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Lattice points under a hyperbola (**)

1 - Introduction

Let r, k be fixed integers greater than unity. A natural number n is *r-free* when in the canonical decomposition $n = p_1^{a_1} \dots p_t^{a_t}$ we have $\alpha_i \leq r - 1$. A natural number n is *k-full* when in the canonical decomposition we have $\alpha_i \geq k$.

Let Q_r denote the set of all r-free integers and $q_r(n)$ denote the characteristic function of the set Q_r, i.e.

$$q_r(n) = \begin{matrix} 1 & \text{if } n \in Q_r \\ 0 & \text{if } n \notin Q_r \end{matrix}$$

Then, we have
$$\sum_{n=1}^{\infty} q_r(n)\, n^{-s} = \frac{\zeta(s)}{\zeta(rs)} \qquad \text{Re } s > 1 .$$

where ζ denotes the Riemann zeta function.

Let $G(k)$ denote the set of all k-full integers. We denote by $f_k(n)$ the characteristic function of the set $G(k)$ and we consider the series

$$F_k(s) = \sum_{n=1}^{\infty} f_k(n)\, n^{-s} .$$

For Re $s > \dfrac{1}{k}$ we have

$$F_k(s) = \prod_{\mu=k}^{2k-1} \zeta(\mu s) \frac{\phi_k(s)}{\zeta((2k+2)\, s)}$$

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where $\phi_2(s) = 1$ and $\phi_k(s)$ has a Dirichlet series with the abscisa of absolute convergence $\frac{1}{2k+3}$ if $k > 2$.

In this paper we study the number of lattice points (u, v), $u > 0$, $v > 0$, under the hyperbola $uv = x$, such that one of the coordinates is r-free and the other is k-full. Here $\gamma_{r,k}(n)$ denotes the arithmetical function, whose associated Dirichlet series is

$$\sum_{n=1}^{\infty} \frac{\gamma_{r,k}(n)}{n^s} = \zeta(s) \prod_{\mu=k}^{2k-1} \zeta(\mu s) \frac{\phi_k(s)}{\zeta(rs)\,\zeta((2k+2)s)} \qquad \operatorname{Re} s = \sigma > 1 . \tag{1.1}$$

If $2 \leqslant r < k$ then

$$\gamma_{r,k}(n) = \sum_{uv=n} q_r(u) f_k(v) . \tag{1.2}$$

When $k \leqslant r \leqslant 2k-1$, then (1.1) becomes

$$\sum_{n=1}^{\infty} \frac{\gamma_{r,k}(n)}{n^s} = \prod_{\substack{\mu=k \\ \mu \neq r}}^{2k-1} \zeta(\mu s) \frac{\zeta(s)\phi_k(s)}{\zeta((2k+2)s)} . \tag{1.3}$$

Moreover

$$\frac{\phi_k(s)}{\zeta((2k+2)s)} = \sum_{n=1}^{\infty} \frac{g_k(n)}{n^s} = G_k(s) \tag{1.4}$$

is a Dirichlet series, absolutely convergent for $\operatorname{Re} s = \sigma > \frac{1}{2k+2}$ and $\gamma_{r,k}(n)$ can be expressed by the Dirichlet convolution

$$\gamma_{r,k}(n) = \sum_{n_1 n_2 = n} d(\boldsymbol{a};\, n_1) g_k(n_2) \tag{1.5}$$

where $\boldsymbol{a} = (1, k, \ldots, r-1, r+1, \ldots, 2k-1)$. The divisor function $d(\boldsymbol{a}, n)$, $\boldsymbol{a} = (a_1, \ldots, a_t)$ counts the number of ways of expressing n as the product $n = n_1^{a_1} \ldots n_t^{a_t}$.

Similar situations occur when $r \geqslant 2k$. We shall give estimates for the summatory function of $\gamma_{r,k}(n)$.

Theorem 1. *Let r, k be natural numbers, $2 \leqslant r < k$, then*

$$\sum_{n \leqslant x} \gamma_{r,k}(n) = \frac{F_k(1)}{\zeta(r)} x + O(x^{\frac{1}{r}}\, \delta(x^{\frac{2k-r-2}{2(r+1)(k-1)}})) \tag{1.6}$$

with $\delta(x) = \exp\{-A \log^{\frac{3}{5}} x\, (\log\log x)^{-\frac{1}{5}}\}$ where A is a positive constant.

