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On conformally flat totally real submanifolds (**)

1 - Introduction

Let $\widetilde{M}^{2m}$ be a *2m-dimensional Kähler manifold* with Riemannian metric g, complex structure J and Riemannian connection $\widetilde{\nabla}$. The curvature tensor, the Ricci tensor and the scalar curvature of $\widetilde{M}^{2m}$ are denoted by $\widetilde{R}$, $\widetilde{S}$, $\widetilde{\tau}$, respectively. The Bochner curvature tensor $\widetilde{B}$ of $\widetilde{M}$ is given by

$$\widetilde{B}=\widetilde{R}-\frac{1}{2(n+2)}(\varphi+\psi)(\widetilde{S})+\frac{\widetilde{\tau}}{8(n+1)(n+2)}(\varphi+\psi)(g)$$

where the operators φ and ψ are defined by

$$\varphi(Q)(x,y,z,u)=g(x,u)\,Q(y,z)-g(x,z)\,Q(y,u)+g(y,z)\,Q(x,u)-g(y,u)\,Q(x,z)$$

$$\psi(Q)(x,y,z,u)=g(x,Ju)\,Q(y,Jz)-g(x,Jz)\,Q(y,Ju)-2g(x,Jy)\,Q(z,Ju)$$

$$+g(y,Jz)\,Q(x,Ju)-g(y,Ju)\,Q(x,Jz)-2g(z,Ju)\,Q(x,Jy)$$

for a tensor field Q of type $(0,2)$, and x, y, z, u are vector fields of $\widetilde{M}^{2m}$.

Let M be a *submanifold of* $\widetilde{M}$. The Gauss and the Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y=\nabla_X Y+\sigma(X,Y) \qquad \widetilde{\nabla}_X\xi=-A_\xi X+D_X\xi$$

for vector fields X, Y tangent to M and ξ normal to M, where ∇ is the Rieman-

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nian connection on M, D is the normal connection, σ is the second fundamental form of M and $A_\xi X$ is the tangential component of $\widetilde{\nabla}_X \xi$. It is well known that $g(\sigma(X, Y), \xi) = g(A_\xi X, Y)$. The mean curvature vector H is defined by $H = \frac{1}{n} \operatorname{tr} \sigma$. If $H = 0$, M is called minimal. In particular, if $\sigma = 0$, M is said to be totally geodesic. A normal vector field ξ is said to be parallel, if $D_X \xi = 0$ for each vector field X on M.

An n-dimensional submanifold M^n of $\widetilde{M}^{2m}$ is said to be a *totally real submanifold* of $\widetilde{M}^{2m}$, if for each point $p \in M^n$, $JT_pM^n \subset T_p(M^n)^\perp$. Then $n \leqslant m$. In the following we suppose that *M^n is a totally real submanifold of $\widetilde{M}^{2m}$* and $m = n$. In this case it is not difficult to find

$$\sigma(X, Y) = JA_{JX}Y = JA_{JY}X \tag{1.1}$$

$$D_X JY = J\nabla_X Y. \tag{1.2}$$

See e.g. [**10**].

Let $\widetilde{R}$ be the curvature tensor of $\widetilde{M}^{2n}$. Then using (1.1), the Gauss equation can be written as

$$\{\widetilde{R}(X, Y)Z\}^t = R(X, Y)Z - [A_{JX}, A_{JY}]Z$$

where t denotes the tangential component.

Let $\overline{\nabla}$ denote the connection of van der Waerden-Bortolotti. Then M is said to be a *parallel submanifold* of $\widetilde{M}$ if $\overline{\nabla}\sigma = 0$. More generally M is called a *semiparallel submanifold* of $\widetilde{M}$, if $\overline{R}(X, Y).\sigma = 0$, where

$$(\overline{R}(X, Y).\sigma)(Z, U) = R^\perp(X, Y)\sigma(Z, U) - \sigma(R(X, Y)Z, U) - \sigma(Z, R(X, Y)U)$$

$R^\perp$ being the curvature tensor of the normal connection D. The investigation of semiparallel submanifolds initiated with J. Deprez [**2**]. For a semiparallel manifold by (1.1) and (1.2) we obtain

$$R(X, Y)A_{JZ}U = A_{JZ}R(X, Y)U + A_{JU}R(X, Y)Z. \tag{1.3}$$

On the other hand the *submanifolds with semiparallel mean curvature vector* are defined by $R^\perp(X, Y)H = 0$ [**3**]. Note that the class of submanifolds with semiparallel mean curvature includes the semiparallel submanifolds and the submanifolds with parallel mean curvature vector.

