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Reciprocal and variational principles on thermoelasticity for porous piezolectric materials (**)

1 - Introduction

The problem of the interaction of electromagnetical fields with elastic dielectrics was the subject of important investigations (cf. C. Truesdell and R. Toupin [16], Parkus [14], Grot [5], Nowacki [12], Maugin [10]).

The theory of infinitesimal deformations and weak fields superimposed on a finite deformation and strong electromagnetic field has been defined in the fundamental work of Toupin [15]. In this context the photoelastic effect is only an example of the several phenomena connected with the sphere of applications of this theory.

Goodman and Cowin [4] presented a continuum theory for the flowing of granular materials. Using the fundamental concept of *distributed body* introduced in [4], Nunziato and Cowin [13] have defined a theory for porous solids in which the skeletal (matrix material) is elastic and the interstices are void of material. In this theory the (*bulk*) mass density of the material turns out to be the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom. The linear theory of elastic material with voids was established by Cowin and Nunziato in [3]. The dynamic theory of (prestressed) thermoelastic dielectrics with voids seems to be an adeguate tool to describe the behaviour of numerous

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kinds of (prestressed) dielectric materials (piezoceramics, piezoelectric pressed powders, ecc.).

The context of this work is based on the dynamical theory of elastic dielectrics developed by Toupin [15], the theory of elastic materials with voids presented by Cowin and Nunziato [13], [3] and the theory of incremental thermoelasticity presented in [7], [8]. This study is a $v^2 c^{-2}$ approximation (v is the velocity field in the material and c is the velocity of light in vacuum) to a relativistic invariant theory (cf. [3]).

Starting from the constitutive equations established for the problem in concern in [8], we shall deal with reciprocal and variational theorems (considering the case of a quasi-static electric field).

We establish the reciprocal relations by applying the method given in [9]; these relations are then used to prove a variational principle for the initial-boundary value problem in concern.

2 - Notation and basic equations

Let Ω be the region of the physical 3-dimensional space $\boldsymbol{R}^3$ occupied by a continuum body with voids in a reference (stressed) state, and $I = [0, +\infty)$ be the time interval in concern. Position and time will be denoted by $(\boldsymbol{x}, t) \in \Omega \times I$, while the motion of the body is referred to a fixed orthonormal frame in $\boldsymbol{R}^3$.

We shall denote the tensor components of order $p \geqslant 1$ by latin subscripts, ranging over $\{1, 2, 3\}$. Summation over repeated subscripts is implied. Superposed dots or subscripts preceded by a comma will mean partial derivative with respect to the time or the corresponding coordinates. Occasionally, we shall use bold-face character and typical notations for vectors and operations upon them.

As is well-known (see, e.g., [8], [1]), the behaviour of a porous elastic body submitted to quasi-static electric field, is governed by the following *local balance equations*:

$$
(1) \quad \begin{aligned}
&S_{ji,j} + \varrho f_i = \varrho \ddot{u}_i && \text{balance of momentum} \\
&h_{i,i} + g + \varrho l = \varrho \chi \ddot{\psi} && \text{balance of equilibrated stress} \\
&T_0 \dot{\eta} = q_{i,i} + \varrho r && \text{energy equation} \\
&d_{i,i} = F \qquad e_i = -\varphi_{,i} && \text{quasi-static electric field.}
\end{aligned}
$$

In these equations, we mean:

$\boldsymbol{S}, \boldsymbol{f}$	stress tensor and body force
$\boldsymbol{h}, g, l$	equilibrated stress vector, extrinsic and intrinsic equilibrated body force, respectively
η, q, r	entropy, heat flux and heat supply, respectively
$\boldsymbol{d}, e$	electric displacement and electric field.

Moreover, we consider as independent variables of this theory the displacement $\boldsymbol{u}$, the change in volume fraction from the reference configuration ψ, the temperature ϑ and the electric potential φ. All the above fields, of course, represent incremental quantities with respect to the referential values.

Finally, ϱ, χ, T_0 and F are bulk mass density, equilibrated inertia, temperature in the reference state and volume density of free charge.

