all distributions.

Let us consider the axiomatic **Ax** for a product $\times$ on $\mathcal{D}'$.

Ax1. The map $(T, S) \to T \times S$ from $\mathcal{D}' \times \mathcal{D}'$ into $\mathcal{D}'$ is bilinear.

Ax2. For each $S \in \mathcal{D}'$ the map $T \to T \times S$ from $\mathcal{D}'$ into $\mathcal{D}'$ is continuous for the usual topology in $\mathcal{D}'$.

Ax3. For each $T, S \in \mathcal{D}'$ and each $k \in \{1, 2, \dots, N\}$ we have

$$D_k(T \times S) = (D_k T) \times S + T \times (D_k S)$$

where D_k means the usual k-partial derivative operator.

Ax4. For each $T \in \mathcal{D}'$, $1 \times T = T$ where 1 means the distribution which corresponds to the constant function equal to 1 all over $\boldsymbol{R}^N$.

Ax5. For each $T, S \in \mathcal{D}'$, $(T \times 1) \cdot S = T \times S$ where the dot means the classical distribution product.

Note that, if there exists such a product, we have:

Proposition 1. *For each* $T \in \mathcal{D}'$, *we have* $D_k(T \times 1) = (D_k T) \times 1$.

Proof. By **Ax3** we have $D_k(T \times 1) = (D_k T) \times 1 + T \times D_k 1 = (D_k T) \times 1$.

Proposition 2. *For each* $T \in \mathcal{D}'$, *we have* $T \times 1 \in C^\infty$.

Proof. Because **Ax5** is true for all $S \in \mathcal{D}'$.

Remark. We also can replace **Ax3** with Prop. 1 and prove **Ax3**. In fact, using Prop. 1 and **Ax5** we have

$$D_k(T \times S) = D_k[(T \times 1) \cdot S] = D_k(T \times 1) \cdot S + (T \times 1) \cdot D_k S$$

$$= [(D_k T) \times 1] \cdot S + (T \times 1) \cdot D_k S = (D_k T) \times S + T \times D_k S\,.$$

Lemma 1. *If a product* $\times$ *verifies* **Ax** *there exist* $\alpha_0 \in \mathcal{D}'$ *with* $\int \alpha_0 = 1$ *such that for all* $T \in \mathcal{D}'$, $T \times 1 = \alpha_0 * T$.

Note. All integrals are taken all over $\boldsymbol{R}^N$ when they are not specified.

Proof. Let $L\colon \mathcal{D}' \to \mathcal{D}'$ be defined by $L(T) = T \times 1$. By **Ax1** and **Ax2** L is linear continuous and verifies $L(D_k T) = D_k L(T)$ on account of Prop. 1. Thus, by a note which follows Theorem X, cap. VI of Schwartz [4], there exist $\alpha_0 \in \mathcal{E}'$ (space of distribuitons of bounded support) such that for all $T \in \mathcal{D}'$, $L(T) = \alpha_0 * T$. Then, $T \times 1 = \alpha_0 * T$ and by Prop. 2 $\alpha_0 * T \in C^\infty$ for all $T \in \mathcal{D}'$.

Putting $T = \delta$ we conclude that $\alpha_0' \in C^\infty$ and so $\alpha_0 \in \mathcal{D}$. Also $1 \times 1 = \alpha_0 * 1$ and by **Ax4**, $\int \alpha_0 = 1$.

Theorem 1. *A product in $\mathcal{D}'$ verifies* **Ax** *if and only if there exist $\alpha_0 \in \mathcal{D}$ with $\int \alpha_0 = 1$, such that $T \times S = (\alpha_0 * T) \cdot S$.*

Proof. By **Ax5** $T \times S = (T \times 1) \cdot S$ and by Lemma 1 there exist $\alpha_0 \in \mathcal{D}$ with $\int \alpha_0 = 1$ such that $T \times S = (\alpha_0 * T) \cdot S$. The reciprocal is easily verified.

We also prove

Proposition 3. *A product in $\mathcal{D}$ satisfying* **Ax** *is separately continuous, and for all T, $S \in \mathcal{D}'$ and $a \in \boldsymbol{R}^N$ we have*

$$\tau_a(T \times S) = (\tau_a T) \times (\tau_a S) \quad \text{and} \quad \operatorname{supp}(T \times S) \subset \operatorname{supp} S .$$

Proof. By **Ax2** it is sufficient to prove that, for each $T \in \mathcal{D}'$ the map $S \to T \times S$ from $\mathcal{D}'$ into $\mathcal{D}'$ is continuous. This map is sequentially continuous because if $S_n \overset{\mathcal{D}'}{\to} S$ when $n \to \infty$ then $T \times S_n \overset{\mathcal{D}'}{\to} T \times S$ when $n \to \infty$. In fact, by Theorem 1 there exists $\alpha_0 \in \mathcal{D}$ such that $T \times S_n = (\alpha_0 * T) \cdot S_n \overset{\mathcal{D}'}{\to} (\alpha_0 * T) \cdot S$ when $n \to \infty$. Also recall that $\mathcal{D}'$ is a bornological and locally convex space and all sequentially continuous linear operators, defined in a bornological space with values in a locally convex space are continuous.

Again by Theorem 1 we have:

$$\tau_a(T \times S) = \tau_a[(\alpha_0 * T) \cdot S]$$

$$= \tau_a(\alpha_0 * T) \cdot \tau_a S = [\alpha_0 * (\tau_a T)] \cdot (\tau_a S) = (\tau_a T) \times (\tau_a S)$$

$$\operatorname{supp}(T \times S) = \operatorname{supp}[(\alpha_0 * T) \cdot S] \subset \operatorname{supp} S .$$

Next we recall some concepts and define the integral of an operator. For details the reader must see [3].

2 - Some basic operations in $L(\mathcal{D})$

Let us denote by $L(\mathcal{D})$ the algebra of all continuous endomorphism $\phi : \mathcal{D} \to \mathcal{D}$, where the usual composition product will be indicated by a dot.

An operator $\phi \in L(\mathcal{D})$ is said to vanish on an open set Ω iff $\phi(x) = 0$ for all

$x \in \mathcal{D}$ the support of which is contained in Ω. We denote by supp ϕ the complement of the largest open set in which ϕ vanishes.

We call *natural representation* of a function $\beta \in C^\infty$ the operator $\phi \in L(\mathcal{D})$ defined by $\phi(x) = \beta x$ for all $x \in \mathcal{D}$ (C^∞ means the algebra of indefinitely differentiable complex functions defined on $\boldsymbol{R}^N$ endowed with the usual topology). The representation $\varrho: C^\infty \to L(\mathcal{D})$, in which $\varrho(\beta)$ denotes the operator that maps $x \in \mathcal{D}$ onto $\beta x \in \mathcal{D}$, is the natural extension to C^∞ of the regular representation of the algebra $\mathcal{D}$.

The concept of support, we have defined, is coherent with the natural representation of the C^∞-functions in the sense that, if $\beta \in C^\infty$ then we have supp $\varrho(\beta) =$ supp β.