Let S and τ denote the Ricci tensor and the scalar curvature of M, respectively. Then as it is well known for $n > 3$ M is *conformally flat*, if and only if the Weil conformal curvature tensor C of M vanishes, where

$$C = R - \frac{1}{n-2}\varphi(S) + \frac{\tau}{2(n-1)(n-2)}\varphi(g).$$

In Sections 2 and 3 we prove

Theorem 1. *Let M^n be a conformally flat totally real submanifold of a Kähler manifold $\widetilde{M}^{2n}$, $n > 3$. Assume also that the mean curvature vector of M^n is semiparallel. If M^n is not minimal at a point p, then, in a neighborhood of p, M^n is either flat or a product $M_1^{(n-1)}(c) \times I$, where $M_1^{n-1}(c)$ is an $n-1$-dimensional manifold of constant sectional curvature $c \neq 0$ and I is a segment.*

Theorem 2. *Let M^n be a conformally flat totally real semiparallel submanifold of a Kähler manifold $\widetilde{M}^{2n}$, $n > 3$. If M^n is not totally geodesic at a point p, then, in a neighborhood of p, M^n is flat or $M^n = M_1^{n-1}(c) \times I$.*

In Section 4 we deal with products of Kähler manifolds with vanishing Bochner curvature tensor.

2 - Proof of Theorem 1

First we prove

Proposition. *Let M^n be a conformally flat totally real submanifold of a Kähler manifold $\widetilde{M}^{2n}$, $n > 3$, such that the mean curvature vector H is semiparallel at a point p. If M^n is not minimal at p, it is quasi-Einstein at p with $S(JH_p, JH_p) = 0$.*

Proof. Let $\{e_i\}$ $i = 1, \ldots, n$ be an orthonormal basis of T_pM, such that $S_{1,1}(e_i) = \lambda_i e_i$ for $i = 1, \ldots, n$, where $S_{1,1}$ is the Ricci tensor of type $(1,1)$. From $C = 0$ and $R(e_i, e_j)JH = 0$ we obtain

$$(\lambda_i + \lambda_j - \frac{\tau}{n-1})g(e_j, JH) = 0. \tag{2.1}$$

If $g(e_j, JH) = 0$ for each j, M^n is minimal at p. Let e.g. $g(e_1, JH) \neq 0$. Then (2.1) implies

$$\lambda_1 + \lambda_i - \frac{\tau}{n-1} = 0 \tag{2.2}$$

for $i = 2, \dots n$. Hence $\lambda_i = \lambda_j$ for $i, j = 2, \dots, n$, i.e. M^n is *quasi-Einstein* at p. Moreover (2.2) implies $\lambda_1 = 0$. If there exists $i > 1$, such that $g(e_i, JH) \neq 0$, it follows that M^n is Einstein at p, with $\tau = 0$, so $S = 0$ at p. If $g(e_i, JH) = 0$ for $i = 2, \dots, n$ it follows that $(JH)_p$ is proportional to e_1, thus proving our assertion.

Now we can prove Theorem 1. Since H does not vanishes at p, then this holds also in a neighborhood of p. Then Theorem 1 follows from our Proposition and a theorem of Kurita, see [4].

If H has constant length, then M is minimal or H does not vanishes. Hence we have:

Corollary 1. *Let M^n be a conformally flat totally real submanifold of a Kähler manifold $\tilde{M}^{2n}$. Assume also that the mean curvature vector H of M be semiparallel and with constant length. Then one of the following holds:*

M^n is minimal

M^n is locally flat or a product $M_1^{n-1}(c) \times I$, $c \neq 0$.

In particular the result is true when H is parallel.

3 - Proof of Theorem 2

As in Section 2 let $\{e_i\}$ $i = 1, \dots, n$ be a basis of T_pM such that $S_{1,1}(e_i) = \lambda_i e_i$ for $i = 1, \dots, n$. Under the assumptions of Theorem 2 we prove some lemmas.

Lemma 1. *Let there exist $i \neq k$, such that $g(A_{Je_i} e_i, e_k) \neq 0$. Then $\lambda_i = \lambda_j$ for all $j \neq k$ and $\lambda_k = 0$.*

Proof. We put in (1.3) $X = e_j$, $Y = e_k$, $Z = U = e_i$ for $j \neq i, k$ and we obtain

$$(\lambda_j + \lambda_k - \frac{\tau}{n-1}) g(A_{Je_i} e_i, e_k) = 0$$

which implies

$$\lambda_j + \lambda_k - \frac{\tau}{n-1} = 0 . \tag{3.1}$$

Now we put in (1.3) $X = e_k$, $Y = Z = U = e_i$ and we find

$$(3.2) \qquad (\lambda_i + \lambda_k - \frac{\tau}{n-1})(2A_{Je_i}e_k + g(A_{Je_i}e_i, e_k)e_i - g(A_{Je_i}e_i, e_i(e_k) = 0$$

which implies

$$(3.3) \qquad \lambda_i + \lambda_k - \frac{\tau}{n-1} = 0.$$

From (3.1) and (3.3) it follows $\lambda_i = \lambda_j$. Then (3.1) implies $\lambda_k = 0$.