Theorem 2. *Let be* $\boldsymbol{a} = (1, k, \ldots, 2k-1)$. *We suppose that*

$$\sum_{n \leqslant x} d(\boldsymbol{a}; n) = b_{1,x}\, x + \sum_{\mu=k}^{2k-1} b_{\mu,k}\, x^{\frac{1}{\mu}} + \Delta_k^*(x) \tag{1.7}$$

where

$$b_{1,k} = \prod_{\nu=k}^{2k-1} \zeta(\nu) \qquad b_{\mu,k} = \prod_{\substack{\nu=k \\ \nu \neq \mu}}^{2k-1} \zeta\left(\frac{\nu}{\mu}\right)\zeta\left(\frac{1}{\mu}\right)$$

and $\Delta_k^*(x) \ll x^{\eta_k} \log^{\beta_k} x$. *Then we have:*

1. *If* $\eta_k \geqslant 0$ *and* $r \geqslant 2k+2$ *we get*

$$\sum_{n \leqslant x} \gamma_{r,k}(n) = b_{1,k}\, G_{r,k}(1)\, x + \sum_{\mu=k}^{2k-1} b_{\mu,k}\, G_{r,k}\left(\frac{1}{\mu}\right) x^{\frac{1}{\mu}} + \Delta_k(x) \tag{1.8}$$

being

$$G_{r,k}\left(\frac{1}{\mu}\right) = \sum_{n=1}^{\infty} \frac{g_{r,k}(n)}{n^{\frac{1}{\mu}}} = \frac{\phi_k\left(\frac{1}{\mu}\right)}{\zeta\left(\frac{r}{\mu}\right)\zeta\left(\frac{2k+2}{\mu}\right)}$$

and

$$\Delta_k(x) \ll \begin{cases} x^{\eta_k} \log^{\beta_k+1} x & \textit{if } \eta_k = (2k+2)^{-1} \\ x^{\eta_k} \log^{\beta_k} x & \textit{if } \eta_k > (2k+2)^{-1} \\ x^{\frac{1}{2k+2}} \log^{\beta_k} x & \textit{if } \eta_k < (2k+2)^{-1}. \end{cases}$$

2. *If* $r = 2k, 2k+1$ *we have again* (1.8) *with*

$$\Delta_k(x) \ll \begin{cases} x^{\eta_k} \log^{\beta_k+1} x & \textit{if } \eta_k = r^{-1} \\ x^{\eta_k} \log^{\beta_k} x & \textit{if } \eta_k > r^{-1} \\ x^{\frac{1}{r}} \log^{\beta_k} x & \textit{if } \eta_k < r^{-1}. \end{cases}$$

Theorem 3. *Let be* $\boldsymbol{a} = (1, k, \ldots, r-1, r+1, \ldots, 2k-1)$. *We assume*

$$\sum_{n \leqslant x} d(\boldsymbol{a}; n) = b_{1,k}^*\, x + \sum_{\substack{\mu=k \\ \mu \neq r}}^{2k-1} b_{\mu,k}^*\, x^{\frac{1}{\mu}} + O(x^{\eta_k} \log^{\beta_k} x) \tag{1.9}$$

with

$$b_{1,k}^* = \frac{1}{\zeta(r)} \prod_{\nu=k}^{2k-1} \zeta(\nu) \qquad b_{\mu,k}^* = \prod_{\substack{\nu=k \\ \nu \neq \mu}}^{2k-1} \frac{\zeta\left(\frac{\nu}{\mu}\right)\zeta\left(\frac{1}{\mu}\right)}{\zeta\left(\frac{r}{\mu}\right)}.$$