2.1 - The change of variable in $L(\mathcal{D})$

Let h be a C^∞-diffeomorphism of $\boldsymbol{R}^N$ and consider the operator $S_h: \mathcal{D} \to \mathcal{D}$ defined by $S_h(x) = x \circ h$ for all $x \in \mathcal{D}$. Then, we will say that the *operator* $\phi \odot h = S_h \cdot \phi \cdot S_{h^{-1}}$ results from ϕ through the change of variable h. For this operation we have

Proposition 4. *Let h, s be C^∞-diffeomorphisms of $\boldsymbol{R}^N$, $\lambda \in \boldsymbol{C}$, ϕ, $\psi \in L(\mathcal{D})$ and $\beta \in C^\infty$. Then*

$$(\lambda\phi) \odot h = \lambda(\phi \odot h) \qquad (\phi + \psi) \odot h = \phi \odot h + \psi \odot h$$

$$(\phi \cdot \psi) \odot h = (\phi \odot h) \cdot (\psi \odot h) \qquad (\phi \odot h) \odot s = \phi \odot (h \circ s)$$

$$\operatorname{supp}(\phi \odot h) = h^{-1}(\operatorname{supp} \phi) \qquad \varrho(\beta) \odot h = \varrho(\beta \circ h)$$

Thus, given a C^∞-diffeomorphism h of $\boldsymbol{R}^N$, $\phi \to \phi \odot h$ is an automorphism of $L(\mathcal{D})$.

Considering $h: \boldsymbol{R}^N \to \boldsymbol{R}^N$ defined by $h(t) = t - a$ with $a \in \boldsymbol{R}^N$ we can define the a-translation of $\phi \in L(\mathcal{D})$ by $\bar{\tau}_a\ \phi = \phi \odot h$.

2.2 - Partial derivative of an operator $\phi \in L(\mathcal{D})$

For each $k \in \{1, 2, \ldots, N\}$ we call *k-partial derivative* of $\phi \in L(\mathcal{D})$, the operator $\overline{D}_k \phi = D_k \cdot \phi - \phi \cdot D_k$, where D_k is the usual k-partial derivative operator, that is, the commutator $[D_k, \phi]$.

The operator $\overline{D}_k: L(\mathcal{D}) \to L(\mathcal{D})$ is linear, verifies the usual law of the derivative of a product $\overline{D}_k(\phi \cdot \psi) = (\overline{D}_k \phi) \cdot \psi + \phi \cdot (\overline{D}_k \psi)$, for all ϕ, $\psi \in L(\mathcal{D})$, contracts the support of ϕ, i.e. supp $\overline{D}_k \phi \subset$ supp ϕ and is compatible with the natural representation of the C^∞-functions, i.e. $D_k \varrho(\beta) = \varrho(D_k \beta)$ for all $\beta \in C^\infty$.

2.3 - The physical value of an operator $\phi \in L(\mathcal{D})$

The map $\phi \to \tilde{\xi}(\phi)$ of $L(\mathcal{D})$ onto $\mathcal{D}'$, which associates to ϕ the distribution defined by

$$\langle \tilde{\xi}(\phi), x \rangle = \int \phi(x) \qquad \text{for all } x \in \mathcal{D}$$

is linear and verifies

Proposition 5. *For $\phi, \psi \in L(\mathcal{D})$, $h: \boldsymbol{R}^N \to \boldsymbol{R}^N$ linear bijective, $a \in \boldsymbol{R}^N$ and $\beta \in C^\infty$, we have:*

a. $\tilde{\xi}(\phi \cdot \psi) = \tilde{\xi}(\phi) \cdot \psi$ *($T \cdot \psi$ with $T \in \mathcal{D}'$ and $\psi \in L(\mathcal{D})$ denotes the distribution ${}^t\psi(T)$)*

b. $\tilde{\xi}(\overline{D}_k \phi) = D_k \tilde{\xi}(\phi)$ *(D_k denote the k-partial derivative on $\mathcal{D}'$)*

c. $\tilde{\xi}(\phi \odot h) = \tilde{\xi}(\phi) \circ h$ *($T \circ h$ with $T \in \mathcal{D}$ denotes the distribution T after the change of variable h).*

d. $\tilde{\xi}(\overline{\tau}_a \phi) = \tau_a \tilde{\xi}(\phi)$ *($\tau_a T$ with $T \in \mathcal{D}$ and $a \in \boldsymbol{R}^N$ means the a-translation of the distribution T)*

e. $\tilde{\xi}(\varrho(\beta)) = \beta$ *(taken as a distribution).*

We call *physical value* of the operator $\phi \in L(\mathcal{D})$ the distribution $\tilde{\xi}(\phi)$.

2.4 - The integral of an operator $\phi \in L(\mathcal{D})$

The operator $\phi \in L(\mathcal{D})$ is said to be *integrable* on $\boldsymbol{R}^N$, when its physical value $\tilde{\xi}(\phi)$ is integrable on $\boldsymbol{R}^N$ in the sense of Silva [5], [6]. In this case we will write $\int \phi = \int \tilde{\xi}(\phi)$ (in this paper all integrals of distributions are taken in the sense of Silva).

The consistence of this definition follows from

Proposition 6. *If $\beta \in C^\infty$ is Silva-integrable on $\boldsymbol{R}^N$ and $\phi = \varrho(\beta) \in L(\mathcal{D})$, then $\int \phi = \int \beta$.*

Proof. $\int \phi = \int \tilde{\xi}(\phi) = \int \beta$.

The operation $\phi \to \int \phi$ is linear. Moreover, we have

Proposition 7. *If* $h: \boldsymbol{R}^N \to \boldsymbol{R}^N$ *is defined by* $h(t) = At + b$, *where* A *is an* $N \times N$ *regular matrix,* $b \in \boldsymbol{R}^N$ *and* $\phi \in L(\mathcal{D})$ *is integrable on* $\boldsymbol{R}^N$, *then* $(\phi \odot h) \cdot |\det A|$ *is integrable on* $\boldsymbol{R}^N$ *and* $\int \phi = \int (\phi \odot h) |\det A|$.

Proof. $\tilde{\xi}((\phi \odot h) |\det A|) = \tilde{\xi}(\phi \odot h) |\det A| = (\tilde{\xi}(\phi) \circ h) |\det A|$ by Prop. 5 c. By Theorem 14.2 of Silva [5], $\int \tilde{\xi}((\phi \odot h) |\det A|) = \int \tilde{\xi}(\phi)$, which proves that $(\phi \odot h) |\det A|$ is integrable on $\boldsymbol{R}^N$ and also that $\int (\phi \odot h) |\det A| = \int \phi$.

Also it is easy to prove

Proposition 8. *If* $\phi, \psi \in L(\mathcal{D})$ *with* $\phi - \psi \in \ker \tilde{\xi}$ *and one of them is integrable on* $\boldsymbol{R}^N$, *then the other is also integrable on* $\boldsymbol{R}^N$ *and* $\int \phi = \int \psi$.

2.5 - The mean value operator $s_\alpha : L(\mathcal{D}) \to L(\mathcal{D})$

Given $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, we define the *operator* $s_\alpha : L(\mathcal{D}) \to L(\mathcal{D})$ by $s_\alpha(\phi) = \psi$, where ψ is given by $(\psi(x))(y) = \int \phi_t(\alpha(y-t)x(t))\,dt$ for all $x \in \mathcal{D}$ and all $y \in \boldsymbol{R}^N$. Here ϕ_t denotes the operator ϕ when it acts on functions of t in $\mathcal{D}$. Thus $(\psi(x))(y) = \int \phi((\tau_y \check{\alpha})x)$.

s_α is a linear operator and if $\gamma \in \mathcal{D}$ with $\int \gamma = 1$, then $s_\gamma \circ s_\alpha = s_\gamma$. In particular, taking $\gamma = \alpha$ we have $s_\alpha \circ s_\alpha = s_\alpha$, which proves that s_α is a projector and so the principal left (right) ideal generated by s_α is idempotent. The proof that $s_\gamma \circ s_\alpha = s_\gamma$ is essentially the same we have done in [3] for proposition 1.3.3f, p. 303.