Lemma 2. *Let there exists i, such that $g(A_{Je_i}e_i, e_i) \neq 0$. Then $\lambda_i = 0$ and $\lambda_j = \lambda_k$ for all $j, k \neq i$.*

Proof. If we have

$$\lambda_i + \lambda_k - \frac{\tau}{n-1} = 0$$

for any $k = 1, \ldots n$ the assertion follows immediately. Let us assume that there exists a k such that

$$\lambda_i + \lambda_k - \frac{\tau}{n-1} \neq 0.$$

As in Lemma 1 we find (3.2) and hence we have

$$2A_{Je_i}e_k + g(A_{Je_i}e_i, e_k)e_i - g(A_{Je_i}e_i, e_i)e_k = 0$$

which implies $g(A_{Je_k}e_k, e_i) \neq 0$. Using Lemma 1 we obtain $\lambda_i = 0$ and $\lambda_j = \lambda_k$ for $j, k \neq i$.

Lemma 3. *Let M be minimal at p. Then M is totally geodesic at p or there exists k, such that $\lambda_k = 0$ and $\lambda_i = \lambda_j$ for $i, j \neq k$.*

Proof. If there exists i, such that $A_{Je_i}e_i \neq 0$, the assertion follows from Lemmas 1 and 2. So let $A_{Je_i}e_i = 0$ for any $i = 1, \ldots, n$. Suppose that M is not totally geodesic at p. Then $g(A_{Je_i}e_j, e_k) \neq 0$ for some $i \neq j \neq k \neq i$. We put in (1.3) $X = U = e_s$, $Y = e_j$, $Z = e_k$ for $s \neq j, k$ and we obtain

$$(\lambda_s + \lambda_j - \frac{\tau}{n-1})(A_{Je_j}e_k + g(A_{Je_s}e_k, e_j)e_s) = 0$$

which implies

$$\lambda_s + \lambda_j - \frac{\tau}{n-1} = 0$$

Analogously

$$\lambda_s + \lambda_k - \frac{\tau}{n-1} = 0 \qquad \lambda_t + \lambda_k - \frac{\tau}{n-1} = 0$$

for $t \neq i, k$. Hence it follows $\lambda_l = 0$ for any $l = 1, \dots, n$, which proves the Lemma.

Now we are in position to prove Theorem 2. If M is not minimal at p, Theorem 2 follows from Theorem 1. Let M be minimal at p. Then the assertion follows from Lemma 3 and [4].

4 - Submanifolds of Bochner flat Kähler products

Let $\tilde{M}^{2n}$ be a Kähler manifold with vanishing Bochner curvature tensor and constant holomorphic sectional curvature. Then $\tilde{M}^{2n}$ either has constant holomorphic sectional curvature or is locally a product of two Kähler manifolds of constant holomorphic sectional curvature μ and $-\mu$, respectively, $\mu > 0$, [5]. Totally real submanifolds of Kähler manifolds of constant holomorphic sectional curvature have been studied by many authors, see e.g. [1], [9], [10]. Now we consider the case of Kähler products with vanishing Bochner curvature tensor.

Theorem 3. *Let M^n be a totally real semiparallel submanifold with commutative second fundamental form and mean curvature vector of constant length of a Kähler product $\tilde{M}^{2k}(\mu) \times \tilde{M}^{2(n-k)}(-\mu)$, $\mu \neq 0$, $n > 3$, $k \geqslant n-k \geqslant 1$. Then M^n is a product $M^k(\frac{\mu}{4}) \times M^{n-k}$, where $M^k(\frac{\mu}{4})$ is a manifold of constant curvature $\frac{\mu}{4}$ and is totally geodesic in $\tilde{M}^{2k}(\mu)$. If in addition $n-k > 1$, then M^{n-k} is totally geodesic in $\tilde{M}^{2(n-k)}(\mu)$ and has constant sectional curvature $-\frac{\mu}{4}$.*

Proof. Since M^n has commutative second fundamental form (i.e.