Then, if $\eta'_k \geqslant 0$, $\beta'_k \geqslant 0$, $k \leqslant r \leqslant 2k-1$ *we have*

$$\sum_{n \leqslant x} \gamma_{r,k}(n) = b^*_{1,k}\, G_k(1)\, x + \sum_{\substack{\mu = k \\ \mu \neq r}}^{2k-1} b^*_{\mu,k}\, G_k\Big(\frac{1}{\mu}\Big)\, x^{\frac{1}{\mu}} + \Delta'_k(x)$$

where $G_k(s)$ *is defined by* (1.4) *and*

$$\Delta'_k(x) \ll \begin{cases} x^{\eta_k} \log^{\beta_k+1} x & \text{if } \eta'_k = (2k+2)^{-1} \\ x^{\eta_k} \log^{\beta_k} x & \text{if } \eta'_k > (2k+2)^{-1} \\ x^{\frac{1}{2k+2}} \log^{\beta_k} x & \text{if } \eta'_k < (2k+2)^{-1}. \end{cases}$$

2 - Proof of Theorems

We know ([7], p. 278) that for the function $f_k(n)$ we have

$$\sum_{n \leqslant x} f_k(n) = \sum_{\mu = k}^{2k-1} c_{\mu,k}\, x^{\frac{1}{\mu}} + \Delta_k(x) \tag{2.1}$$

where
$$c_{\mu,k} = \prod_{\substack{\nu = k \\ \nu \neq \mu}}^{2k-1} \frac{\zeta(\frac{\nu}{\mu})}{\zeta(\frac{2k+2}{\mu})}\, \phi_k\Big(\frac{1}{\mu}\Big)$$

for $k \geqslant 3$, (when $k = 2$, $\phi_2(\frac{1}{\mu}) = 1$).

We put $\lambda_k = \inf\{\varrho_k \,|\, \Delta_k(x) \ll x^{\varrho_k}\}$. The investigation on powerful numbers (i.e. elements of $G(k)$) began in 1935 when P. Erdös and G. Szekeres [2] proved by elementary methods that $\varrho_k \leqslant (k+1)^{-1}$ for $k \geqslant 2$. This was improved in 1958 by P. Bateman and E. Grosswald [1], who proved that $\varrho_2 \leqslant \frac{1}{6}$, $\varrho_3 \leqslant \frac{7}{46}$, $\varrho_k \leqslant (k+2)^{-1}$ for $k \geqslant 2$ and $\varrho_k \leqslant \max\big(\frac{r}{k(r+2)}, \frac{1}{k+r+1}\big)$, $r = [\sqrt{2k}]$ $k \geqslant 4$. E. Krätzel [6], [7] proved that $\Delta_3(x) \ll x^{\frac{1}{8}}\delta(x)$, $\Delta_4(x) \ll x^{\frac{106}{913}+\varepsilon}$, $\Delta_5(x) \ll x^{\frac{65}{622}+\varepsilon}$, $\varrho_k = \frac{6k+9}{8k^2+26k+36}$, $6 \leqslant k \leqslant 8$, $\varrho_k \leqslant (k+H(k))^{-1}$, $k < \sqrt{\frac{8}{3}}H(k) < (1+\sqrt{\frac{7}{3}})k$ for sufficiently large k. For a historical outline of the development of the problem see E. Krätzel [7].

A. Ivić [3], [4] obtained sharp bounds for ϱ_3, ϱ_4, ϱ_5 and proved that

$\varrho_k \leqslant (2k)^{-1}$, $k > 2$ when the Lindelöf hypothesis $\zeta(\frac{1}{2} + it) \ll t^{\varepsilon}$ is assumed. The result $\varrho_k \leqslant \frac{1}{2}k$, $8 \leqslant k \leqslant 10$ is proved by Ivić in [5].

Now, we prove two lemmas which are related with r-free and k-full integers.

Lemma 1. *Let be $k \geqslant 2$ and $\Delta_k(x)$ defined in (2.1). If $\Delta_k(x) \ll x^{\varrho_k}$, $\varrho_k < (2k-1)^{-1}$, then*

$$\sum_{n \leqslant x} \frac{f_k(n)}{n^s} = \sum_{v=k}^{2k-1} \frac{c_{vk}}{1-vs} x^{\frac{1}{v}-s} + F_k(s) + O(x^{\varrho_k - s}).$$

for $\sigma > \varrho_k$.

