Riv. Mat. Univ. Parma (5) 4 (1995), 219-230

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A dynamic problem in linear thermoviscoelasticity (**)

1 - Introduction

A new model for a linearized theory of thermoviscoelasticity, compatible with the theory of fading memory has been exhibited in [13]. In that paper the necessary approximations in order to obtain a completely linear theory are introduced and necessary and sufficient conditions on the costitutive equations for the validity of thermodynamic principles are derived.

A first model of linear thermoviscoelastic material has been established by Gurtin in [10]. This author extends the results (obtained by Day in [5]) on the linearized isothermal theory of viscoelasticity as a consequence of the compatibility with thermodynamic principles and of the hypothesis of invariance under time-reversal.

Subsequently, various authors [14], [1], [2], [11], [12] studied the evolutive problem for Gurtin's model and deduced very interesting theorems of existence, uniqueness, asymptotic stability and continuous dependence. They follow methods, based on the works of Dafermos in [4], which require hypotheses on the constitutive equations, that appear reasonable and fully compatible with the experimental results, but are partially different from the Second Law of Thermodynamics.

The main purpose of this paper is to obtain some results on the existence and uniqueness of the solutions for the dynamic problem under more general hypotheses, which are consequence of the Second Law of Thermodynamics for cyclic processes.

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(**) Received September 7, 1995. AMS classification 75 U 05. This work has been perfomed under the auspices of GNFM and partially supported by MURST 40%: Metodi matematici nella meccanica dei continui.

Sections 2, 3, are devoted, respectively, to the description of the constitutive equations and to the analysis of a linearized statement of the Second Law in the form of the Clausius property, following a technique proposed in [7], [9].

In Section 4, the thermodynamic requirements are shown to be sufficient to prove an existence and uniqueness theorem for a boundary-initial value problem of a linear anisotropic inhomegeneous thermoviscolastic material.

2 - Preliminaries

We consider an inhomogeneous, anisotropic thermoviscoelastic material occupying a bounded domain $\Omega \subset \boldsymbol{R}^3$, with smooth boundary $\partial\Omega$, in a reference configuration, with zero stress and constant and uniform absolute base temperature Θ_0 and we denote by x the position vector of any point of the body.

Let $t \in \boldsymbol{R}$ be the time. We indicate by $\boldsymbol{u}(x, t)$ the displacement vector, $\Theta(x, t)$ the absolute temperature and $\vartheta(x, t) = \Theta(x, t) - \Theta_0$ the temperature variation field. The mass density ϱ is taken to constant and particularly $\varrho = 1$. The quantities $\boldsymbol{T}(x, t)$, $\boldsymbol{g}(x, t)$, $\boldsymbol{q}(x, t)$ and $h(x, t)$ represent the Cauchy stress tensor, the temperature gradient at x, the heat flux vector and the rate at which heat is absorbed for unit of volume, respectively. Whenever no ambiguity arises, the dependence on x is omitted.

As usual, given a function f on $\boldsymbol{R}$, the past history f^t of f up to time t is defined as $f^t(s) = f(t-s)$, $s \in \boldsymbol{R}^+$.

We consider materials for which the physical state at a time t is given by the history of the displacement gradient and by the history of the temperature difference up to time t, i.e., by the set $(\nabla\boldsymbol{u}(t), \nabla\boldsymbol{u}^t, \vartheta(t), \vartheta^t)$, while the thermodynamic process is defined by the triplet $(\nabla\dot{\boldsymbol{u}}, \dot{\vartheta}, \boldsymbol{g})$, where the superposed dot represents the derivative with respecto to t.

Under hypotheses of small deformations and small variations of the temperature with respect to the given reference configuration, the *constitutive relations* for $\boldsymbol{T}$, h and $\boldsymbol{q}$ are described by the following linearized equations [13]:

$$\boldsymbol{T}(t) = \boldsymbol{G}_0 \nabla\boldsymbol{u}(t) + \int_0^\infty \boldsymbol{G}'(s)\nabla\boldsymbol{u}^t(s)\,ds + \boldsymbol{M}_0\vartheta(t) + \int_0^\infty \boldsymbol{M}'(s)\vartheta^t(s)\,ds \tag{2.1}$$

$$\begin{aligned} h(t) &= \Theta_0[\boldsymbol{B}_\infty \cdot \nabla\dot{\boldsymbol{u}}(t) + d_\infty \dot{\vartheta}(t) + \boldsymbol{D}_0 \cdot \nabla\boldsymbol{u}(t)] \\ &+ \Theta_0[\int_0^\infty \boldsymbol{D}'(s)\cdot\nabla\boldsymbol{u}^t(s)\,ds + a_0\vartheta(t) + \int_0^\infty a'(s)\vartheta^t(s)\,ds] \end{aligned} \tag{2.2}$$

$$\boldsymbol{q}(t) = -\boldsymbol{K}\boldsymbol{g}(t) \tag{2.3}$$

where G_0 and $G'(s)$, $s \geq 0$ are fourth order tensors; M_0, D_0, B_∞, K and $M'(s)$, $D'(s)$, $s \geq 0$ are second order tensors; d_∞, a_0 and $a'(s)$, $s \geq 0$ are scalar fields. The constitutive equation (2.3) represents the usual Fourier's law.