Moreover, for s_α we have

Proposition 9. *Let* $\phi \in L(\mathcal{D})$, $h: \boldsymbol{R}^N \to \boldsymbol{R}^N$ *unimodular (linear with* $|\det h'| = 1$), $a \in \boldsymbol{R}^N$ *and* $\alpha \in \mathcal{D}$ *be such that* $\int \alpha = 1$ *with* $\alpha \circ h = \alpha$. *Then*

a. $s_\alpha(\overline{D}_k \phi) = \overline{D}_k(s_\alpha \phi)$
b. $s_\alpha(\phi \odot h) = s_\alpha(\phi) \odot h$
c. $s_\alpha(\overline{\tau}_a \phi) = \overline{\tau}_a(s_\alpha \phi)$
d. $\tilde{\xi} \circ s_\alpha = \tilde{\xi}$
e. $\ker s_\alpha = \ker \tilde{\xi}$
f. $\operatorname{supp} s_\alpha(\phi) = \operatorname{supp} \tilde{\xi}(\phi)$
g. $(s_\alpha(\phi))(x) = \alpha * \tilde{\xi}(\phi) x$ for all $x \in \mathcal{D}$.

Proof. See [3] for a, b, c, d, e and f. We shall prove g. Putting $\tilde{\xi}(\phi) = T$ we have $\phi = \psi + \xi$, where ψ is such that $\tilde{\xi}(\psi) = T$ and $\xi \in \ker \tilde{\xi}$. Let $\beta \in \mathcal{D}$ be such that $\int \beta = 1$. Taking $\psi(x) = \beta \ \langle T, x\rangle$, we have $\tilde{\xi}(\psi) = T$. Then we get:

$$(s_a \phi)(x) = (s_a \psi)(x) + (s_a \xi)(x) = (s_a \psi)(x) \qquad \text{because } \ker s_a = \ker \tilde{\xi}$$

$$(s_a(\psi)(x))(y) = \int \psi_t(\alpha(y-t)\,x(t))\,dt = \int \beta(t)\langle T_\lambda, \alpha(y-\lambda)\,x(\lambda)\rangle\,dt$$

$$= \langle T_\lambda, \alpha(y-\lambda)\,x(\lambda)\rangle = \langle T_\lambda x(\lambda), \alpha(y-\lambda)\rangle = (\alpha * Tx)(y).$$

Thus $$s_a(\phi)(x) = s_a(\psi)(x) = \alpha * Tx = \alpha * \tilde{\xi}(\phi)\,x.$$

As we refer in the introduction for usual functions;

Proposition 10. *If $\phi \in L(\mathcal{D})$ is integrable on $\boldsymbol{R}^N$, then $s_a \phi$ is also integrable on $\boldsymbol{R}^N$ and $\int s_a \phi = \int \phi$.*

Proof. $\int s_a \phi = \int \tilde{\xi}(s_a \phi) = \int \tilde{\xi}(\phi) = \int \phi$ applying Prop. 9 d.

Moreover

Proposition. 11. *If $\phi \in L(\mathcal{D})$ and $\psi = \varrho(\beta)$, $\beta \in C^\infty$, then we have $s_a(\phi \cdot \psi) = (s_a \phi) \cdot \psi$.*

Proof. Let $x \in \mathcal{D}$ and $y \in \boldsymbol{R}^N$. Then

$$s_a(\phi \cdot \psi)(x)(y) = \int \phi \cdot \psi((\tau_y \check{\alpha})\,x) = \int \phi(\beta(\tau_y \check{\alpha})\,x) = s_a(\phi)(\beta x)(y)$$

which means that $s_a(\phi \cdot \psi)(x) = s_a(\phi)(\beta x) = ((s_a \phi) \cdot \psi)(x)$ and so we get $s_a(\phi \cdot \psi) = (s_a \phi) \cdot \psi$.

Thus, we have not in general $s_a(\phi \cdot \psi) = s_a(\phi) \cdot s_a(\psi)$; the space E which we define in the following will remove this problem.

3 - The space E and the basic operations

We call E the quotient space $L(\mathcal{D})/\ker \tilde{\xi}$ whose elements $[\phi]$, $[\psi]$, ... are classes of operators in $L(\mathcal{D})$ and we define the sum of two elements in E and the product of a complex number $\lambda \in \boldsymbol{C}$ by an element of E as usually

$$[\phi] + [\psi] = [\phi + \psi], \qquad \text{for all } [\phi], [\psi] \in E$$
$$\lambda[\phi] = [\lambda \phi], \qquad \text{for all } \lambda \in \boldsymbol{C} \text{ all } [\phi] \in E.$$

We also can define the support of an element of E by $\operatorname{supp}[\phi] = \operatorname{supp} \tilde{\xi}(\phi)$

and this definition does not depend of the element $\phi \in L(\mathcal{D})$ which represents the class $[\phi]$ because for all $\xi \in \ker\widetilde{\zeta}$ we have

$$\operatorname{supp}[\phi + \xi] = \operatorname{supp} \widetilde{\zeta}(\phi + \xi) = \operatorname{supp}(\widetilde{\zeta}(\phi) + \widetilde{\zeta}(\xi)) = \operatorname{supp} \widetilde{\zeta}(\phi) = \operatorname{supp}[\phi].$$

Note that if $\theta: L(\mathcal{D}) \to L(\mathcal{D})$ is additive and verifies $\theta(\ker\widetilde{\zeta}) \subset \ker\widetilde{\zeta}$, then θ can be defined on E by $\theta([\phi]) = [\theta(\phi)]$ for all $[\phi] \in E$ because

$$\theta([\phi + \xi]) = [\theta(\phi + \xi)] = [\theta(\phi) + \theta(\xi)] = [\theta(\phi)] = \theta([\phi]) \quad \text{for all } \xi \in \ker\widetilde{\zeta}$$

and so $\theta([\phi])$ does not depend of the element ϕ that represents $[\phi]$.

Thus, we can define on E the following *operations* we have defined in $L(\mathcal{D})$

a. $[\phi] \to [\phi] \odot h = [\phi \odot h]$ where $h: \boldsymbol{R}^N \to \boldsymbol{R}^N$ is linear and bijective

b. $[\phi] \to \overline{\tau}_a[\phi] = [\overline{\tau}_a \phi]$ where $a \in \boldsymbol{R}^N$

c. $[\phi] \to \overline{D}_k[\phi] = [\overline{D}_k \phi]$ where $k \in \{1, 2, \dots, N\}$

d. $[\phi] \to s_\alpha[\phi] = [s_\alpha \phi]$ where $\alpha \in \mathcal{D}$ and $\int \alpha = 1$.

because for all $\xi \in \ker\widetilde{\zeta}$ we have

a′. $\widetilde{\zeta}(\xi \odot h) = \widetilde{\zeta}(\xi) \circ h = 0$ (Prop. 5 c)

b′. $\widetilde{\zeta}(\overline{\tau}_a \xi) = \tau_a \widetilde{\zeta}(\xi) = 0$ (Prop. 5 d)

c′. $\widetilde{\zeta}(\overline{D}_k \xi) = D_k \widetilde{\zeta}(\xi) = 0$ (Prop. 5 b)

d′. $\widetilde{\zeta}(s_\alpha(\xi)) = (\widetilde{\zeta} \circ s_\alpha)(\xi) = \widetilde{\zeta}(\xi) = 0$ (Prop. 9 d).

The integral of a class $[\phi] \in E$ can also be defined: $[\phi] \in E$ is said to be integrable on $\boldsymbol{R}^N$, iff ϕ is integrable on $\boldsymbol{R}^N$ and in this case we will write $\int[\phi] = \int \phi = \int \widetilde{\zeta}(\phi)$.