Proof. By partial summation and (2.1) we obtain

$$\sum_{n\leqslant x} \frac{f_k(n)}{n^s} = \sum_{v=k}^{2k-1} \frac{c_{vk}}{1-vs} x^{\frac{1}{v}-s} - \sum_{v=k}^{2k-1} \frac{vs}{1-vs} c_{vk} + s\int_1^{\infty} \frac{\Delta_k(t)}{t^{s+1}} dt + O(x^{\varrho_k - s}). \tag{2.2}$$

Assuming that $\operatorname{Re} s > 1$ we have, as $x \to \infty$

$$F_k(s) = -\sum_{v=k}^{2k-1} \frac{vs}{1-vs} c_{vk} + s\int_1^{\infty} \frac{\Delta_k(t)}{t^{s+1}} dt. \tag{2.3}$$

We observe that the integral of (2.3) is an analytic function for $\operatorname{Re} s = \sigma > \varrho_k$. Then by analytic continuation, formula (2.3) holds also for $\operatorname{Re} s = \sigma > \varrho_k$. From (2.2), (2.3) we derive Lemma 1.

Lemma 2. *Let be $\alpha > 0$, $\alpha \neq 1$ and $r \geqslant 2$. Then we have*

$$\sum_{n \leqslant x} \frac{q_r(n)}{n^{\alpha}} = \frac{\zeta(\alpha)}{\zeta(r\alpha)} + \frac{1}{\zeta(r)} \frac{x^{1-\alpha}}{1-\alpha} + O(x^{\frac{1}{r}-\alpha}\delta(x)).$$

If $\alpha = 1$ and $r \geqslant 2$, then

$$\sum_{n \leqslant x} \frac{q_r(n)}{n} = \frac{\gamma + \log x}{\zeta(r)} + r\frac{\zeta'(r)}{\zeta^2(r)} + O(x^{\frac{1}{r}-1}\delta(x))$$

being γ the Euler constant and $\delta(x)$ is the function defined in Theorem 1.

Proof. (First part). By the definition of $q_r(n)$ we have

$$S = \sum_{n \leqslant x} \frac{q_r(n)}{n^{\alpha}} = \sum_{dm^r \leqslant x} \frac{\mu(m)}{(dm^r)^{\alpha}}.$$

Let be $z = x^{\frac{1}{r}}$, and $0 < \varrho = \varrho(x) < 1$ which will be chosen later. If $dm^r \leqslant x$, then

$m > \varrho z$ and $d > \varrho^{-r}$ cannot occur at the same time, so we divide the sum S writing

$$S = \sum_{\substack{dm^r \leq x \\ m \leq \varrho z}} \frac{\mu(m)}{m^{ra} d^a} + \sum_{\substack{dm^r \leq x \\ d \leq \varrho^{-r}}} \frac{\mu(m)}{m^{ra} d^a} - \sum_{\substack{m \leq z\varrho \\ d \leq \varrho^{-r}}} \frac{\mu(m)}{m^{ra} d^a} = S_1 + S_2 - S_3 .$$

We study each S_i separately.

$$S_1 = \sum_{m \leq z\varrho} \frac{\mu(m)}{m^r} \sum_{d \leq xm^{-r}} \frac{1}{d^a} = \frac{x^{1-a}}{1-a} \sum_{m \leq z\varrho} \frac{\mu(m)}{m^r} + \zeta(a) \sum_{m \leq z\varrho} \frac{\mu(m)}{m^{ra}} + O(x^{-a} z\varrho)$$

$$= \frac{1}{\zeta(r)} \frac{x^{1-a}}{1-a} + \frac{\zeta(a)}{\zeta(ra)} + O(x^{1-a} \frac{\delta(\varrho z)}{(\varrho z)^{r-1}}) + O(\frac{\delta(\varrho z)}{(z\varrho)^{ra-1}}) + O(x^{-a} \varrho z).$$

For S_2 we have

$$S_2 = \sum_{d \leq \varrho^{-r}} \frac{1}{d^a} \sum_{m \leq (\frac{x}{d})^{\frac{1}{r}}} \frac{\mu(m)}{m^{ra}}$$

$$= \frac{1}{\zeta(ra)} \sum_{d \leq \varrho^{-r}} \frac{1}{d^a} + O(\frac{\delta(\varrho z)}{x^{(ra-1)r^{-1}}} \sum_{d \leq \varrho^{-r}} \frac{1}{d^{a-(ra-1)r^{-1}}})$$