As usual for material with memory, a fading memory principle for the Boltzmann functions G', M', D', a' is required. We assume that G', M', D', a' belong to $L^1(\boldsymbol{R}^+) \cap L^2(\boldsymbol{R}^+)$. This implies that, if we denote with f_0 and $f'(s)$ a general pair of constitutive functions, then the relaxation function $f(t) = f_0 + \int_0^t f'(s)\,ds$ is well defined, for every $t \geq 0$ and $f_\infty = \lim_{t \to \infty} f(t)$ exists.

In the context of a linearized thory, we assume that the reference configuration is a natural undeformed state, which implies $M_\infty = \boldsymbol{0}$. Moreover, requiring that in the reference configuration the rate at which heat is absorbed by the unity of volume h is equal to zero, we derive $a_\infty = 0$.

In [13] it is shown that the constitutive equations (2.1)-(2.3) describe a model of linear thermoviscoelasticity more general than the one produced by Navarro in [14].

3 - Thermodynamic restrictions

In this section we recall the restrictions on the constitutive equations (2.1)-(2.2), due to the two laws of thermodynamics [3]:

(3.1) $$\oint [h(t) + \boldsymbol{T}(t) \cdot \nabla \dot{\boldsymbol{u}}(t)]\,dt = 0 \qquad \text{First Law}$$

(3.2) $$\oint \Big[\frac{h(t)}{[\Theta_0 + \vartheta(t)]} + \frac{\boldsymbol{q}(t) \cdot \boldsymbol{g}(t)}{[\Theta_0 + \vartheta(t)]^2}\Big]\,dt \geq 0 \qquad \text{Second Law}$$

for any cyclic process. The complete statement of the Second Law, however, specifies that the inequality refers to an irreversible process, whereas the equality occurs in irreversible processes only (see for instance [6]).

Under linear approximation (i.e. small variations of the temperature with respect to a given reference temperature Θ_0 and small $\boldsymbol{g}$), $(\Theta_0 + \vartheta)^{-1}$ may be substituted by its Taylor linear polinomial $\frac{1}{\Theta_0}(1 - \frac{\vartheta}{\Theta_0})$. Thus (3.2), after a comparison with (3.1), becomes

(3.3) $$\frac{1}{\Theta_0^2} \oint [h(t)\,\vartheta(t) + \Theta_0 \boldsymbol{T}(t) \cdot \nabla \dot{\boldsymbol{u}}(t) - \boldsymbol{q}(t) \cdot \boldsymbol{g}(t)]\,dt \geq 0\,.$$

Substituting the constitutive equations (2.1)-(2.3) into (3.3), we obtain the dissipation inequality

$$\frac{1}{\Theta_0}\oint\{[\boldsymbol{G}_0\nabla\boldsymbol{u}(t)+\int_0^\infty \boldsymbol{G}'(s)\nabla\boldsymbol{u}^t(s)\,\mathrm{d}s+(\boldsymbol{M}_0+\boldsymbol{B}_\infty)\vartheta(t)+\int_0^\infty \boldsymbol{M}'(s)\vartheta^t(s)\,\mathrm{d}s]\cdot\nabla\dot{\boldsymbol{u}}(t)\}\,\mathrm{d}t$$

$$(3.4)\qquad +\frac{1}{\Theta_0}\oint[d_\infty\dot{\vartheta}(t)+\boldsymbol{D}_0\cdot\nabla\boldsymbol{u}(t)+\int_0^\infty \boldsymbol{D}'(s)\cdot\nabla\boldsymbol{u}^t(s)\,\mathrm{d}s+a_0\vartheta(t)]\dot{\vartheta}(t)\,\mathrm{d}t$$

$$+\frac{1}{\Theta_0}\oint\{\int_0^\infty a'(s)\vartheta^t(s)\,\mathrm{d}s\,\dot{\vartheta}(t)+\frac{1}{\Theta_0}\boldsymbol{K}\boldsymbol{g}(t)\cdot\boldsymbol{g}(t)\}\,\mathrm{d}t\geqslant 0\,.$$

In order to deduce thermodynamic restrictions for the constitutive functionals, we consider histories of the strain tensor and of the temperature variation field oscillating in the time with the law:

$$(3.5)\qquad \begin{aligned}\nabla\boldsymbol{u}^t(s)&=\nabla\boldsymbol{u}_1\cos\omega(t-s)-\nabla\boldsymbol{u}_2\sin\omega(t-s)=\mathrm{Re}[\nabla\boldsymbol{u}\,\mathrm{e}^{i\omega(t-s)}]\\ \vartheta^t(s)&=\vartheta_1\cos\omega(t-s)-\vartheta_2\sin\omega(t-s)=\mathrm{Re}[\vartheta\,\mathrm{e}^{i\omega(t-s)}]\end{aligned}$$

where $\boldsymbol{u}=\boldsymbol{u}_1+i\boldsymbol{u}_2$, $\vartheta=\vartheta_1+i\vartheta_2$ with $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ϑ_1, ϑ_2 constant values in the time, ω a positive constant frequency, and we choose a cyclic irreversible process $(\nabla\dot{\boldsymbol{u}},\dot{\vartheta},\boldsymbol{g})$ of duration $\tau=\frac{2\pi}{\omega}$, $\omega>0$, defined by:

$$(3.6)\qquad \begin{aligned}&\nabla\dot{\boldsymbol{u}}(t)=-\omega[\nabla\boldsymbol{u}_1\sin\omega t+\nabla\boldsymbol{u}_2\cos\omega t]\\ &\dot{\vartheta}(t)=-\omega[\vartheta_1\sin\omega t+\vartheta_2\cos\omega t]\qquad \boldsymbol{g}(t)=\boldsymbol{g}_1\sin\omega t+\boldsymbol{g}_2\cos\omega t\,.\end{aligned}$$