Obviously this definition is independent of the operator ϕ which represents $[\phi]$. Also,

Proposition 12. *If $h: \boldsymbol{R}^N \to \boldsymbol{R}^N$ is defined by $h(t) = At + b$, where A is a regular $N \times N$ matrix, $b \in \boldsymbol{R}^N$ and $[\phi] \in E$ is integrable on $\boldsymbol{R}^N$, then $([\phi] \odot h) \mid \det A \mid$ is integrable on $\boldsymbol{R}^N$ and $\int[\phi] = \int([\phi] \odot h) \mid \det A \mid$.*

Proof. $([\phi] \odot h) \mid \det A \mid = [(\phi \odot h) \mid \det A \mid]$ and $(\phi \odot h) \mid \det A \mid$ is integrable on $\boldsymbol{R}^N$ by Prop. 7. Thus, $([\phi] \odot h) \mid \det A \mid$ is integrable on $\boldsymbol{R}^N$ and, taking account of Prop. 7, we have

$$\int([\phi] \odot h)\,|\det A| = \int(\phi \odot h)\,|\det A| = \int \phi = \int[\phi].$$

The properties of operations a, b, c are similar to the correspondent ones on $L(\mathcal{D})$ and with the help of **2.1**, **2.2** and **2.3** we can prove the following three propositions

Proposition 13. *Let $h, s: \boldsymbol{R}^N \to \boldsymbol{R}^N$ be linear bijective maps, $\lambda \in \boldsymbol{C}$ and $[\phi]$, $[\psi] \in E$. Then*

$$\lambda[\phi] \odot h = \lambda([\phi] \odot h) \qquad ([\phi] + [\psi]) \odot h = [\phi] \odot h + [\psi] \odot h$$
$$([\phi] \odot h) \odot s = [\phi] \odot (h \circ s) \qquad \operatorname{supp}([\phi] \odot h) = h^{-1}(\operatorname{supp}[\phi])$$

Proposition 14. *Let $[\phi], [\lambda] \in E$, $\lambda \in \boldsymbol{C}$ and $a, b \in \boldsymbol{R}^N$. Then*

$$\bar{\tau}_a(\lambda[\phi]) = \lambda\bar{\tau}_a[\phi] \qquad \bar{\tau}_a([\phi] + [\psi]) = \bar{\tau}_a[\phi] + \bar{\tau}_a[\psi]$$
$$\bar{\tau}_a\bar{\tau}_b[\phi] = \bar{\tau}_{a+b}[\phi] \qquad \operatorname{supp}\tilde{\tau}_a[\phi] = \operatorname{supp}[\phi] + a\,.$$

Proposition. 15. *Let $[\phi], [\psi] \in E$, $\lambda \in \boldsymbol{C}$ and $k \in \{1, 2, \dots, N\}$. Then*

$$\overline{D}_k([\phi] + [\psi]) = \overline{D}_k[\phi] + \overline{D}_k[\psi] \qquad \overline{D}_k(\lambda[\phi]) = \lambda\overline{D}_k[\phi]$$
$$\operatorname{supp}\overline{D}_k[\phi] \subset \operatorname{supp}[\phi]\,.$$

Unfortunately, the natural definition of product $[\phi][\psi] = [\phi \cdot \psi]$ of two elements of E is not consistent, because if ξ, $\eta \in \ker\tilde{\zeta}$ we obtain $(\phi + \xi) \cdot (\psi + \eta) = \phi \cdot \psi + (\phi \cdot \eta + \xi \cdot \psi + \xi \cdot \psi)$ and $\phi \cdot \eta + \xi \cdot \psi + \xi \cdot \eta$ is not always in $\ker\tilde{\zeta}$. However, there are many ways of defining a consistent product on E.

In fact, let G be a group of unimodular transformations $h: \boldsymbol{R}^N \to \boldsymbol{R}^N$ and $\alpha \in \mathcal{D}$ be G-invariant with $\int \alpha = 1$. We define the *(G, α)-product* $[\phi] \dot{}_\alpha [\psi]$ of two elements in E by the formula

$$[\phi] \dot{}_\alpha [\psi] = [\phi \cdot s_\alpha(\psi)]\,.$$

This definition is consistent; in fact

Proposition 16. *If $[\phi]$, $[\psi] \in E$ and ξ, $\eta \in \ker\tilde{\zeta}$, then we obtain* $[\phi + \xi] \dot{}_\alpha [\psi + \eta] = [\phi] \dot{}_\alpha [\psi]$.

Proof. Using Prop. 9 **e** we can write

$$[\phi + \xi] \underset{\alpha}{\cdot} [\psi + \eta] = [(\phi + \xi) \cdot s_\alpha(\psi + \eta)] = [(\phi + \xi) \cdot s_\alpha(\psi)].$$

But $[(\phi + \xi) \cdot s_\alpha(\psi)] = [\phi \cdot s_\alpha(\psi) + \xi \cdot s_\alpha(\psi)] = [\phi \cdot s_\alpha(\psi)] = [\phi] \underset{\alpha}{\cdot} [\psi]$

because by Prop. 5 a we have $\tilde{\zeta}(\xi \cdot s_\alpha(\psi)) = \tilde{\zeta}(\xi) \cdot s_\alpha(\psi) = 0$.

This product is bilinear and verifies

Proposition 17. *If* $[\phi], [\psi] \in E$, $\alpha \in \mathcal{D}$ *is G-invariant with* $\int \alpha = 1$, $h \in G$, $k \in \{1, 2, \dots, N\}$ *and* $a \in \boldsymbol{R}^N$ *then*

a. $$\overline{D}_k([\phi] \underset{\alpha}{\cdot} [\psi]) = (\overline{D}_k[\phi]) \underset{\alpha}{\cdot} [\psi] + [\phi] \underset{\alpha}{\cdot} (\overline{D}_k[\psi])$$

b. $$\overline{\tau}_a([\phi] \underset{\alpha}{\cdot} [\psi]) = (\overline{\tau}_a[\phi]) \underset{\alpha}{\cdot} (\overline{\tau}_a[\psi])$$

c. $$([\phi] \underset{\alpha}{\cdot} [\psi]) \odot h = ([\phi] \odot h) \underset{\alpha}{\cdot} ([\psi] \odot h)$$

d. $$\operatorname{supp}([\phi] \underset{\alpha}{\cdot} [\psi]) \subset \operatorname{supp}[\psi].$$

Note that, as we have said in the introduction, we are now able to derive

Proposition 18. *If* $\alpha, \gamma \in \mathcal{D}$ *with* $\int \alpha = \int \gamma = 1$, *then we have*

(1) $$s_\gamma \overline{D}_k[\phi] = \overline{D}_k s_\gamma[\phi]$$

(2) $$\int s_\gamma[\phi] = \int[\phi]$$

(3) $$s_\gamma([\phi] \underset{\alpha}{\cdot} [\psi]) = (s_\gamma[\phi]) \underset{\alpha}{\cdot} (s_\gamma[\psi]).$$

Indeed, the operation on E defined by the mean value operator s_α is given by $s_\alpha[\phi] = [s_\alpha \phi] = [\phi]$ and so s_α is the identical operator on E for any $\alpha \in \mathcal{D}$ with $\int \alpha = 1$.

4 - The multiplicative products in $\mathcal{D}'$

Thus, the bijection $\zeta: E \to \mathcal{D}'$ defined by $\zeta([\phi]) = \tilde{\zeta}(\phi)$ for all $[\phi] \in E$ allows us to define a class of products of distributions with physical significance.