The system of field equations is completed by the following *constitutive equations*:

$$
\begin{aligned}
S_{ij} &= (C_{ijrs} + T_{is}\delta_{jr})u_{r,s} + M_{ij}\psi + M_{ijk}\psi_{,k} - B_{ijk}e_k - \beta_{ij}\vartheta \\
h_i &= M_{jki}u_{j,k} + B_i\psi + D_{ij}\psi_{,j} + P_{ik}e_k - N_i\vartheta \\
g &= -M_{ij}u_{i,j} - C\psi - B_i\psi_{,i} - A_k e_k + m\vartheta \\
\eta &= \beta_{ij}u_{i,j} + m\psi + N_i\psi_{,i} + c_k e_k + P\vartheta \\
d_i &= B_{jki}u_{j,k} - A_i\psi - P_{ki}\psi_{,k} + \varepsilon_{ik}e_k + c_i\vartheta \\
q_i &= K_{ji}\vartheta_{,j}
\end{aligned} \tag{2}
$$

where T_{ij} denotes the Cauchy pre-stress tensor.

We note that the incremental stress S_{ij} is not symmetric unless $T_{is}\,\delta_{jr}$ vanishes.

If we set $A_{jirs} = C_{ijrs} + T_{is}\delta_{jr}$, we achieve the symmetric relations of the constitutive coefficients, which were discussed by Ciarletta and Scalia (see [2]) in connection with the problem of a thermoelastic dielectric with voids, i.e.

$$
\begin{aligned}
&A_{jirs} = A_{rsji} \qquad M_{ijk} = M_{jik} = M_{ikj} \qquad M_{ij} = M_{ji} \qquad D_{ij} = D_{ji} \\
&B_{ijk} = B_{jik} \qquad \varepsilon_{ij} = \varepsilon_{ji} \qquad \beta_{ij} = \beta_{ji} \qquad K_{ij} = K_{ji}\,.
\end{aligned} \tag{3}
$$

According to the classical interpretation of system (1), (2), we assume

i. $u_i, \psi \in C^{2,2}(\bar{\Omega} \times I) \qquad \vartheta \in C^{2,1}(\bar{\Omega} \times I) \qquad \varphi \in C^{2,0}(\bar{\Omega} \times I)$

ii. $\varrho, \chi \in C^0(\bar{\Omega})$

iii. $S_{ij}, h_i, q_i, d_i \in C^{1,0}(\bar{\Omega} \times I) \qquad \eta \in C^{0,1}(\bar{\Omega} \times I)$

iv. $f_i, g, l, r \in C^{1,0}(\bar{\Omega} \times I)$

v. the constitutive coefficients are continuously differentiable on $\overline{\Omega}$.

Let us denote four pairs of disjoint and complementary subsets of the (smooth) boundary $\partial\Omega$ by $\{\Sigma_i, \Sigma_{i+1}\}$ with $i = 1, 3, 5, 7$. The surface traction $\boldsymbol{t}$, the heat flux q, the surface equilibrated stress h, and the normal component of the electric diplacement $\boldsymbol{d}$ are defined by

$$\boldsymbol{t} = \boldsymbol{S}\boldsymbol{n} \qquad q = \boldsymbol{q}\cdot\boldsymbol{n} \qquad h = \boldsymbol{h}\cdot\boldsymbol{n} \qquad d = \boldsymbol{d}\cdot\boldsymbol{n} \qquad \text{on } \partial\Omega\times I$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$.

We consider the following initial-boundary conditions

$$\begin{aligned}
&\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad &&\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}) \qquad &&\eta(\boldsymbol{x}, 0) = \eta_0(\boldsymbol{x}) \\
&\psi(\boldsymbol{x}, 0) = \psi_0(\boldsymbol{x}) &&\dot{\psi}(\boldsymbol{x}, 0) = \dot{\psi}_0(\boldsymbol{x}) && \\
&\boldsymbol{d}(\boldsymbol{x}, 0) = \boldsymbol{d}_0(\boldsymbol{x}) &&\varphi(\boldsymbol{x}, 0) = \varphi_0(\boldsymbol{x}) &&
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \Sigma_1\times I \qquad && \boldsymbol{t} = \hat{\boldsymbol{t}} \quad \text{on } \Sigma_2\times I \\
&\psi = \hat{\psi} \quad \text{on } \Sigma_3\times I && h = \hat{h} \quad \text{on } \Sigma_4\times I \\
&\vartheta = \hat{\vartheta} \quad \text{on } \Sigma_5\times I && q = \hat{q} \quad \text{on } \Sigma_6\times I \\
&\varphi = \hat{\varphi} \quad \text{on } \Sigma_7\times I && d = -\sigma \quad \text{on } \Sigma_8\times I
\end{aligned} \tag{5}$$

where right-hand terms stand for (sufficiently smooth) assigned fields.