Moreover, we denote with $\hat{f}_c(\omega)=\int_0^\infty f(s)\cos\omega s\,\mathrm{d}s$, $\hat{f}_s(\omega)=\int_0^\infty f(s)\sin\omega s\,\mathrm{d}s$ the *half-range Fourier cosine and sine transforms* of the function f, so that we have $\hat{f}(\omega)=\hat{f}_c(\omega)-i\hat{f}_s(\omega)$.

Substituting (3.5)-(3.6) into (3.4), after the integration over the period of oscillation, we obtain

$$\frac{\pi}{\Theta_0}\{\nabla\boldsymbol{u}_2\cdot[(\boldsymbol{G}_0+\hat{\boldsymbol{G}}'_c(\omega))^T-(\boldsymbol{G}_0+\hat{\boldsymbol{G}}'_c(\omega))]\nabla\boldsymbol{u}_1-\nabla\boldsymbol{u}_1\cdot\hat{\boldsymbol{G}}'_s(\omega)\nabla\boldsymbol{u}_1-\nabla\boldsymbol{u}_2\cdot\hat{\boldsymbol{G}}'_s(\omega)\nabla\boldsymbol{u}_2$$

$$(3.7)\qquad \begin{aligned}&+[\boldsymbol{M}_0+\boldsymbol{B}_\infty+\hat{\boldsymbol{M}}'_c(\omega)-\frac{1}{\omega}\hat{\boldsymbol{D}}'_s(\omega)]\cdot(\vartheta_2\nabla\boldsymbol{u}_1-\vartheta_1\nabla\boldsymbol{u}_2)\\ &+[-\hat{\boldsymbol{M}}'_s(\omega)+\frac{1}{\omega}(\boldsymbol{D}_0+\hat{\boldsymbol{D}}'_c(\omega))]\cdot(\vartheta_1\nabla\boldsymbol{u}_1+\vartheta_2\nabla\boldsymbol{u}_2)\\ &+\frac{1}{\omega}[a_0+\hat{a}'_c(\omega)](\vartheta_1^2+\vartheta_2^2)+\frac{1}{\omega\Theta_0}(\boldsymbol{K}\boldsymbol{g}_1\cdot\boldsymbol{g}_1+\boldsymbol{K}\boldsymbol{g}_2\cdot\boldsymbol{g}_2)\}>0\end{aligned}$$

where $\boldsymbol{G}^T$ denotes the transposed of $\boldsymbol{G}$.

After straightforward calculations (3.7) may be written

$$\begin{aligned}(3.8)\qquad & \operatorname{Im}\{(\boldsymbol{G}_0+\widehat{\boldsymbol{G}}'(\omega))\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^*+(\boldsymbol{M}_0+\boldsymbol{B}_\infty+\widehat{\boldsymbol{M}}'(\omega))\cdot\nabla\boldsymbol{u}^*\vartheta\}\\ & +\frac{1}{\omega}\operatorname{Re}\{\boldsymbol{D}_0+\widehat{\boldsymbol{D}}'(\omega))\cdot\nabla\boldsymbol{u}\vartheta^*+(a_0+\widehat{a}'(\omega))\vartheta\vartheta^*+\frac{1}{\Theta_0}\boldsymbol{K}\boldsymbol{g}\cdot\boldsymbol{g}^*\}>0\end{aligned}$$

where the symbol * denotes the *complex conjugate*.

As a consequence of (3.7), in isothermal conditions (with $\vartheta = 0$, $\boldsymbol{g} = \boldsymbol{0}$) the classical conditions on the stress-strain relaxation tensor $\boldsymbol{G}$ hold:

$$(3.9)\qquad \boldsymbol{G}_0=\boldsymbol{G}_0^T \qquad \boldsymbol{G}_\infty=\boldsymbol{G}_\infty^T \qquad \operatorname{Im}\{(\boldsymbol{G}_0+\widehat{\boldsymbol{G}}'(\omega))\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^*\}>0$$

obtained in [8] for linear viscoelastic materials; while, for thermal processes (with $\nabla\boldsymbol{u} = \boldsymbol{0}$, $\boldsymbol{g} = \boldsymbol{0}$) it holds condition

$$(3.10)\qquad \operatorname{Re}\{a_0+\widehat{a}'(\omega)\}>0$$

obtained in [9] for the rigid conductors.

Finally, if $\vartheta = 0$, but $\boldsymbol{g}_1 = \boldsymbol{g}_2 = \boldsymbol{g}$, it follows the positive definiteness of the thermal conductivity

$$(3.11)\qquad \boldsymbol{K}>0\,.$$

Letting $\omega \to \infty$ in (3.8), by virtue of (3.9), it follows

$$(3.12)\qquad \boldsymbol{M}_0=-\boldsymbol{B}_\infty\,.$$

This condition is equivalent to require that the instantaneous derivative of the free energy with respect to the strain is the stress and the instantaneous derivative of the free energy with respect to the temperature is the negative of the entropy.