Definition 1. Given a group of unimodular transformations G of $\boldsymbol{R}^N$ and a function $\alpha \in \mathcal{D}$, G-invariant with $\int \alpha = 1$, we define the *(G, α)-product* $T \underset{\alpha}{\cdot} S$ of two distributions T, $S \in \mathcal{D}'$ by

$$T \underset{\alpha}{\cdot} S = \zeta(\zeta^{-1}(T) \underset{\alpha}{\cdot} \zeta^{-1}(S)).$$

It is easy to prove

Proposition 19. *The map $\zeta: E \to \mathcal{D}'$ is an isomorphism for the structure defined by the operations $0_1, 0_2, 0_3, 0_4, 0_5, 0_6$ and 0_7 and the correspondent ones in $\mathcal{D}'$.*

0_1. *Addition:* $([\phi],[\psi]) \to [\phi] + [\psi]$ *from* $E \times E$ *onto* E

0_2. *Right product induced by* $\psi \in \varrho(C^\infty)$: $[\phi] \to [\phi \cdot \psi]$ *from* E *into* E

0_3. *k-partial derivation:* $[\phi] \to \overline{D}_k[\phi]$ *from* E *into* E

0_4. *Translation defined by* $a \in \boldsymbol{R}^N$: $[\phi] \to \overline{\tau}_a[\phi]$ *from* E *onto* E

0_5. *Change of variable defined by* $h \in G$: $[\phi] \to [\phi] \odot h$ *from* E *onto* E

0_6. *(G, α)-product:* $([\phi],[\psi]) \to [\phi] \underset{\alpha}{\cdot} [\psi]$ *from* $E \times E$ *onto* E

0_7. *Integration:* $[\phi] \to \int[\phi]$ *defined on the subset of integrable elements of E.*

Note that the correspondent in $\mathcal{D}'$ to operation 0_2 is the usual product induced by $\beta = \varrho(\psi) \in C^\infty$. In fact, for all $x \in \mathcal{D}$ taking account of Prop. 5 a, we have

$$\langle \zeta[\phi \cdot \psi], x \rangle = \langle \tilde{\xi}(\phi \cdot \psi), x \rangle = \langle \tilde{\xi}(\phi) \cdot \psi, x \rangle$$

$$= \langle \tilde{\xi}(\phi), \psi(x) \rangle = \langle \tilde{\xi}(\phi), \beta x \rangle = \langle \tilde{\xi}(\phi)\beta, x \rangle$$

and so $\zeta[\phi \cdot \psi] = \tilde{\xi}(\phi)\beta = \tilde{\xi}(\phi)\varrho(\psi)$. Also Silva integral is the correspondent in $\mathcal{D}'$ to operation 0_7. The correspondent in $\mathcal{D}'$ to the mean value operator $[\phi] \to s_a[\phi] = [\phi]$ is obviously the identical operator in $\mathcal{D}'$ and so, given $T \in D'$, $s_a T$ is the distribution T itself. Thus, we use ordinary distributions instead of classes (elements of E) in the description of a physical system, interpreting the product as the (G, α)-product and the integrals as Silva integrals.

Proposition 20. *Each (G, α)-product verifies the axioms* **Ax** *for the product $\times$ introduced in Section* 1.

Proof. Let $T, S \in \mathcal{D}'$. Taking $\phi \in \zeta^{-1}(T)$ and $\psi \in \zeta^{-1}(S)$, we have $\tilde{\zeta}(\phi) = T$ and $\tilde{\zeta}(\psi) = S$ and by Definition 1

$$T \underset{\alpha}{\cdot} S = \zeta([\phi] \underset{\alpha}{\cdot} [\psi] = \zeta([\phi \cdot s_a \psi]) = \tilde{\zeta}(\phi \cdot s_a \psi) = \tilde{\zeta}(\phi) \cdot s_a(\psi) = T \cdot s_a(\psi)$$

(see Prop. 5 a). Thus, we can compute the product $T \underset{\alpha}{\cdot} S$ by $T \underset{\alpha}{\cdot} S = T \cdot s_a(\psi)$, where $\psi \in L(\mathcal{D})$ is such that $\tilde{\zeta}(\psi) = S$.

Also, for all $x \in \mathcal{D}$, using Prop. 9 g at the third step, we have

$$\langle T \underset{\alpha}{\cdot} S, x\rangle = \langle T \cdot s_a(\psi), x\rangle = \langle T, s_a(\psi)(x) = \langle T, \alpha * Sx\rangle$$

$$= \langle T_u, \langle S_t x(t), \alpha(u-t)\rangle\rangle = \langle T_u, \langle S_t, \alpha(u-t)\, x(t)\rangle\rangle$$

$$= \langle T_u \otimes S_t, \alpha(u-t)\, x(t)\rangle = \langle T_t \otimes S_u, \alpha(t-u)\, x(u)\rangle = \langle S_u \otimes T_t, \alpha(t-u)\, x(u)\rangle$$

$$= \langle S_u, \langle T_t, \alpha(t-u)\, x(u)\rangle\rangle = \langle S_u, \langle T_t, \alpha(t-u)\rangle x(u)\rangle = \langle S_u, \langle T_t, \check{\alpha}(u-t)\rangle x(u)\rangle$$

$$= \langle S, (\check{\alpha} * T)x\rangle = \langle (\check{\alpha} * T) \cdot S, x\rangle. \quad \text{Thus}$$

Proposition 21. *For any S, T of $\mathcal{D}'$, we have $T \underset{\alpha}{\cdot} S = (\check{\alpha} * T) \cdot S$.*

Taking $\alpha_0 = \check{\alpha}$ in Theorem 1, we prove Prop. 20.

Taking $G = \{I\}$, where $I: \boldsymbol{R}^N \to \boldsymbol{R}^N$ is the identical transformation, we can see that the $(\{I\}, \alpha)$-products (Definition 1) are exactly the same we have defined in Sec. 1.

Now, we list the principal properties of the (G, α)-product.

Proposition 22. *Let G be a group of unimodular transformations of $\boldsymbol{R}^N$, $\alpha \in \mathcal{D}$, G-invariant with $\int \alpha = 1$, $n \in \boldsymbol{N}$, $a \in \boldsymbol{R}^N$, $h \in G$, $k \in \{1, 2, \dots, N\}$ and $T, S \in \mathcal{D}'$. Then*

a. *$T \underset{\alpha}{\cdot} S$ is a bilinear function of T and S separately continuous*

b. $\tau_a(T \underset{\alpha}{\cdot} S) = (\tau_a T) \underset{\alpha}{\cdot} (\tau_a S)$

c. $(T \underset{\alpha}{\cdot} S) \circ h = (T \circ h) \underset{\alpha}{\cdot} (S \circ h)$

d. $D_k(T \underset{\alpha}{\cdot} S) = (D_k T) \underset{\alpha}{\cdot} S + T \underset{\alpha}{\cdot} (D_k S)$

e. $\operatorname{supp}(T \underset{\alpha}{\cdot} S) \subset \operatorname{supp} S$.

Proof. a, b and e are direct consequence of Prop. 20 and Prop. 3, d is **Ax3**, c is a direct consequence of Prop. 19 and Prop. 17 c.

5 - Examples and comments

Let us denote by $T \cdot S$ the α-product in the sense of [3]. The product we have defined here in the sense of Definition 1 is not an extension of the product defined in [3] but, when $T \in \mathcal{D}'$ and $S \in \mathcal{D}'_n$, we have $T \cdot S = T \underset{\alpha}{\cdot} S$. (We recall than $\mathcal{D}'_n$ means the space of distributions with nowhere dense support).