3 - Reciprocity

Let a_1 and a_2 be scalar fields on $\Omega\times I$ that are continuous in time. We denote by $a_1 * a_2$ the time convolution of a_1 and a_2

$$a_1 * a_2(\boldsymbol{x}, t) = \int_0^t a_1(\boldsymbol{x}, t-s)\, a_2(\boldsymbol{x}, s)\, \mathrm{d}s\,.$$

We introduce the functions:

$$L(t) = 1 \qquad \xi(t) = L * L(t) = t \qquad t\in I$$

and for any continuous function $a(\boldsymbol{x}, t)$ on $\Omega\times I$ we denote by $\bar{a}$ the function on $\Omega\times I$, defined by

$$\bar{a}(\boldsymbol{x}, t) = L * a(\boldsymbol{x}, t) = \int_0^t a(\boldsymbol{x}, s)\, \mathrm{d}s\,.$$

The energy equation $(1)_3$ and initial condition $(4)_5$ are, together, equivalent to the integro-differential equation

(6) $$T_0\eta = \bar{q}_{i,i} + \mathcal{W} \quad \text{on } \Omega\times I, \qquad \text{where } \mathcal{W} = T_0\eta_0 + \varrho\bar{r}.$$

Now, let $\mathcal{U}^{(\alpha)} = (\boldsymbol{u}^{(\alpha)}, \psi^{(\alpha)}, \vartheta^{(\alpha)}, \varphi^{(\alpha)})$, $\alpha = 1, 2$ denote two regular solutions, corresponding to different sets of data

$$\Gamma^{(\alpha)} = \{f_i^{(\alpha)}, l^{(\alpha)}, \boldsymbol{u}_0^{(\alpha)}, \boldsymbol{v}_0^{(\alpha)}, \eta_0^{(\alpha)}, \psi_0^{(\alpha)}, \dot{\psi}_0^{(\alpha)}, \boldsymbol{d}_0^{(\alpha)}, \varphi_0^{(\alpha)},$$

$$\mathcal{W}^{(\alpha)}, \hat{\boldsymbol{u}}^{(\alpha)}, \hat{\boldsymbol{t}}^{(\alpha)}, \hat{\psi}^{(\alpha)}, \hat{h}^{(\alpha)}, \hat{\vartheta}^{(\alpha)}, \hat{q}^{(\alpha)}, \hat{\varphi}^{(\alpha)}, \sigma^{(\alpha)}\}.$$

Moreover, define $S_{ij}^{(\alpha)}, h_i^{(\alpha)}, g^{(\alpha)}, \eta^{(\alpha)}, q_i^{(\alpha)}, d_i^{(\alpha)}$ by means equations (2), $(1)_5$ for each $\alpha = 1, 2$.

We have the lemma

Lemma 1. *Let $\mathcal{U}^{(\alpha)}$ be solutions corresponding to different sets of data $\Gamma^{(\alpha)}$ ($\alpha = 1, 2$), and assume that the symmetry relations (3) hold. Then we obtain*