Letting $\omega \to 0^+$ in (3.8), with $\boldsymbol{g} = \boldsymbol{0}$, it follows

$$(3.13)\qquad \boldsymbol{D}_\infty=\boldsymbol{0} \qquad a_\infty\geqslant 0\,.$$

Because of (3.12) the inequality (3.8) becomes

$$\begin{aligned}(3.14)\qquad & \operatorname{Im}\{(\boldsymbol{G}_0+\widehat{\boldsymbol{G}}'(\omega))\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^*\}\\ & +\frac{1}{\omega}\operatorname{Re}\{(i\omega\widehat{\boldsymbol{M}}'^*(\omega)+\boldsymbol{D}_0+\widehat{\boldsymbol{D}}'(\omega))\cdot\nabla\boldsymbol{u}\,\vartheta^*+(a_0+\widehat{a}'(\omega))\vartheta\vartheta^*+\frac{1}{\Theta_0}\boldsymbol{K}\boldsymbol{g}\cdot\boldsymbol{g}^*\}>0\,.\end{aligned}$$

We can summarize our results in the following

Theorem 1. *For a linear thermoviscoelastic material, satisfying* (2.1)-(2.3), *the Second Law of Thermodynamics holds, if and only if the relaxation functions satisfy* (3.9)-(3.14).

Paralleling the argument explained in [9], it is easily shown that (3.9)-(3.14) are also sufficient conditions for the validity of the Second Law for thermoviscoelastic materials in the form (2.1)-(2.3).

4 - Existence and uniqueness results

The ensuing *evolution problem* in the space-temporal domain $\Omega \times \boldsymbol{R}^+$ is given by:

$$\begin{gathered} \ddot{\boldsymbol{u}}(x, t) = \nabla \cdot \boldsymbol{T}(x, t) + \boldsymbol{b}(x, t) \qquad h(x, t) = -\nabla \cdot \boldsymbol{q}(x, t) + r(x, t) \\ \boldsymbol{u}(x, t) = \boldsymbol{0}, \quad \vartheta(x, t) = 0 \quad \text{on } \partial\Omega \\ \boldsymbol{u}^t(x, s) = \boldsymbol{u}^0(x, s-t), \quad \vartheta^t(x, s) = \vartheta^0(x, s-t), \quad s \geq t . \end{gathered} \tag{4.1}$$

The first equation represents the local form of *balance of linear momentum*, where $\boldsymbol{b}$ is the specific body force, while the second is the *balance of the energy*, where r is the specific heat supply field.

Substituting the constitutive equations (2.1)-(2.3) into problem (4.1) and separating initial past histories from the solution, one obtains the *initial-boundary value problem:*

$$\begin{aligned} \ddot{\boldsymbol{u}}(x, t) = \nabla \cdot [\boldsymbol{G}_0(x) \nabla \boldsymbol{u}(x, t) + \int_0^t \boldsymbol{G}'(x, s) \nabla \boldsymbol{u}^t(x, s) \, ds + \boldsymbol{M}_0(x) \vartheta(x, t)] \\ + \nabla \cdot [\int_0^t \boldsymbol{M}'(x, s) \vartheta^t(x, s) \, ds] + \nabla \cdot \boldsymbol{T}_0(x, t) + \boldsymbol{b}(x, t) \end{aligned} \tag{4.2}$$

$$\begin{aligned} \Theta_0[-\boldsymbol{M}_0(x) \cdot \nabla \dot{\boldsymbol{u}}(x, t) + d_\infty(x) \dot{\vartheta}(x, t) + \boldsymbol{D}_0(x) \cdot \nabla \boldsymbol{u}(x, t)] \\ + \Theta_0[\int_0^t \boldsymbol{D}'(x, s) \cdot \nabla \boldsymbol{u}^t(x, s) \, ds + a_0(x) \vartheta(x, t) + \int_0^t a'(x, s) \vartheta^t(x, s) \, ds] \\ = \nabla \cdot [\boldsymbol{K}(x) \nabla \vartheta(x, t)] + c_0(x, t) + r(x, t) \end{aligned} \tag{4.3}$$

$$\boldsymbol{u}(x, t) = \boldsymbol{0}, \qquad \vartheta(x, t) = 0 \qquad \text{on } \partial\Omega \tag{4.4}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), \qquad \vartheta(x, 0) = \vartheta_0(x), \qquad \dot{\boldsymbol{u}}(x, 0) = \dot{\boldsymbol{u}}_0(x) \tag{4.5}$$

where $\boldsymbol{u}_0(x)=\boldsymbol{u}^0(x,0)$, $\vartheta_0(x)=\vartheta^0(x,0)$, $\dot{\boldsymbol{u}}_0(x)=-\lim_{s\to 0^+}\frac{\partial}{\partial s}\boldsymbol{u}^0(x,s)=\boldsymbol{v}_0(x)$

$$\boldsymbol{T}_0(x,t)=\int_t^\infty \boldsymbol{G}'(x,s)\nabla\boldsymbol{u}^0(x,s-t)\,\mathrm{d}s+\int_t^\infty \boldsymbol{M}'(x,s)\vartheta^0(x,s-t)\,\mathrm{d}s$$

$$c_0(x,t)=-\int_t^\infty \boldsymbol{D}'(x,s)\cdot\nabla\boldsymbol{u}^0(x,s-t)\,\mathrm{d}s-\int_t^\infty a'(x,s)\vartheta^0(x,s-t)\,\mathrm{d}s\,.$$

The boundary homogeneous Dirichlet conditions (4.4) state that the boundary is fixed and it is maintained at constant reference temperature Θ_0.