If $S \notin \mathcal{D}'_n$ the two products may be different. For instance, if we take as G the orthogonal group, as we usually do in non-relativistic physics, we can compute in dimension N

$$\delta \underset{\alpha}{\cdot} \delta = \alpha(0)\,\delta \qquad H \underset{\alpha}{\cdot} \delta = \frac{1}{2^N}\,\delta \qquad \delta \underset{\alpha}{\cdot} (D_k \delta) = \alpha(0)\,D_k \delta \qquad (D_k \delta) \underset{\alpha}{\cdot} \delta = 0$$

where δ is the Dirac measure, H the Heaviside function, $k \in \{1, 2, \dots, N\}$ and these products coincide with the products in the sense of [3]. However, $\delta \underset{\alpha}{\cdot} H = H\alpha$ and $\delta \cdot H$ does not exist because $H \notin C^\infty \oplus \mathcal{D}'_n$.

If $T \in \mathcal{D}'$ and $\beta \in C^\infty$, then $T \underset{\alpha}{\cdot} \beta = \beta(\alpha * T)$ and $\beta \underset{\alpha}{\cdot} T = T(\alpha * \beta)$. The distribution $T \cdot \beta$ is the same in classical sense or in the sense of [3], while $\beta \cdot T$ is not always defined. Recall that the product in the sense of [3] is consistent with the classical product when the C^∞-function β is on the right. The product in the sense of this paper has a *limit consistence* with classical products, that is, $T \underset{\alpha}{\cdot} \beta$ and $\beta \underset{\alpha}{\cdot} T$ are near the classical $T \cdot \beta$ if we cloose α near δ. In a concrete physical situation there are always the possibility of choosing α in such a way that we cannot distinguish experimentally these differences from a macroscopic or average point of view.

In [3], p. 311, we considered a point with mass m moving in the real line towards the origin in the negative part of the x axis with a constant velocity v. At the instant $t = 0$ it collides with an obstacle situated at the origin. Supposing that the collision was completely inelastic, we computed the work done by the force field $F(t) = mx'(t) = -m\delta(t)$

$$w = \int_{\mathbf{R}} F(t)\,x'(t)\,\mathrm{d}t = mv^2 \int_{\mathbf{R}} H(-t)\,\delta(t)\,\mathrm{d}t = -\frac{1}{2}\,mv^2$$

In the sense of this paper we can also compute

$$w = -mv^2 \int_{\mathbf{R}} \delta(t) \underset{\alpha}{\cdot} H(-t)\,\mathrm{d}t = -mv^2 \int_{\mathbf{R}} H(-t)(\check{\alpha}(t) * \delta(t))\,\mathrm{d}t$$

$$= -mv^2 \int_{\mathbf{R}} H(-t)\,\alpha(-t)\,\mathrm{d}t = -mv^2 \int_{-\infty}^{0} \alpha(-t)\,\mathrm{d}t = -\frac{1}{2}\,mv^2$$

and both results coincide with the value of the cinetic energy of the motion before the instant $t = 0$. This is not a coincidence. As in the heuristic calculations of quantum physics the product is commutative under the simbol of integral. More precisely

Theorem 2. *Let (G, α) be as in Definition 1 and let the map $t \to -t$ of $\boldsymbol{R}^N$ onto $\boldsymbol{R}^N$ belong to G. Then if $T, S \in \mathcal{D}'$ and $\alpha(x-t)\,[T(t) \otimes S(x)]$ is Silva integrable on $\boldsymbol{R}^{2N}$, $T \underset{\alpha}{\cdot} S$ and $S \underset{\alpha}{\cdot} T$ are Silva integrable on $\boldsymbol{R}^N$ and we have*

$$\int T \underset{\alpha}{\cdot} S = \int S \underset{\alpha}{\cdot} T .$$

Note. When T or S has bounded support, $\alpha(x-t)\,[T(t) \otimes S(x)]$ is always Silva-integrable on R^{2N}.

Proof. Since

$$T \underset{\alpha}{\cdot} S = (\alpha * T) \cdot S = [\int \alpha(x-t)\, T(t)\, \mathrm{d}t]\, S(x) = \int \alpha(x-t)\, T(t)\, S(x)\, \mathrm{d}t$$

applying theorem 14.1 of Silva [4] we conclude that $T \underset{\alpha}{\cdot} S$ is integrable and

$$\int T \underset{\alpha}{\cdot} S = \int [\int \alpha(x-t)\, T(t)\, S(x)\, \mathrm{d}t]\, \mathrm{d}x = \int \int \alpha(x-t)\, T(t)\, S(x)\, \mathrm{d}t\, \mathrm{d}x.$$

On the other hand

$$S \underset{\alpha}{\cdot} T = (\alpha * S)\, T = [\int \alpha(t-x)\, S(x)\, \mathrm{d}t] \cdot T(t) = \int \alpha(t-x)\, S(x)\, T(t)\, \mathrm{d}x$$

and for the same reason $S \underset{\alpha}{\cdot} T$ integrable and

$$\int S \underset{\alpha}{\cdot} T = \int [\int \alpha(t-x)\, S(x)\, T(t)\, \mathrm{d}x]\, \mathrm{d}t = \int \int \check{\alpha}(x-t)\, S(x)\, T(t)\, \mathrm{d}x\, \mathrm{d}t .$$

Hence $\int T \underset{\alpha}{\cdot} S = \int S \underset{\alpha}{\cdot} T$ because α is G-invariant and so $\check{\alpha} = \alpha$.

Note also that by Prop. 21, when we multiply distributions we can replace the hypotesis $\alpha \in \mathcal{D}$ with $\alpha \in C^\infty$ (if necessary for a more general probabilistic interpretation of α), if we restrict ourselfs to the product of a distribution $T \in \mathcal{E}'$ by a distribution $S \in \mathcal{D}'$.

6 - A liner Cauchy problem

Let us consider the Cauchy problem

$$X' = ig\,\delta' X \qquad X(t_0) = 1 \tag{6.1}$$

where i is the imaginary unit, δ' is the derivative of Dirac measure δ concentrated at the origin of $\boldsymbol{R}$, g and t_0 are real numbers with $g \neq 0$ and $t_0 < 0$.

In the classical framework $\delta' \in \mathcal{D}'^1(\boldsymbol{R})$ and so we must have $X \in C^1(\boldsymbol{R})$, otherwise the product $\delta' X$ has no meaning. Then, $\operatorname{supp} X' = \{0\}$ or $X' = 0$. The first condition is impossible because X' is a continuous function. The second conditions is also impossible because X ought to be the null function on $\boldsymbol{R}$ which is against $X(t_0) = 1$. Thus, we would seek for solutions of (6.1) in $C^1(\boldsymbol{R})$, that don't exist.

In the context of this paper we can ask for solutions X of (6.1) in $\mathcal{D}'(\boldsymbol{R})$ which are continuous in a neighbourhood of t_0 (in order that initial condition $X(t_0) = 1$ makes sense).

Cosider now a pair (G, α) as in Definition 1 and the problems

$$X' = ig\,\delta' \mathbin{\dot{}_{\alpha}} X \qquad X(t_0) = 1 \tag{6.2}$$

$$X' = igX \mathbin{\dot{}_{\alpha}} \delta' \qquad X(t_0) = 1\,. \tag{6.3}$$

By Prop. 21 $X' = ig\,\delta' \mathbin{\dot{}_{\alpha}} X$ is equivalent to $X' = ig(\check{\alpha} * \delta')X$ or $X' = ig(\check{\alpha})'X$ and so $X = e^{ig(\alpha'(-t_0) - \alpha'(-t))}$ is the only solution of (6.2).