(7) $$E_{\alpha\beta}(r, s) = E_{\beta\alpha}(s, r).$$

where

$$E_{\alpha\beta}(r,s) = \int_\Omega \{\varrho[f_i^{(\alpha)}(r) - \ddot{u}_i^{(\alpha)}(r)]u_i^{(\beta)}(s) + \varrho[l^{(\alpha)}(r) - \chi\ddot{\psi}^{(\alpha)}(r)]\psi^{(\beta)}(s)\}\,d\Omega$$

$$-\int_\Omega \{F^{(\alpha)}(r)\varphi^{(\beta)}(s) + \frac{1}{T_0}[\mathcal{W}^{(\alpha)}(r)\vartheta^{(\beta)}(s) - \bar{q}_i^{(\alpha)}(r)\vartheta_{,i}^{(\beta)}(s)]\}\,d\Omega$$

$$+\int_{\partial\Omega} [t_i^{(\alpha)}(r)u_i^{(\beta)}(s) + h^{(\alpha)}(r)\psi^{(\beta)}(s) + d^{(\alpha)}(r)\varphi^{(\beta)}(s) - \frac{1}{T_0}\bar{q}^{(\alpha)}(r)\vartheta^{(\beta)}(s)]\,d\Sigma$$

for all $r, s \in I$ and $\alpha, \beta = 1, 2$.

In Lemma 1 and in the sequel the argument x is understood.

Proof. Consider the expression

$$F_{\alpha\beta}(r,s) = S_{ji}^{(\alpha)}(r)u_{i,j}^{(\beta)}(s) + h_i^{(\alpha)}(r)\psi_{,i}^{(\beta)}(s)$$
$$-g^{(\alpha)}(r)\psi^{(\beta)}(s) - \eta^{(\alpha)}(r)\vartheta^{(\beta)}(s) + d_i^{(\alpha)}(r)\varphi_{,i}^{(\beta)}(s).$$

From the constitutive equations and the symmetry relations we deduce

(8) $$F_{\alpha\beta}(r, s) = F_{\beta\alpha}(s, r).$$

Since we have

$$F_{\alpha\beta}(r,s)=[S_{ji}^{(\alpha)}(r)\,u_i^{(\beta)}(s)]_{,j}-S_{ji,j}^{(\alpha)}(r)\,u_i^{(\beta)}(s)+[h_i^{(\alpha)}(r)\,\psi^{(\beta)}(s)]_{,i}-h_{i,i}^{(\alpha)}(r)\,\psi^{(\beta)}(s)$$

$$+[d_i^{(\alpha)}(r)\,\varphi^{(\beta)}(s)]_{,i}-d_{i,i}^{(\alpha)}(r)\,\varphi^{(\beta)}(s)-g^{(\alpha)}(r)\,\psi^{(\beta)}(s)-\eta^{(\alpha)}(r)\,\vartheta^{(\beta)}(s)$$

we can write

$$F_{\alpha\beta}(r,s)=[S_{ij}^{(\alpha)}(r)\,u_{,j}^{(\beta)}(s)]_{,i}+\varrho[f_i^{(\alpha)}(r)-\ddot{u}_i^{(\alpha)}(r)]\,u_i^{(\beta)}(s)+[h_i^{(\alpha)}(r)\,\psi^{(\beta)}(s)]_{,i}$$

$$+\varrho[l^{(\alpha)}(r)-\chi\ddot{\psi}^{(\alpha)}(r)]\,\psi^{(\beta)}(s)$$

$$-[\frac{1}{T_0}\,\bar{q}_i^{(\alpha)}(r)\,\vartheta^{(\beta)}(s)]_{,i}+[d_i^{(\alpha)}(r)\,\varphi^{(\beta)}(s)]_{,i}+\frac{1}{T_0}\,\bar{q}_i^{(\alpha)}(r)\,\vartheta_{,i}^{(\beta)}(s)-\mho^{(\alpha)}(r)\,\vartheta^{(\beta)}(s)$$

taking into account equations $(3)_1$, $(3)_2$, (6). If we integrate $F_{\alpha\beta}(r,s)$ over Ω and use the divergence theorem and the field equation $(1)_5$, we finally obtain

$$\int_\Omega F_{\alpha\beta}(r,s)\,\mathrm{d}\Omega=E_{\alpha\beta}(r,s)\,.$$

Then, the lemma follows by (8).