As $\sqrt[r]{\frac{x}{d}} \geq \varrho z$, then $\delta((\frac{x}{d})^{\frac{1}{r}}) \leq \delta(\varrho z)$ and

$$S_2 = \frac{1}{\zeta(ra)} \sum_{d \leq \varrho^{-r}} \frac{1}{d^a} + O(\frac{\delta(\varrho z)}{x^{a-\frac{1}{r}}} \varrho^{-r(1-\frac{1}{r})}).$$

In the same form we deduce for the sum S_3 the asymptotic formula

$$S_3 = \frac{1}{\zeta(ra)} \sum_{d \leq \varrho^{-r}} \frac{1}{d^a} + O(\frac{\delta(\varrho z)}{(\varrho z)^{ra-1}}).$$

Then

$$S = \frac{\zeta(a)}{\zeta(ra)} + \frac{1}{\zeta(r)} \frac{x^{1-a}}{1-a} + O(x^{1-a} \delta(\varrho z)(\varrho z)^{1-r})$$

$$+ O(\delta(\varrho z)(\varrho z)^{1-ra}) + O(x^{-a} \varrho z) + O(x^{\frac{1}{r}-a} \delta(\varrho z) \varrho^{1-r}).$$

We take $\varrho = \varrho(x) = (\delta(x^{\frac{1}{2r}}))^{\frac{1}{r}}$, then

$$\exp\{-\frac{A}{r^{\frac{8}{5}}} \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\} \leq \varrho(x) \leq \exp\{-\frac{A}{2r^{\frac{8}{5}}} \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\}$$

and as $\varrho \geq x^{-\frac{1}{2r}}$ then $\varrho z \geq x^{\frac{1}{2r}}$ and $\delta(\varrho z) \leq \delta(x^{\frac{1}{2r}}) = \varrho^r$, hence we deduce that the O-terms are like $x^{\frac{1}{r}-a} \delta(x)$ and the first part of Lemma 2 is proved.

To prove the second part, if $\alpha = 1$ and $r \geqslant 2$, then we divide the sum $S = \sum_{n \leqslant x} \frac{q_r(n)}{n}$ as before, but in this case we have

$$S_1 = \sum_{m \leqslant \varrho z} \frac{\mu(m)}{m^r} \sum_{d \leqslant \frac{x}{m^r}} \frac{1}{d}.$$

Since we know that $\sum_{d \leqslant x} d^{-1} = \log x + \gamma + O(\frac{1}{x})$, the second part of Lemma 2 is proved.

Moreover, if $\alpha = 0$, $r \geqslant 2$, then (A. Ivic [4], p. 394)

$$\sum_{n \leqslant x} q_r(n) = \frac{x}{\zeta(r)} + O(x^{\frac{1}{r}} \delta(x)).$$

Proof of Theorem 1. Let

$$S = \sum_{mn \leqslant x} q_r(n) f_k(m) = \sum_{n \leqslant x} q_r(n) \sum_{m \leqslant \frac{x}{n}} f_k(m) = S_1 + S_2$$

where for any $Y \leqslant x$, by Lemmas 1 and 2

$$S_1 = \sum_{m \leqslant Y} f_k(m) \sum_{n \leqslant \frac{x}{m}} q_r(n) = \sum_{m \leqslant Y} f_k(m) \{ \frac{x}{m} \frac{1}{\zeta(r)} + O((\frac{x}{m})^{\frac{1}{r}} \delta(\frac{x}{m}))\}$$

$$= \frac{x}{\zeta(r)} \sum_{m \leqslant Y} \frac{f_k(m)}{m} + O(x^{\frac{1}{r}} \delta(\frac{x}{Y}) \sum_{m \leqslant Y} \frac{f_k(m)}{m^{\frac{1}{r}}})$$

$$= \frac{F_k(1)}{\zeta(r)} x + \frac{x}{\zeta(r)} \sum_{v=k}^{2k-1} \frac{\gamma_{vk}}{1-v} Y^{\frac{1}{v}-1} + O(xY^{\varrho_k - 1}) + O(x^{\frac{1}{r}} \delta(\frac{x}{Y})).$$

Now we study the sum

$$S_2 = \sum_{Y < m \leqslant x} f_k(m) \sum_{n \leqslant \frac{x}{m}} q_r(n) = \sum_{n \leqslant \frac{x}{Y}} q_r(n) \sum_{Y < m \leqslant \frac{x}{n}} f_k(m).$$