In order to achieve an existence and uniqueness result relative to problem (4.2)-(4.5), we impose the *assumptions*:

H_1. The relaxation functions $\boldsymbol{G}(x,s)$, $\boldsymbol{M}(x,s)$ are continuous in $\overline{\Omega}\times[0,\infty)$ and differentiable on $\Omega\times(0,\infty)$, $\boldsymbol{D}(x,s)$, $a(x,s)$ are continuous in $\overline{\Omega}\times[0,\infty)$, the tensor $\boldsymbol{K}(x)$ is continuous in $\overline{\Omega}$ and differentiable on Ω and $d_\infty(x)$ is a continuous function in Ω.

H_2. The tensor $\boldsymbol{G}_\infty(x)$ is a positive definite tensor in Ω, i.e. a positive constant g_∞ exists such that, for every $\boldsymbol{v}\in C_0^\infty(\Omega)$

$$\int_\Omega \boldsymbol{G}_\infty(x)\nabla\boldsymbol{v}(x)\cdot\nabla\boldsymbol{v}(x)\,\mathrm{d}x\geqslant g_\infty\int_\Omega\nabla\boldsymbol{v}(x)\cdot\nabla\boldsymbol{v}(x)\,\mathrm{d}x\,.$$

H_3. As a consequence to (3.14) it is possible to prove that there exists always a positive constant $\mu(\omega)$ such that, for every $\omega>0$

$$\begin{aligned}
&\int_\Omega \mathrm{Im}\,[(\boldsymbol{G}_0(x)+\widehat{\boldsymbol{G}}'(x,\omega))\nabla\boldsymbol{u}(x)\cdot\nabla\boldsymbol{u}^*(x)]\,\mathrm{d}x\\
&+\frac{1}{\omega}\int_\Omega \mathrm{Re}\,[(i\omega\widehat{\boldsymbol{M}}'^*(x,\omega)+\boldsymbol{D}_0(x)+\widehat{\boldsymbol{D}}'(x,\omega))\cdot\nabla\boldsymbol{u}(x)\,\vartheta^*(x)]\,\mathrm{d}x\\
&+\frac{1}{\omega}\int_\Omega \mathrm{Re}\,[(a_0(x)+\widehat{a}'(x,\omega))\,\vartheta(x)\,\vartheta^*(x)]\,\mathrm{d}x\\
&+\frac{1}{\Theta_0}\int_\Omega \boldsymbol{K}(x)\nabla\vartheta(x)\cdot\nabla\vartheta^*(x)\,\mathrm{d}x\geqslant\mu(\omega)\{\|\boldsymbol{u}(x)\|_1^2+\|\vartheta(x)\|_1^2\}
\end{aligned}\tag{4.6}$$

where $\|\cdot\|_1$ denotes the norm in the space $H_0^1(\Omega)$.

In the sequence a function will be called *causal*, if it equals 0 for negative t. A function defined on $\boldsymbol{R}^+$ can be identified with a function on $\boldsymbol{R}$, which vanishes identically on $(-\infty,0)$.

First we consider the Fourier transformed problem of (4.2)-(4.5) for causal functions, which is defined for each $\omega \in \boldsymbol{R}$:

$$
\begin{aligned}
(4.7) \qquad -\omega^2 \hat{\boldsymbol{u}}(x, \omega) = \nabla \cdot [(\boldsymbol{G}_0(x) + \hat{\boldsymbol{G}}'(x, \omega)) \nabla \hat{\boldsymbol{u}}(x, \omega)] \\
+ \nabla \cdot [(\boldsymbol{M}_0(x) + \hat{\boldsymbol{M}}'(x, \omega)) \hat{\vartheta}(x, \omega)] + \hat{\boldsymbol{f}}(x, \omega)
\end{aligned}
$$

$$
\begin{aligned}
(4.8) \qquad \Theta_0[(-i\omega \boldsymbol{M}_0(x) + \boldsymbol{D}_0(x) + \hat{\boldsymbol{D}}'(x, \omega)) \cdot \nabla \hat{\boldsymbol{u}}(x, \omega)] \\
+ \Theta_0[(i\omega d_\infty(x) + a_0(x) + \hat{a}'(x, \omega)) \hat{\vartheta}(x, \omega)] = \nabla \cdot [\boldsymbol{K}(x) \nabla \hat{\vartheta}(x, \omega)] + \hat{l}(x, \omega)
\end{aligned}
$$

$$
(4.9) \qquad \hat{\boldsymbol{u}}(x, \omega) = \boldsymbol{0}, \qquad \hat{\vartheta}(x, \omega) = 0 \qquad \text{on } \partial\Omega
$$

where
$$
\hat{\boldsymbol{f}}(x, \omega) = \hat{\boldsymbol{b}}(x, \omega) + \nabla \cdot \hat{\boldsymbol{T}}_0(x, \omega) + i\omega \boldsymbol{u}_0(x) + \boldsymbol{v}_0(x)
$$
$$
\hat{l}(x, \omega) = \hat{r}(x, \omega) + \hat{c}_0(x, \omega) + \Theta_0[d_\infty(x) \vartheta_0(x) - \boldsymbol{M}_0(x) \cdot \nabla \boldsymbol{u}_0(x)].
$$