Lemma 1 forms the basis of the

Reciprocal Theorem. *Let $\mathcal{U}^{(\alpha)}$ be solutions corresponding to different sets of data $\Gamma^{(\alpha)}$ ($\alpha=1,2$), and assume*

i. *the symmetry relations* (3) *hold*

ii. *K_{ij} is a symmetric tensor.*

Then, we obtain $I_{12}(t)=I_{21}(t)$, $t\in I$ where

$$I_{\alpha\beta}=\int_\Omega \mathcal{F}_i^{(\alpha)}*u_i^{(\beta)}+\mathcal{L}^{(\alpha)}*\psi^{(\beta)}-\xi*[\frac{1}{T_0}\,\mho^{(\alpha)}*\vartheta^{(\beta)}+F^{(\alpha)}*\varphi^{(\beta)}]\,\mathrm{d}\Omega$$

$$+\int_{\partial\Omega}\xi*[t_i^{(\alpha)}*u_i^{(\beta)}+h^{(\alpha)}*\psi^{(\beta)}-\frac{1}{T_0}\,\bar{q}^{(\alpha)}*\vartheta^{(\beta)}+d^{(\alpha)}*\varphi^{(\beta)}]\,\mathrm{d}\Sigma$$

and

$$(9)\qquad \mathcal{F}_i^{(\alpha)}=\varrho[\xi*f_i^{(\alpha)}+u_{i0}^{(\alpha)}+tv_{i0}^{(\alpha)}]\qquad \mathcal{L}^{(\alpha)}=\varrho[\xi*l^{(\alpha)}+\psi_0^{(\alpha)}+t\dot{\psi}_0^{(\alpha)}]\,.$$

Proof. If we put in (7) $r=\tau$, $s=t-\tau$ ($\tau\in[0,t]$) and integrate from 0 to t

with respect to τ, we get

$$\int_0^t E_{\alpha\beta}(\tau, t-\tau)\,d\tau = \int_0^t E_{\alpha\beta}(t-\tau, \tau)\,d\tau . \tag{10}$$

If we remark that for (any) function $a = a(\boldsymbol{x}, t)$ the following equation holds

$$\xi * \ddot{a}(t) = a(t) - [a(0) + t\dot{a}(0)],$$

we easily obtain the desired result, since the convolution of the relation (10) with ξ gives

$$\xi * \int_0^t E_{\alpha\beta}(\tau, t-\tau)\,d\tau$$

$$= I_{\alpha\beta} - \int_\Omega \{\varrho[u_i^{(\alpha)} * u_i^{(\beta)} + \chi\psi^{(\alpha)} * \psi^{(\beta)}] - \frac{1}{T_0}\,\xi * K_{ji}\bar{\vartheta}_{,j}^{(\alpha)} * \vartheta_{,i}^{(\beta)}\}\,d\Omega.$$

4 - Variational principle

Following Gurtin [6], it is a simple matter to prove

Lemma 2. *The functions u_i, ψ_i, S_{ij}, h_i, e_i and η satisfy the equations of motion* (1) *on $\Omega \times I$ and the initial conditions* (4), *if and only if*

$$\xi * S_{ji,j} + \mathcal{F}_i = \varrho u_i \qquad \xi * h_{i,i} + \xi * g + \mathcal{L} = \varrho\chi\psi$$

$$\xi * (T_0\eta) = \xi * \bar{q}_{i,i} + \xi * \mathcal{W} \qquad \xi * d_{i,i} = \xi * F$$

where $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{W}$ are defined in (6), (9).

If we introduce the notations

$$A\mathcal{U}_i = \varrho u_i - \xi * S_{ji,j} \qquad (i = 1, 2, 3) \qquad A\mathcal{U}_4 = \varrho\chi\psi - \xi * h_{i,i} - \xi * g$$

$$A\mathcal{U}_5 = \xi * \frac{\bar{q}_{i,i}}{T_0} - \xi * \eta \qquad A\mathcal{U}_6 = -\xi * d_{i,i}$$

the field equations (1), (2) along with the initial conditions (4), thanks to Lemma 2, can be equivalently written in form of one only vector equation

$$A\mathcal{U} = \mathcal{O} \qquad \text{on } \Omega \times I$$

where A is a linear operator, $\mathcal{U}$ is an admissible vector field and $\mathcal{D} = (\mathcal{D}_i, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6)$ with

$$\mathcal{D}_i = \mathcal{F}_i, \qquad \mathcal{D}_4 = \mathcal{L}, \qquad \mathcal{D}_5 = -\xi * \frac{W}{T_0} \qquad \mathcal{D}_6 = -\xi * F\,.$$