By Prop. 22 e $\operatorname{supp}(igX \mathbin{\dot{}_{\alpha}} \delta') \subset \{0\}$ and so (6.3) leads us to $X' = 0$ or to $\operatorname{supp} X' = \{0\}$. But $X' = 0$ is impossible because it implies $X(t) = 1$ for all $t \in \boldsymbol{R}$, which is incompatible with $X' = igX \mathbin{\dot{}_{\alpha}} \delta'$. If $\operatorname{supp} X' = \{0\}$, we have

$$X' = c_0\delta + c_1\delta' + \ldots + c_n\delta^{(n)} \qquad \text{with} \qquad c_0, c_1, \ldots, c_n \in \boldsymbol{C}\,.$$

Then

$$X = c_0 H + c_1\delta + \ldots + c_n\delta^{(n-1)}$$

$$X \mathbin{\dot{}_{\alpha}} \delta' = (\alpha * X)\delta' = (\alpha * (c_0 H + c_1\delta + \ldots + c_n\delta^{(n-1)})) \cdot \delta'\,.$$

Putting $\beta = \alpha * (c_0 H + c_1\delta + \ldots + c_n\delta^{(n-1)})$ we have $\beta \in C^\infty$ and

$$X \mathbin{\dot{}_{\alpha}} \delta' = \beta \cdot \delta' = (\beta\delta') - \beta'\delta = \beta(0)\delta' - \beta'(0)\delta\,.$$

Now $X' = igX \dot{_\alpha} \delta'$ is equivalent to

$$c_0 \delta + c_1 \delta' + \ldots + c_n \delta^{(n)} = ig\beta(0)\delta' - ig\beta'(0)\delta$$

and so $c_0 = -ig\beta'(0)$, $c_1 = ig\beta(0)$, $c_2 = 0, \ldots, c_n = 0$ and

$$X = -ig\beta'(0)H + ig\beta(0)\delta$$

which is incompatible with $X(t_0) = 1$ and $t_0 < 0$ and proves that (6.3) is impossible.

In quantum physics the *scattering operator* can be defined in an heurtistc form by the Cauchy problem

$$S'(t) = igH(t)S(t) \qquad S(t_0) = I$$

where $g \in \mathbf{R}$, $H(t)$ is the Hamiltonean interaction (distribution valued operator) and I is the identical operator on a Fock space. If we call $S_{t_0}(t)$ the heuristic solution of this problem the scattering operator is defined by $S_{-\infty}(+\infty)$.

After these considerations, Colombean [1] p. 70, with a drastic simplification, which consists in considering $\mathbf{C}$ as a Fock space, considers the problem (6.1) with another approach and the scattering operator is computed in example 2 p. 75 with the result $X_{-\infty}(+\infty) = 1$.

In our setting $X_{t_0}(t) = e^{ig(\alpha'(-t_0) - \alpha'(-t))}$ and we also have $X_{-\infty}(+\infty) = 1$. Note that in Colombean approach $X_{t_0}(t) = e^{ig\delta(t)}$ is not a distribution but an element of a space $G \supset \mathcal{D}'$.

7 - The Burger's equation and the velocity of certain shock waves

Let us consider a one dimensional fluid moving on the real line in the absence of external forces. Let $u(x, t)$ be the velocity of the particle in the position x at the instant t. The law of motion $x(t)$ of a particle in the fluid is clearly a solution of the differential equation

$$\frac{dx}{dt} = u(x(t), t)$$

and we have
$$\frac{d^2x}{dt^2} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = 0$$

on account of the absence of external forces.

Thus, we can consider the equation

$$\frac{\partial u}{\partial x} u + \frac{\partial u}{\partial t} = 0 \tag{7.1}$$

and ask for pure shock wave solutions, that is, solutions of the form

$$u(x, t) = u_1 + (u_2 - u_1) H(x - vt) \tag{7.2}$$

where v is the velocity of the shoch wave, u_1 and u_2 are complex constants with $u_1 \neq u_2$ and H is the Heaviside function on $\boldsymbol{R}$.

In the sense of this paper we can associate to equation (7.1) two equations:

$$\frac{\partial u}{\partial x} \dot{\underset{\alpha}{}} u + \frac{\partial u}{\partial t} = 0 \tag{7.3}$$

$$u \dot{\underset{\alpha}{}} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \tag{7.4}$$

where $\alpha \in \mathcal{D}(\boldsymbol{R}^2)$ is G-invariant with $\int \alpha = 1$. We take as G the orthogonal group on $\boldsymbol{R}^2$.

Let $\delta(x - vt)$ be the distribution defined by

$$\langle \delta(x - vt), \quad \varphi(x, t) \rangle = \int_{\boldsymbol{R}} \varphi(vt, t)\, dt \qquad \varphi \in \mathcal{D}(\boldsymbol{R}^2).$$

Then

$$\frac{\partial u}{\partial x} = (u_2 - u_1)\, \delta(x - vt) \qquad \frac{\partial u}{\partial t} = -v(u_2 - u_1)\, \delta(x - vt)$$

and (7.3) is equivalent to

$$(u_2 - u_1)\, \delta(x - vt) \dot{\underset{\alpha}{}} [u_1 + (u_2 - u_1) H(x - vt)] - v(u_2 - u_1)\, \delta(x - vt) = 0\,.$$

We have

$$\delta(x - vt) \dot{\underset{\alpha}{}} u_1 = u_1 [\alpha(x, t) * \delta(x - vt)] = u_1 \int_{\boldsymbol{R}} \alpha(x - vz, t - z)\, dz$$

$$\delta(x - vt) \dot{\underset{\alpha}{}} H(x - vt) = H(x - vt)[\alpha(x, t) * \delta(x - vt)] = H(x - vt) \int_{\boldsymbol{R}} \alpha(x - vz, t - z)\, dz\,.$$

Thus (7.3) is equivalent to

$$(u_2 - u_1)\, u_1 \int_R \alpha(x - vz,\, t - z)\, dz$$
(7.5)
$$+ (u_2 - u_1)^2 H(x - vt) \int_R \alpha(x - vz,\, t - z)\, dz - v(u_2 - u_1)\, \delta(x - vt) = 0\,.$$

At the instant $t = 0$ we have for all $x < 0$ $(u_2 - u_1)\, u_1 \int_R \alpha(x - vz,\, -z)\, dz = 0$

and so $$u_1 \int_{-\infty}^{0} \left[\int_R \alpha(x - vz,\, -z)\, dz\right] dx = 0$$

or $$u_1 \int_{x<0}\int \alpha(x - vz,\, -z)\, dx\, dz = 0\,.$$

Making the change of variable $(x, z) \to (r, s)$ defined by

$$x - vz = r \qquad -z = s$$

we have $u_1 \int\int_{r - vs < 0} \alpha(r, s)\, dr\, ds = 0$, equivalent to $u_1 = 0$.

Returning to (7.5) at the instant $t = 0$ we have for all $x > 0$ $u_2^2 \int \alpha(x - vz,\, -z)\, dz = 0$ and by the same change of variable we conclude that $u_2 = 0$, which is impossible because $u_1 \neq u_2$.

Now, considering equation (7.4), noting that $u(x, t) \in \mathcal{D}'(\boldsymbol{R}^2)$, $\dfrac{\partial u}{\partial x} \in \mathcal{D}'_n(\boldsymbol{R}^2)$ and remembering what we have said in Sec. **5**, we conclude that (7.2) is a solution of (7.4), if and only if $v = \dfrac{1}{2}(u_1 + u_2)$ (see [3], p. 314). This is in agreement with physical reality. Note also that in [3] we cannot consider equation (7.3) because $u \notin \mathcal{D}'_n(\boldsymbol{R}^2)$.