Definition. A pair $(\hat{\boldsymbol{u}}, \hat{\vartheta}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is called a *weak solution* of (4.7)-(4.9) if

$$
\begin{aligned}
&\int_\Omega \{-\omega^2 \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{w}}^* + [(\boldsymbol{G}_0 + \hat{\boldsymbol{G}}') \nabla \hat{\boldsymbol{u}} + (\boldsymbol{M}_0 + \hat{\boldsymbol{M}}') \hat{\vartheta}] \cdot \nabla \hat{\boldsymbol{w}}^* \} \, dx \\
(4.10) \qquad &+ \int_\Omega \{\Theta_0[(-i\omega \boldsymbol{M}_0 + \boldsymbol{D}_0 + \hat{\boldsymbol{D}}') \cdot \nabla \hat{\boldsymbol{u}}]\} \, dx \\
&+ \int_\Omega \{[(i\omega d_\infty + a_0 + \hat{a}') \hat{\vartheta}] \hat{\alpha}^* + \boldsymbol{K} \nabla \hat{\vartheta} \cdot \nabla \hat{\alpha}^* \} \, dx = \int_\Omega \{\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{w}}^* + \hat{l} \hat{\alpha}^* \} \, dx
\end{aligned}
$$

holds for every $(\hat{w}^*, \hat{\alpha}^*) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

Lemma 1. *Under the hypotheses* H_1-H_3 *problem* (4.7)-(4.9) *has one and only one weak solution* $(\hat{\boldsymbol{u}}(\cdot, \omega), \hat{\vartheta}(\cdot, \omega)) \in H_0^1(\Omega) \times H_0^1(\Omega)$, *for every* $(\hat{\boldsymbol{f}}(\cdot, \omega), \hat{l}(\cdot, \omega)) \in L^2(\Omega) \times L^2(\Omega)$ *and* $\omega \in \boldsymbol{R}$.

Proof. (*Existence*) By virtue of well-know theorems on elliptic problems [8], (4.7)-(4.9) has one and only one solution, if and only if the associated operator is coercive for every $\omega \in \boldsymbol{R}$.

If $\omega \neq 0$, we consider the quadratic form related to the system (4.7)-(4.9)

$$
(4.11) \qquad F(\boldsymbol{u}, \vartheta, \omega) = i\omega A(\boldsymbol{u}, \vartheta, \omega) + B(\boldsymbol{u}, \vartheta, \omega)
$$

where

$$
\begin{aligned}
A(\boldsymbol{u}, \vartheta, \omega) = &- \int_\Omega \{\omega^2 \boldsymbol{u}(x) \cdot \boldsymbol{u}^*(x)\} \, dx \\
&- \int_\Omega \{[\boldsymbol{G}_0(x) + \hat{\boldsymbol{G}}'(x, \omega)] \nabla \boldsymbol{u}(x) \cdot \nabla \boldsymbol{u}^*(x) + [\boldsymbol{M}_0(x) + \hat{\boldsymbol{M}}'(x, \omega)] \vartheta(x) \cdot \nabla \boldsymbol{u}^*(x)\} \, dx
\end{aligned}
$$

$$
\begin{aligned}
B(\boldsymbol{u}, \vartheta, \omega) = &\int_\Omega \{[-i\omega \boldsymbol{M}_0(x) + \boldsymbol{D}_0(x) + \hat{\boldsymbol{D}}'(x, \omega)] \cdot \nabla \boldsymbol{u}(x) \vartheta^*(x)\} \, dx \\
&+ \int_\Omega \{[i\omega d_\infty(x) + a_0(x) + \hat{a}'(x, \omega)] \vartheta(x) \vartheta^*(x) + \frac{1}{\Theta_0} \boldsymbol{K}(x) \nabla \vartheta(x) \cdot \nabla \vartheta^*(x)\} \, dx.
\end{aligned}
$$

Therefore

$$|F(\boldsymbol{u}, \vartheta, \omega)| \geq \operatorname{Re}\{F(\boldsymbol{u}, \vartheta, \omega)\}$$

$$= \operatorname{Re}\{i\omega A(\boldsymbol{u}, \vartheta, \omega) + B(\boldsymbol{u}, \vartheta, \omega)\} \geq \mu(\omega)\{\|\boldsymbol{u}(x)\|_1^2 + \|\vartheta(x)\|_1^2\}$$

by virtue of (4.6).

If $\omega = 0$, the transformed problem becomes

$$\nabla \cdot [\boldsymbol{G}_\infty(x) \nabla \hat{\boldsymbol{u}}(x, 0)] + \hat{\boldsymbol{f}}(x, 0) = \boldsymbol{0} \qquad \nabla \cdot [\boldsymbol{K}(x) \nabla \hat{\vartheta}(x, 0)] + \hat{l}(x, 0) = 0 \tag{4.12}$$

$$\hat{\boldsymbol{u}}(x, 0) = \boldsymbol{0}, \qquad \hat{\vartheta}(x, 0) = 0 \qquad \text{on } \partial\Omega \tag{4.13}$$

and the positive definiteness of $\boldsymbol{G}_\infty(x)$ and $\boldsymbol{K}(x)$ imply that the associated operator is *coercive*.