We know that
$$\sum_{m \leqslant x} f_k(m) = \sum_{v=k}^{2k-1} c_{vk} x^{\frac{1}{v}} + O(x^{\varrho_k}).$$

Then
$$S_2 = \sum_{v=k}^{2k-1} c_{vk} \{ x^{\frac{1}{v}} \sum_{n \leqslant \frac{x}{Y}} \frac{q_r(n)}{n^{\frac{1}{v}}} - Y^{\frac{1}{v}} \sum_{n \leqslant \frac{x}{Y}} q_r(n)\}$$

$$+ O(x^{\varrho_k} \sum_{n \leqslant \frac{x}{Y}} \frac{q_r(n)}{n^{\varrho_k}}) + O(Y^{\varrho_k} \sum_{n \leqslant \frac{x}{Y}} q_r(n)).$$

By Lemma 2 we obtain

$$\sum_{n\leqslant \frac{x}{Y}} \frac{q_r(n)}{n^{\frac{1}{v}}} = \frac{1}{\zeta(r)} \frac{(\frac{x}{Y})^{1-\frac{1}{v}}}{1-\frac{1}{v}} + \frac{\zeta(\frac{1}{v})}{\zeta(\frac{r}{v})} + O((\frac{x}{Y})^{\frac{1}{r}-\frac{1}{v}}\delta(\frac{x}{Y}))$$

and

$$\sum_{n\leqslant \frac{x}{Y}} \frac{q_r(n)}{n^{\varrho_k}} = O((\frac{x}{Y})^{1-\varrho_k}).$$

Then

$$S_2 = \frac{x}{\zeta(r)} \sum_{v=k}^{2k-1} \frac{c_{vk}}{1-\frac{1}{v}} Y^{\frac{1}{v}-1} + \sum_{v=k}^{2k-1} c_{vk}\, x^{\frac{1}{v}} \frac{\zeta(\frac{1}{v})}{\zeta(\frac{r}{v})} - \frac{x}{\zeta(r)} \sum_{v=k}^{2k-1} c_{vk}\, Y^{\frac{1}{v}-1}$$

$$+O(x^{\frac{1}{r}}\delta(\frac{x}{Y}) \sum_{v=k}^{2k-1} c_{vk} Y^{\frac{1}{v}-\frac{1}{r}}) + O(x Y^{\varrho_k - 1}).$$

Therefore

$$S = S_1 + S_2 = \frac{F_k(1)}{\zeta(r)} x + \sum_{v=k}^{2k-1} c_{vk}\, x^{\frac{1}{v}} \frac{\zeta(\frac{1}{v})}{\zeta(\frac{r}{v})} + O(x Y^{\varrho_k - 1}) + O(x^{\frac{1}{r}} \delta(\frac{x}{Y})).$$

Now, we take $Y = x^{\alpha}$ with $\alpha = \frac{r}{r+1}\frac{2k-1}{2k-2}$, then

$$S = \frac{F_k(1)}{\zeta(r)} x + O(x^{\frac{1}{r}} \delta(x^{\frac{2k-r-2}{2(r+1)(k-1)}})).$$

Proof of Theorem 2. Let be $r \geqslant 2k+2$. We see that the function $\gamma_{r,k}(n)$ can be written as Dirichlet convolution

$$\gamma_{r,k}(n) = \sum_{n_1 n_2 = n} d(\boldsymbol{a}; n_1)\, g_{r,k}(n_2) \tag{2.4}$$

being $\boldsymbol{a} = (1, k, \dots, 2k-1)$ and $g_{r,k}(n)$ the arithmetic function such that

$$G_{r,k}(s) = \sum_{n=1}^{\infty} \frac{g_{r,k}(n)}{n^s} = \frac{\phi_k(s)}{\zeta(rs)\,\zeta((2k+2)s)}$$

in $\operatorname{Re} s > \max\{(2k+2)^{-1}, r^{-1}\} = (2k+2)^{-1}$.