8 - The (G, α)-convolution product in $\mathcal{S}'$

We denote by $\mathcal{S}$ the space of all C^∞-rapidly decreasing functions defined on $\boldsymbol{R}^N$ and complex valued. $\mathcal{S}'$ means the space of distributions of slow growth. The usual convolution product will be denoted by $*$.

Definition 2. Given a group of unimodular transformations G of $\boldsymbol{R}^N$ and a function $\alpha \in \mathcal{D}$, G-invariant with $\int \alpha = 1$, we define the (G, α)-*convolution product* $f \underset{\alpha}{*} g$ of two distributions $f, g \in \mathcal{S}'$ by

$$f \underset{\alpha}{*} g = g * [(\mathcal{F}^{-1}\alpha) \cdot f]$$

where $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is the *Fourier transform* defined by $(\mathcal{F}\varphi)(x) = \int e^{-2\pi i\langle x, t\rangle} \varphi(t)\, dt$ for all $\varphi \in \mathcal{S}$.

Recall that, if we define the *extension* of $\mathcal{F}$ to $\mathcal{S}'$ as usually by

$$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle \qquad \text{for all } f \in \mathcal{S}' \text{ and } \varphi \in \mathcal{S},$$

we have $\mathcal{F}f \in \mathcal{S}'$. Hence, $\alpha * \mathcal{F}f \in \mathcal{O}_M$, space of C^∞ functions of slow growth, (see [4] p. 245) and so $\mathcal{F}^{-1}(\alpha * \mathcal{F}f) \in \mathcal{O}_c'$, space of rapidly decreasing distributions. Thus, $(\mathcal{F}^{-1}\alpha) \cdot f \in \mathcal{O}_c'$ and $f \underset{\alpha}{\circledast} g \in \mathcal{S}'$ for $g \in \mathcal{S}'$.

We shall prove the principal properties of the (G, α)-convolution product.

Proposition 23. *Let (G, α) be as in Definition 2, $a \in \boldsymbol{R}^N$, $k \in \{1, 2, \dots, N\}$, δ the Dirac measure concentrated at the origin of $\boldsymbol{R}^N$ and $f, g \in \mathcal{S}'$. Then*

a. $\delta \underset{\alpha}{\circledast} g = g$ ***b.*** $\tau_a(f \underset{\alpha}{\circledast} g) = f \underset{\alpha}{\circledast} \tau_a g$ ***c.*** $D_k(f \underset{\alpha}{\circledast} g) = f \underset{\alpha}{\circledast} D_k g$

d. $\mathcal{F}(f \underset{\alpha}{\circledast} g) = (\mathcal{F}f) \underset{\alpha}{\odot} (\mathcal{F}g)$ ***e.*** $\mathcal{F}^{-1}(f \underset{\alpha}{\odot} g) = \mathcal{F}^{-1} f \underset{\alpha}{\circledast} \mathcal{F}^{-1} g$.

Moreover, if the map $t \to -t$ belongs to G, we also have

f. $\mathcal{F}^{-1}(f \underset{\alpha}{\circledast} g) = (\mathcal{F}^{-1}f) \underset{\alpha}{\odot} (\mathcal{F}^{-1}g)$ ***g.*** $\mathcal{F}(f \underset{\alpha}{\odot} g) = (\mathcal{F}f) \underset{\alpha}{\circledast} \mathcal{F}g$.

Proof:

a. $\delta \underset{\alpha}{\circledast} g = g * [(\mathcal{F}^{-1}\alpha) \cdot \delta] = g * [(\mathcal{F}^{-1}\alpha)(0) \cdot \delta] = g * \delta = g$

because
$$(\mathcal{F}^{-1}\alpha)(0) = \int \alpha(t)\, dt = 1\,.$$

b. $\tau_a(f \underset{\alpha}{\circledast} g) = \tau_a[g * [(\mathcal{F}^{-1}\alpha) \cdot f]] = \tau_a g * (\mathcal{F}^{-1}\alpha) \cdot f = f \underset{\alpha}{\circledast} \tau_a g$.

c. $D_k(f \underset{\alpha}{\circledast} g) = D_k[g * [(\mathcal{F}^{-1}\alpha) \cdot f]] = D_k g * (\mathcal{F}^{-1}\alpha) \cdot f = f \underset{\alpha}{\circledast} D_k g$.

d. $\mathcal{F}(f \underset{\alpha}{\circledast} g) = \mathcal{F}[g * [(\mathcal{F}^{-1}\alpha) \cdot f]] = \mathcal{F}g \cdot \mathcal{F}[(\mathcal{F}^{-1}\alpha) \cdot f]$

$$= \mathcal{F}g \cdot (\alpha * \mathcal{F}f) = \mathcal{F}f \underset{\alpha}{\odot} \mathcal{F}g\,.$$

e. $\mathcal{F}[\mathcal{F}^{-1}f \underset{\alpha}{\circledast} \mathcal{F}^{-1}g] = f \underset{\alpha}{\odot} g$ by ***d***, and ***e*** follows.

Note that the (G, α)-product of two distributions of slow growth is a distribution of slow growth.

f. $\mathcal{F}[\mathcal{F}^{-1}f \underset{\alpha}{\cdot} \mathcal{F}^{-1}g] = \mathcal{F}[(\alpha * \mathcal{F}^{-1}f) \cdot \mathcal{F}^{-1}g] = [(\mathcal{F}\alpha) \cdot f] * g$

$$= [(\mathcal{F}^{-1}\alpha) \cdot f] * g = f \underset{\alpha}{*} g$$

because $\mathcal{F}\alpha = \mathcal{F}^{-1}\alpha$ once α is G-invariant.

g. $\mathcal{F}^{-1}[\mathcal{F}f \underset{\alpha}{*} \mathcal{F}g] = f \underset{\alpha}{\cdot} g$ by ***f***, and ***g*** follows.

References

[1] J. F. Colombean, *An elementary introduction to new generalized functions*, North-Holland, Amsterdam, 1985.

[2] C. O. R. Sarrico, *A note on a family of distributional products important in the applications*, Note Mat. 7 (1987), 151-158.

[3] C. O. R. Sarrico, *About a family of distributional products important in the applications*, Portugal. Math. 45 (1988), 295-316.

[4] L. Schwartz, *Theorie des distributions*, Hermann, Paris 1966.

[5] J. Sebastião Silva, *Integrals and orders of growth of distributions*, reprinted from *Theory of distributions*, Proc. Internat. Summer Inst., Lisbon, Sept. 1964.

[6] J. Sebastião Silva, *Novos elementos para a teoria do integral no campo das distribuições*, Bol. Acad. Ciências Lisboa 35 (1963), 175-184.

Sommario

Le entità fisiche macroscopiche sono comunemente interpretate come distribuzioni di Schwartz (spazio $\mathcal{D}'$) e spesso devono essere considerate come medie di variabili microscopiche o «impulsive». L'operatore s di media o di valore medio, applicato a funzioni C^∞ integrabili, è lineare e verifica le proprietà (1) *e* (2) *della introduzione (D è un operatore differenziale a coefficienti costanti), ma non verifica la proprietà* (3) *relativa all'usuale prodotto di funzioni.*

In questo articolo si definisce uno spazio E in modo che le proprietà (1), (2) *e* (3) *siano soddisfatte. Da sottolineare il fatto che E risulta isomorfo a $\mathcal{D}'$. Ciò dà luogo, in tutto $\mathcal{D}'$, ad un prodotto dipendente da s, descritto dal sistema di assiomi presentato al numero* **1**. *Questo prodotto viene applicato al caso di un problema di Cauchy lineare e di un'equazione di Burger non lineare. Viene infine definito un nuovo tipo di convoluzione per distribuzioni a crescita lenta.*

* * *