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New Neumann and Kapteyn series of confluent hypergeometric functions and modified Bessel functions (**)

1. - Introduction

Neumann and Kapteyn series are important in the theory of Bessel functions. In this note, this concept is generalised to a class of confluent hypergeometric functions. New expansions involving Bessel functions are then formed.

In [3], Lemma 1, put $C(\mu) = \Gamma(a+\mu)(\frac{1}{2}x)^{2\mu}$. Hence

$$(1.1)\qquad \begin{aligned} X_\nu = X_\nu(a; x) &= \sum \frac{(-1)^r \Gamma(a+\frac{1}{2}\nu+r)(\frac{1}{2}x)^{\nu+2r}}{r!\,\Gamma(\nu+r+1)} \\ &= (\frac{1}{2}x)^\nu \frac{\Gamma(a+\frac{1}{2}\nu)}{\Gamma(1+\nu)}\, {}_1F_1(a+\frac{1}{2}\nu; 1+\nu; -\frac{x^2}{4}). \end{aligned}$$

Unless otherwise indicated, it is taken that any indices of summation run over all the non-negative integers, and that any values of parameters for which any expression does not make sense are tacitly excluded.

This function is interesting in that it is a confluent hypergeometric function capable of expansions of Neumann and Kapteyn type.

Furthermore, by suitable specialisation, new expansions including Bessel functions may be deduced. The reader is referred to [3] and the references therein for the general background.

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2. - A Neumann expansion

By appealing to Lemma 1 of [3], it will be seen that

$$\Gamma(a+\frac{1}{2}\nu)\,(\frac{1}{2}x)^{\nu}=\sum\frac{(\nu+2k)\,\Gamma(\nu+k)}{k!}\,X_{\nu+2k}(a;\,x) \tag{2.1}$$

provided that this series converges. In order to investigate this, we note that from (1.1), for sufficiently large values of the integer $k>0$, we have

$$|X_{\nu+2k}(a;\,x)|\leqslant\left|\frac{1}{2}x\right|^{\nu+2k}\left|\frac{\Gamma(a+\frac{1}{2}\nu+k)}{\Gamma(1+\nu+2k)}\,\exp(-\frac{1}{4}x^2)\right|$$

and the series on the right of (2.1) converges with

$$\begin{aligned}&\sum\frac{(\nu+2k)\,\Gamma(\nu+k)\,\Gamma(a+\frac{1}{2}\nu+k)}{k!\;\Gamma(1+\nu+2k)}\,(\frac{1}{2}x)^{2k}\\&=\Gamma(a+\frac{1}{2}\nu)\,{}_1F_2(a+\frac{1}{2}\nu;\,\frac{1}{2}\nu,\,\frac{1}{2}\nu+\frac{1}{2};\,\frac{x^2}{16}).\end{aligned} \tag{2.2}$$

The series (2.2) converges for all finite values of x, so that the Neumann expansion (2.1) converges likewise.

3. - A Kapteyn expansion

Lemma 2 of [3] gives a formal expansion of Kapteyn. With the above values of $C(\mu)$, we see that

$$Y_\nu=Y_\nu(a;\,x)=X_\nu(a;\,\nu x)=\sum\frac{(-1)^r\,\Gamma(a+\frac{1}{2}\nu+r)(\frac{1}{2}\nu x)^{\nu+2r}}{r!\;\Gamma(1+\nu+r)}$$

and

$$\Gamma(a+\frac{1}{2}\nu)(\frac{1}{2}x)^{\nu}=\nu^2\sum\frac{\Gamma(\nu+k)}{(\nu+2k)^{\nu+1}k!}\,X_{\nu+2k}(a;(\nu+2k)\,x)\,. \tag{3.1}$$

As in the corresponding case arising in the theory of Bessel functions, the investigation of the convergence of the expansion (3.1) is not so straightforward as in the case of (2.1).

In order to obtain an estimate of $X_{\nu+2k}(a;(\nu+2k)\,x)$ for large values of k, we consider the corresponding asymptotic expansion of its representation as a confluent hypergeometric function for large values of its argument using an expression given by [2], Vol. 1, p. 278.