From (2.4) and (1.7) we have

$$\sum_{n\leq x}\gamma_{r,k}(n)=\sum_{m\leq x}g_{r,k}(m)\{b_{1,k}(\frac{x}{m})+\sum_{\mu=k}^{2k-1}b_{\mu,k}(\frac{x}{m})^{\frac{1}{\mu}}+\Delta_k^*(\frac{x}{m})\}$$

$$=b_{1,k}\,x\sum_{m\leq x}\frac{g_{r,k}(m)}{m}+\sum_{\mu=k}^{2k-1}b_{\mu,k}x^{\frac{1}{\mu}}\sum_{m\leq x}\frac{g_{r,k}(m)}{m^{\frac{1}{\mu}}}+\sum_{m\leq x}g_{r,k}(m)\Delta_k^*(\frac{x}{m}).$$

For the partial sum of $g_{r,k}(n)$ we have $\sum_{n\leq x}g_{r,k}(n)\ll x^{\frac{1}{2k+2}}$; then we deduce

$$\sum_{n\leq x}\frac{g_{r,k}(n)}{n^{\frac{1}{\mu}}}=G_{r,k}(\frac{1}{\mu})+O(x^{(2k-2)^{-1}-\mu^{-1}}).$$

Moreover $$\sum_{m\leq x}g_{r,k}(m)\Delta_k^*(\frac{x}{m})\ll x^{\eta_k}\log^{\beta_k}x\sum_{m\leq x}\frac{g_{r,k}(m)}{m^{\eta_k}}.$$

As we know that
$$\sum_{m\leq x}\frac{g_{r,k}(m)}{m^{\eta_k}}=\begin{cases}O(\log x) & \text{if } \eta_k=(2k+2)^{-1}\\ O(1) & \text{if } \eta_k>(2k+2)^{-1}\\ O(x^{(2k+2)^{-1}-\eta_k}) & \text{if } \eta_k<(2k+2)^{-1}\end{cases}$$

so, we obtain

$$\sum_{n\leq x}\gamma_{r,k}(n)=b_{1,k}\,G_{r,k}(1)x+\sum_{\mu=k}^{2k-1}b_{\mu,k}\,G_{r,k}(\frac{1}{\mu})x^{\frac{1}{\mu}}+\Delta_k(x)$$

where
$$\Delta_k(x)\ll\begin{cases}x^{\eta_k}\log^{\beta_k+1}x & \text{if } \eta_k=(2k+2)^{-1}\\ x^{\eta_k}\log^{\beta_k}x & \text{if } \eta_k>(2k+2)^{-1}\\ x^{\frac{1}{2k+2}}\log^{\beta_k}x & \text{if } \eta_k<(2k+2)^{-1}.\end{cases}$$

When $r=2k,\ 2k+1$, we have $\operatorname{Re}s=\sigma>\frac{1}{r}$ and then $\sum_{n\leq x}g_{r,k}(n)\ll x^{\frac{1}{r}}$. Therefore, as in the preceding case, Theorem 2 is proved.

Proof of Theorem 3. From (1.5) and (1.8) we can deduce

$$\sum_{n\leq x}\gamma_{r,k}(n)=\sum_{\substack{\mu=k\\ \mu=1,\,\mu\neq r}}^{2k-1}b_{\mu,k}^*\,x^{\frac{1}{\mu}}\sum_{m\leq x}\frac{g_k(m)}{m^{\frac{1}{\mu}}}+\sum_{m\leq x}g_k(m)\Delta_k^*(\frac{x}{m})$$

where $\Delta_k^*(x) = O(x^{\eta_k}\log^{\beta_k}x)$. As before

$$\sum_{m \leq x} \frac{g_k(m)}{m^{\frac{1}{\mu}}} = G_k\left(\frac{1}{\mu}\right) + O\left(x^{\frac{1}{2k+2} - \frac{1}{\mu}}\right)$$

and
$$\sum_{m \leq x} g_k(m)\Delta_k^*\left(\frac{x}{m}\right) \ll x^{\eta_k}\log^{\beta_k}x \sum_{m \leq x} \frac{g_k(m)}{m^{\eta_k}}$$

from here Theorem 3 is deduced.

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Sommario

Oggetto di questo lavoro è ottenere una valutazione del numero dei punti di coordinate intere positive sotto un'iperbole, tale che una delle coordinate sia r-libera e l'altra sia k-piena.

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