(*Uniqueness*) Since the quadratic form F turns out to be coercive, the associated homogeneous problem admits unique solution equal to zero, for every $\omega \in \boldsymbol{R}$.

We can represent the solution of (4.7)-(4.9) as follows

$$\hat{\boldsymbol{u}}(x, \omega) = \int_\Omega \{ \boldsymbol{J}(x, x'; \omega) \hat{\boldsymbol{f}}(x', \omega) + \frac{1}{\Theta_0} \boldsymbol{j}(x, x'; \omega) \hat{l}(x', \omega)\} \, dx' \tag{4.14}$$

where the Green functions $\boldsymbol{J}, \boldsymbol{j}$ (second and first order tensors, respectively), for almost all $x \in \Omega$ and $\omega \in \boldsymbol{R}$, solve the problem

$$\begin{aligned} &-\omega^2 \boldsymbol{J}(x, x'; \omega) - \nabla' \cdot [(\boldsymbol{G}_0(x') + \hat{\boldsymbol{G}}'(x', \omega))^T \nabla' \boldsymbol{J}(x, x'; \omega)] \\ &- \nabla' \cdot [(-i\omega \boldsymbol{M}_0(x') + \boldsymbol{D}_0(x') + \hat{\boldsymbol{D}}'(x', \omega)) \boldsymbol{j}(x, x'; \omega)] = \delta(x - x') \boldsymbol{I} \end{aligned} \tag{4.15}$$

$$\begin{aligned} &[\boldsymbol{M}_0(x') + \hat{\boldsymbol{M}}'(x', \omega)] \cdot \nabla' \boldsymbol{J}(x, x'; \omega) - \frac{1}{\Theta_0} \nabla' \cdot [\boldsymbol{K}(x') \nabla' \boldsymbol{j}(x, x'; \omega)] \\ &+ [i\omega d_\infty(x') + a_0(x') + a'(x', \omega)] \boldsymbol{j}(x, x'; \omega) = \boldsymbol{0} \end{aligned} \tag{4.16}$$

$$\boldsymbol{J}(x, x'; \omega) = \boldsymbol{0} \qquad \boldsymbol{j}(x, x'; \omega) = \boldsymbol{0} \qquad \text{on } \partial\Omega \tag{4.17}$$

where δ is the Dirac's delta and $\boldsymbol{I}$ the identity tensor.

Lemma 2. *Under the hypotheses of Lemma* 1, *for almost all* $x \in \Omega$ *and* $\omega \in \boldsymbol{R}$, *problem* (4.15)-(4.17) *has one and only one solution* $\boldsymbol{J}(x, x'; \omega)$, $\boldsymbol{j}(x, x'; \omega)$, *with the properties*:

i. $\boldsymbol{J}(x, x'; \cdot), \boldsymbol{j}(x, x'; \cdot)$ *are continuous on* $\boldsymbol{R}$

ii. $\boldsymbol{J}(x, x'; \omega) = O(\omega^{-2+\varepsilon})$, $\quad \boldsymbol{j}(x, x'; \omega) = O(\omega^{-1+\varepsilon})$ *as* $\omega \to \infty$.

Proof. The continuity of $\boldsymbol{J}, \boldsymbol{j}$ on ω follows by the continuous dependence of $F(\boldsymbol{u}, \vartheta, \omega)$ with respect to ω (see [15]). Now, multiplying (4.15)-(4.17) by the real functions v, $\varphi \in C_0^\infty(\Omega)$ respectively, and integrating over Ω, we have:

$$\int_\Omega J(x, x'; \omega)[-v(x') - \frac{1}{\omega^2}\nabla'\cdot[(G_0(x') + \hat{G}'(x', \omega))\nabla' v(x')]\,dx'$$

$$-\frac{1}{\omega^2}\int_\Omega J(x, x', \omega)[\nabla'\cdot[(M_0(x') + \hat{M}'(x', \omega))\varphi(x')]]\,dx'$$

$$+\frac{1}{\omega^2}\int_\Omega j(x, x'; \omega)[(-i\omega M_0(x') + D_0(x') + \hat{D}'(x', \omega))\cdot\nabla' v(x')]\,dx'$$

$$+\frac{1}{\omega^2}\int_\Omega j(x, x'; \omega)[(i\omega\, d_\infty(x') + a_0(x') + \hat{a}'(x', \omega))\varphi(x')]\,dx'$$

$$-\frac{1}{\omega^2}\int_\Omega j(x, x'; \omega)[\frac{1}{\Theta_0}\nabla'\cdot(K(x')\nabla'\varphi(x')]\,dx' = \frac{1}{\omega^2}v(x).$$