Hence, we see that:

$$
\begin{aligned}
&[\tfrac{1}{2}(\nu+2k)x]^{-\nu-2k}\,\frac{\Gamma(1+\nu+2k)}{\Gamma(a+\frac{1}{2}\nu+k)}\,X_{\nu+2k}(a;(\nu+2k)x) \\
&\quad = {}_1F_1(a+\tfrac{1}{2}\nu+k;\,1+\nu+2k;\,-\tfrac{1}{4}(\nu+2k)^2x) \\
&\quad \sim \frac{\Gamma(1+\nu+2k)}{\Gamma(1-a+\frac{1}{2}\nu+k)}\,[\tfrac{1}{4}(\nu+2k)^2x^2]^{-a-\frac{1}{2}\nu-k} \\
&\quad + \frac{\Gamma(1+\nu+2k)}{\Gamma(a+\frac{1}{2}\nu+k)}\,\exp[\tfrac{1}{4}(\nu+2k)^2x^2][-\tfrac{1}{4}(\nu+2k)^2x^2]^{a-\frac{1}{2}\nu-k-1}.
\end{aligned}
\tag{3.2}
$$

For $|\arg x| < \frac{1}{4}\pi$, the first term on the right of (3.2) dominates, otherwise, the second term, involving the exponential function, is dominant.

For sufficiently large values of k, $k = K$ say, with $|\arg x| < \frac{1}{4}\pi$,

$$\sum \frac{\Gamma(\nu+k)}{(\nu+2k)^{\nu+1}k!}\,X_{\nu+2k}(a;(\nu+2k)x)$$

converges with

$$\sum \frac{\Gamma(\nu+k)\,\Gamma(a+\frac{1}{2}\nu+k)(\nu+2k)^{-\nu-2a-1}}{\Gamma(1-a+\frac{1}{2}\nu+k)\,k!}. \tag{3.3}$$

If the k^{th} term of the series (3.3) is denoted by T_k, then after some algebra, we may write

$$
\begin{aligned}
\frac{T_k}{T_{k+1}} &= [1+\frac{2-2a-\nu}{k}+O(\frac{1}{k^2})][1+\frac{\nu+2a+1}{k}+O(\frac{1}{k^2})] \\
&= 1+\frac{3}{k}+O(\frac{1}{k^2}).
\end{aligned}
$$

By appealing to the tests given in [1] p. 40, the expansions (3.1) and (3.3) converge for all values of a and ν, provided that $|\arg x| < \frac{1}{4}\pi$. Outside of this range of $|\arg x|$ the second term on the right of (3.2) must be taken into account, and the expansions are then divergent.

4. - New expansions involving the modified Bessel function

If we consider the special case of the previous results in which $\alpha = \frac{1}{2}$, the function $X_\nu(\frac{1}{2}; x)$ can be expressed as a modified Bessel function by employing the representation

$$(4.1) \qquad I_\nu(z) = (\frac{1}{2} z)^\nu e^{-z} \frac{{}_1F_1(\nu + \frac{1}{2}; 2\nu + 1; 2z)}{\Gamma(\nu + 1)}$$

given in [2], 2, p. 5. It then follows from (4.1) and (1.1) that

$$(4.2) \qquad X_\nu(\frac{1}{2}; x) = \pi^{\frac{1}{2}} \exp(-\frac{x^2}{8}) I_{\frac{1}{2}\nu}(\frac{x^2}{8}).$$

If (4.2) is inserted into (2.1) and (3.1), we have, respectively, the interesting results

$$(4.3) \qquad \Gamma(\frac{1}{2} + \frac{1}{2}\nu)(\frac{1}{2}x)^\nu = \pi^{\frac{1}{2}} \exp(-\frac{x^2}{8}) \sum \frac{(\nu + 2k)\Gamma(\nu + k)}{k!} I_{\frac{1}{2}\nu + k}(\frac{x^2}{8})$$

$$(4.4) \qquad \Gamma(\frac{1}{2} + \frac{1}{2}\nu)(\frac{1}{2}x)^\nu = \nu^2 \pi^{\frac{1}{2}} \sum \frac{\Gamma(\nu + k)}{(\nu + 2k)^{\nu + 1} k!} \exp[-(\nu + 2k)^2 \frac{x^2}{8}] I_{\frac{1}{2}\nu + k}((\nu + 2k)^2 \frac{x^2}{8}).$$

The expressions (4.3) and (4.4) do not seem to have been previously mentioned in the literature.

References

[1] T. J. Bromwich, *Infinite Series*, Macmillan, London 1931.

[2] A. Erdélyi, *Higher Transcendental Functions*, McGraw Hill, New York 1953.

[3] H. Exton, *On certain series which generalise the Neumann and Kapteyn series*, Riv. Mat. Univ. Parma 13 (1987), 275-279.

Sommario

Sono indicati alcuni sviluppi in serie del tipo di Neumann e Kapteyn, che fanno intervenire funzioni ipergeometriche confluenti. Opportune specializzazioni permettono di ottenere nuovi sviluppi dello stesso tipo, che fanno intervenire funzioni di Bessel modificate

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