$A_\xi A_\eta = A_\eta A_\xi$, $\forall \xi, \eta \in TM^\perp$, [10], p. 29), the Gauss equation implies

$$\tilde{R}(X, Y, Z, U) = R(X, Y, Z, U)$$

for arbitrary vectors X, Y, Z, U in T_pM. Let X, Y, Z, U be orthogonal. Since $\tilde{B} = 0$ we obtain $R(X, Y, Z, U) = 0$ and hence M^n is conformally flat, see e.g. [7] p. 307. If M^n is totally geodesic, it is straightforward that it is a product $M^k(\frac{\mu}{4}) \times M^{n-k}(-\frac{\mu}{4})$, where $M^k(\frac{\mu}{4})$, resp. $M^{n-k}(-\frac{\mu}{4})$, is totally geodesic in $\tilde{M}^{2k}(\mu)$, resp. $\tilde{M}^{2(n-k)}(-\mu)$.

Let M^n is not totally geodesic. According to Theorem 2 it is locally flat or a product $M_1^{n-1}(c) \times I$. As it is easily seen, if M^n is flat, it follows $\mu = 0$, which is not our case. So M^n is locally $M_1^{n-1}(c) \times I$. Denote by π_1 and π_2 the projections of $\tilde{M}^{2k}(\mu) \times \tilde{M}^{2(n-k)}(-\mu)$, onto $\tilde{M}^{2k}(\mu)$ and $\tilde{M}^{2(n-k)}(-\mu)$, respectively. The induced differentials will be denoted also by π_1 and π_2. Let $F = \pi_1 - \pi_2$. Then we have [6], [8]

$$\begin{aligned}\tilde{R}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) = \frac{\mu}{8}\{ & g(F\tilde{x}, \tilde{u})g(\tilde{y}, \tilde{z}) - g(F\tilde{x}, \tilde{z})g(\tilde{y}, \tilde{u}) \\ & + g(\tilde{x}, \tilde{u})g(F\tilde{y}, \tilde{z}) - g(\tilde{x}, \tilde{z})g(F\tilde{y}, \tilde{u}) + g(J\tilde{x}, \tilde{u})g(JF\tilde{y}, \tilde{z}) \\ & - g(J\tilde{x}, \tilde{z})g(JF\tilde{y}, \tilde{u}) + g(JF\tilde{x}, \tilde{u})g(J\tilde{y}, \tilde{z}) - g(JF\tilde{x}, \tilde{z})g(J\tilde{y}, \tilde{u}) \\ & + 2g(F\tilde{x}, J\tilde{y})g(J\tilde{z}, \tilde{u}) - 2g(\tilde{x}, J\tilde{y})g(JF\tilde{z}, \tilde{u})\}.\end{aligned} \tag{4.1}$$

Let X, Y, Z be orthogonal tangent vectors at a point p of M^n. Then (4.1) and $[A_{JX}, A_{JY}] = 0$ imply

$$R(X, Y, Z, X) = \frac{\mu}{8} g(X, X)g(FY, Z). \tag{4.2}$$

Let $X, Y \in T_p(M_1^{n-1}(c))$. Then we find $R(X, Y)Z = 0$ for any vector $Z \in T_pM$, orthogonal to X and to Y. Hence using (4.2) we obtain $g(FY, Z) = 0$. Consequently for any $Y \in T_pM_1^{n-1}(c)$ it follows $\pi_1 Y = 0$ or $\pi_2 Y = 0$. Suppose now that there exist nonzero vectors $U, V \in T_pM_1^{n-1}(c)$, such that $\pi_1 U = 0$ and $\pi_2 V = 0$. But we must have $\pi_1(U + V) = 0$ or $\pi_2(U + V) = 0$. Let for example $\pi_1(U + V) = 0$. Then $\pi_1 V = 0$, which is a contradiction. Consequently we have either $\pi_1 = 0$ or $\pi_2 = 0$ on $T_pM_1^{n-1}(c)$. Hence we obtain easily that $k = n - 1$ and $M_1^{n-1}(c) \subset \tilde{M}_1^{2(n-1)}(\mu)$, $I \subset \tilde{M}^2(-\mu)$. Since $M_1^{n-1}(c)$ is semiparallel in $\tilde{M}^{2(n-1)}(\mu)$ and $\mu \neq 0$ it follows that $M_1^{n-1}(c)$ is totally geodesic in $M_1^{2(n-1)}(\mu)$, see [3], so $c = \frac{\mu}{4}$.

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Sommario

In una varietà kähleriana si considerano le sottovarietà totalmente reali e conformemente piatte con vettore di curvatura media parallelo e le sottovarietà con seconda forma fondamentale semiparallela.

Sono anche considerate le sottovarietà totalmente reali di una varietà prodotto di varietà kähleriane, avente tensore di Bochner nullo.

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