By virtue of Lebesgue theorem $\hat{G}'$, $\hat{M}'$, $\hat{D}'$, $\hat{a}'$ vanish when $\omega \to \infty$, hence

$$\lim_{\omega\to\infty}\int_\Omega J(x, x'; \omega)[-v(x') - \frac{1}{\omega^2}\nabla'\cdot[G_0(x')\nabla' v(x') + M_0(x')\varphi(x')]]\,dx'$$

$$+\lim_{\omega\to\infty}\frac{1}{\omega^2}\int_\Omega j(x, x'; \omega)[D_0(x')\cdot\nabla' v(x') + a_0(x')\varphi(x') - \frac{1}{\Theta_0}\nabla'\cdot(K(x')\nabla'\varphi(x'))]\,dx'$$

$$+\lim_{\omega\to\infty}\frac{i}{\omega^2}\int_\Omega j(x, x'; \omega)(-M_0(x')\cdot\nabla' v(x') + d_\infty(x')\varphi(x'))\,dx' = \frac{1}{\omega^2}v(x).$$

For the choice of v, φ (4.15) yields

$$\lim_{\omega\to\infty}\omega^\varepsilon\int_\Omega ij(x, x'; \omega)[-M_0(x')\cdot\nabla' v(x') + d_\infty(x')\varphi(x'))\,dx' = 0 \tag{4.18}$$

and finally

$$\lim_{\omega\to\infty}\omega^{-2+\varepsilon}\int_\Omega -J(x, x'; \omega)v(x')\,dx'$$

$$= \omega^\varepsilon\int_\Omega\{J(x, x'; \omega)\nabla'\cdot[G_0(x')\nabla' v(x') + M_0(x')\varphi(x')]\}\,dx' \tag{4.19}$$

$$-\omega^\varepsilon\int_\Omega\{j(x, x'; \omega)[D_0(x')\cdot\nabla' v(x') + a_0(x')\varphi(x') - \frac{1}{\Theta_0}\nabla'\cdot(K(x')\nabla'\varphi(x'))]\}\,dx'.$$

The observation that the test functions are arbitrary completes the proof.

Since $\hat{f}$ and $\hat{l}$ belong to $L^2(\boldsymbol{R}; L^2(\Omega))$, by properties of Green's functions J, j showed in Lemma 2, we conclude that $\hat{u}$ defined through (4.14) belongs to $L^2(R; H_0^1(\Omega))$, and the *inverse Fourier transform* of the function $\hat{u}$ exists:

$$u(x, t) = \frac{1}{2\pi}\int_0^\infty \hat{u}(x, \omega)\,e^{i\omega t}\,d\omega \tag{4.20}$$

Similary we can represent the solution $\hat{\vartheta}$ as follows

$$\hat{\vartheta}(x, \omega) = \int_{\Omega} \{\boldsymbol{c}(x, x'; \omega)\cdot\hat{\boldsymbol{f}}(x', \omega) + \frac{1}{\Theta_0} c(x, x'; \omega)\,\hat{l}(x', \omega)\}\, dx' \tag{4.21}$$

where the vector and the scalar fields $\boldsymbol{c}, c$ are the solutions to the problem

$$\begin{gathered} -\omega^2 \boldsymbol{c}(x, x'; \omega) - \nabla'\cdot[\boldsymbol{G}_0(x') + \hat{\boldsymbol{G}}_0'(x', \omega))^T \nabla' \boldsymbol{c}(x, x'; \omega)] \\ -\nabla'[(-i\omega \boldsymbol{M}_0(x') + \boldsymbol{D}_0(x') + \hat{\boldsymbol{D}}'(x', \omega))\, c(x, x'; \omega)] = \boldsymbol{0} \end{gathered} \tag{4.22}$$

$$\begin{gathered} [\boldsymbol{M}_0(x') + \hat{\boldsymbol{M}}'(x', \omega)]\cdot\nabla' \boldsymbol{c}(x, x'; \omega) \\ +[i\omega\, d_\infty(x') + a_0(x') + \hat{a}'(x', \omega)]\, c(x, x'; \omega) \\ -\frac{1}{\Theta_0}\nabla'\cdot[\boldsymbol{K}(x')\nabla' c(x, x'; \omega)] = \delta(x - x') \end{gathered} \tag{4.23}$$

$$\boldsymbol{c}(x, x'; \omega) = \boldsymbol{0}, \qquad c(x, x'; \omega) = 0 \qquad \text{on } \partial\Omega. \tag{4.24}$$

As a consequence of Lemmas 1, 2, we have:

Theorem 2. *Under the hypotheses of Lemma* 1, *problem* (4.2)-(4.5) *has one and only one solution*

$$(\boldsymbol{u}, \vartheta) \in [L^2(\boldsymbol{R}^+; H_0^1(\Omega)) \cap H^1(\boldsymbol{R}^+; L^2(\Omega))] \times L^2(\boldsymbol{R}^+; H_0^1(\Omega))$$

with data

$(\nabla\cdot \boldsymbol{T}_0 + \boldsymbol{b}, c_0 + r) \in L^2(\boldsymbol{R}^+; L^2(\Omega)) \times L^2(\boldsymbol{R}^+; L^2(\Omega)), (\boldsymbol{u}_0, \vartheta_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $\boldsymbol{v}_0 \in L^2(\Omega)$.

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Sommario

Nell'ambito della teoria linearizzata della termoviscoelasticità si determinano alcune restrizioni sulle equazioni costitutive che risultano essere conseguenza diretta dei principi della termodinamica.

Tali restrizioni ci permettono di dimostrare un teorema di esistenza e unicità per la soluzione debole delle equazioni evolutive di un solido termoviscoelastico lineare.

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