Let $\mathcal{E}_A$ be the domain of definition of A. By means of constitutive equations, to any $\mathcal{V} = (\boldsymbol{u}', \psi', \vartheta', \varphi') \in \mathcal{E}_A$ correspond well defined fields

$$\boldsymbol{t}(\mathcal{V}) = \boldsymbol{S}'_{ji}\, \boldsymbol{n}_j = \boldsymbol{t}' \qquad \boldsymbol{h}(\mathcal{V}) = \boldsymbol{h}' \cdot \boldsymbol{n} = \boldsymbol{h}'$$
$$\boldsymbol{q}(\mathcal{V}) = \boldsymbol{q}' \cdot \boldsymbol{n} = \boldsymbol{q}' \qquad \boldsymbol{d}(\mathcal{V}) = \boldsymbol{d}' \cdot \boldsymbol{n} = \boldsymbol{d}'\,.$$

Denote $\widehat{\mathcal{E}}_A$ the subdomain of all vector fields $\mathcal{V} \in \mathcal{E}_A$ such that

$$(11) \qquad \begin{array}{llll} \boldsymbol{u}' = \boldsymbol{0} & \text{on } \Sigma_1 \times I & \boldsymbol{t}' = \boldsymbol{0} & \text{on } \Sigma_2 \times I \\ \psi' = 0 & \text{on } \Sigma_3 \times I & h' = 0 & \text{on } \Sigma_4 \times I \\ \vartheta' = 0 & \text{on } \Sigma_5 \times I & q' = 0 & \text{on } \Sigma_6 \times I \\ \varphi' = 0 & \text{on } \Sigma_7 \times I & d' = 0 & \text{on } \Sigma_8 \times I \end{array}$$

Lemma 3. *The restriction of A to $\widehat{\mathcal{E}}_A$ is symmetric in convolution, i.e.*

$$\int_\Omega (A\mathcal{V}^{(1)} * \mathcal{V}^{(2)} - A\mathcal{V}^{(2)} * \mathcal{V}^{(1)})\,\mathrm{d}\Omega = 0 \qquad \forall \mathcal{V}^{(1)}, \mathcal{V}^{(2)} \in \widehat{\mathcal{E}}_A\,.$$

Proof. The proof is an immediate consequence of the definition of A and the symmetry properties of the reciprocal theorem.

Let $\mathcal{S} \in \mathcal{E}_A$ be a given (six) vector that meets all the boundary conditions (5), and let $\mathcal{U}$ be a regular solution of the mixed problem; then $\mathcal{V} \equiv \mathcal{U} - \mathcal{S}$ satisfies equation

$$(12) \qquad A\mathcal{V} = \mathcal{D}' = \mathcal{D} - A\mathcal{S}$$

and the homogeneous boundary conditions, so that $\mathcal{V} \in \widehat{\mathcal{E}}_A$.

Recalling that the functional on $\widehat{\mathcal{E}}_A$

$$\tilde{\Phi}(\mathcal{V}) = \int_\Omega (A\mathcal{V} * \mathcal{V} - 2\mathcal{V} * \mathcal{D}')\,\mathrm{d}\Omega$$

is stationary at $\mathcal{V}$ if and only if $\mathcal{V}$ solves the homogeneous problem (11), (12) [**11**], we can finally state the

Variational Principle. *Let $\mathcal{E}_A^*$ be the subdomain of those $\mathcal{U} = (\boldsymbol{u}, \psi, \vartheta, \varphi)$ of $\mathcal{E}_A$ that meet the left-side boundary conditions in equation* (5). *Then, the fun-*

ctional on $\mathcal{E}_A^*$ defined by

$$\Phi(\mathcal{U}) = \int_\Omega \xi * [S_{ji} * u_{i,j} + h_i * \psi_{,i} - g * \psi - \eta * \vartheta + d_i * \varphi_{,i}]\, d\Omega$$

$$+ \int_\Omega [\varrho(u_i * u_i + \chi\psi * \psi) - \frac{1}{T_0}\, \xi * \overline{q}_i * \vartheta_{,i}]\, d\Omega$$

$$-2\int_\Omega [\mathcal{F}_i * u_i + \mathcal{L} * \psi - \xi * \frac{W}{T_0} * \vartheta - \xi * F * \varphi]\, d\Omega$$

$$-2\int_{\Sigma_2} \xi * \boldsymbol{t} * \boldsymbol{u}\, d\Sigma - 2\int_{\Sigma_4} \xi * h * \psi\, d\Sigma$$

$$+ \frac{2}{T_0} \quad -2\int_{\Sigma_6} \xi * \overline{q} * \vartheta \;\, d\Sigma + 2\int_{\Sigma_8} \xi * \sigma * \varphi \;\, d\Sigma$$

has a stationary point at, and only at, the solution of the initial boundary value problem (1), (2), (3), (4), (5).

Proof. The proof can be safely reduced to note that, defining

$$\begin{aligned} \Phi(\mathcal{U}) &= \tilde{\Phi}(\mathcal{U} - \mathcal{S}) \\ &= \int_\Omega [A\mathcal{U} * \mathcal{U} + (A\mathcal{S} * \mathcal{U} - A\mathcal{U} * \mathcal{S}) - 2\mathcal{U} * \mathcal{D} - A\mathcal{S} * \mathcal{S} + 2\mathcal{S} * \mathcal{D}]\, d\Omega\,. \end{aligned} \tag{13}$$

The expression (13) can be explicitly worked out from the above equation neglecting inessential terms.

References

[1] K. AMES and B. STRAUGHAN, *Continuous dependence results of stressed thermoelastic bodies*, Internat. J. Engrg. Sci. **30** (1992).

[2] M. CIARLETTA and A. SCALIA, *A theory of thermoelastic dielectrics with voids*, J. Thermal Stresses **17** (1994), 529-548.

[3] S. C. COWIN and J. W. NUNZIATO, *Linear elastic material with voids*, J. Elasticity. **13** (1983), 125-147.

[4] M. A. GOODMAN and S. C. COWIN, *A continuum theory for granular materials*, Arch. Rational Mech. Anal. **44** (1972), 249-265.

[5] R. A. GROT, *Relativistic Continuum Physics: Electromagnetic Interactions*, Continuum Physic **3**, ed. A. C. Eringen, Accademic Press, New York 1976.

[6] M. E. GURTIN, *Variational principles for linear elastodynamics*, Arch. Rational Mech. Anal. **16** (1964), 34-50.

[7] D. IESAN, *Incremental equations in thermoelasticity*, J. Thermal Stresses **3** (1980), 41-56.

[8] D. IESAN, *Initially stressed thermoelasticity dielectrics*, J. Thermal Stresses **11** (1988), 421-440.

[9] D. IESAN, *On some theorems in thermoelastodynamics*, Rev. Roum. Sci. Techn. 34 (1989), 101-111.

[10] A. G. MAUGIN, *Continuum Mechanics of Electronagnetic Solids*, North-Holland, Amsterdam 1987.

[11] S. G. MIKHLIN, *Variational Methods*, Math. Phys., Macmillan, New York 1964.

[12] W. NOWACKI, *Mathematical models of phenomenological piezoelectricity*, New Problems in Mechanics of Continua, ed. O. Brulin and R. K. T. Hsieh, Univ. Waterloo Press, Ontario 1983.

[13] J. W. NUNZIATO and S. C. COWIN, *A nonlinear theory of elastic materials with voids*, Arch. Rational Mech. Anal. 72 (1979), 175-201.

[14] H. PARKUS, *Magneto-thermoelasticity*, CISM Lecture Notes, **118**, Springer, Berlin 1972.

[15] R. TOUPIN, *A dynamical theory of elastic dielectric*, Internat. J. Engrg. Sci. **1** (1963), 101-126.

[16] C. TRUESDELL and R. TOUPIN, *The Classical Field Theories*, Handbuch der Physik 3, ed. S. Flugge, Springer, Berlin 1960.

Sommario

In questo lavoro consideriamo il problema della termoelasticità incrementale per materiali piezoelettrici porosi. Usando tecniche recenti vengono stabilite proprietà di reciprocità ed una caratterizzazione variazionale.

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