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MARCELLO CICCHESE (*)

A way for constructing order-homogeneous arcs (**)

A *linearly ordered topological space* (abbreviated LOTS) is a linearly ordered set equipped with the usual interval topology. If A, B are subsets of a LOTS X, we write $A < B$ to denote that each point of A precedes each point of B.

Recall that a *jump* in a LOTS X is a pair of consecutive elements of X and a *gap* in X is a pair (A, B) of non-empty subsets of X such that $X = A \cup B$, $A < B$, A has no last point and B has no first point.

For a LOTS X, the following properties are well known:

1. X is connected if and only if it has no jumps and no gaps

2. X is compact if and only if it has no gaps and it has a first and a last point.

Definition 1. A LOTS is said to be an *arc* if it is compact and connected.

Recall that a point x of a topological space X is said to be an *end-point* of X provided that $X - \{x\}$ is connected.

It is known that a topological space is orderable as an arc, if and only it is a *continuum* (compact, connected Hausdorff space) with exactly two end-points, and that an arc is isomorphic to the real interval $[0, 1]$ if it is separable.

Definition 2. An arc X is said to be:

a. a *real arc* if X is isomorphic to $[0, 1]$

b. a *smooth arc* if every closed interval of X consisting of more than one point is isomorphic to X.

Smooth arcs are usually called *order-homogeneous arcs*. Examples of non-real smooth arcs are given, for instance, in [5], [2], [4]. In our paper we give a way for constructing other smooth arcs.

(*) Dip. di Matem., Univ. Parma, Via D'Azeglio 85, 43100 Parma, Italia.
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We will fix some notations:

If X is a topological space, we denote by $|X|$, $d(X)$, $w(X)$ the cardinality, the density, the weight of X.

If X is a LOTS and a, b ($a < b$) are points of X, we denote by $[a, b]$, $[a, b[$, $]a, b]$, $]a, b[$ the intervals of X in an analogous manner as for the real line.

If X is an arc, we denote by x^0 and x^1, the first and the last point of X; we donote also by $[x^0, x^1]$ an arc which has x^0 as first point and x^1 as last point.

The symbol $\approx$ will be used to denote isomorphism of LOTS.

If ϱ is an ordinal number and $\{X_\lambda\}_{\lambda<\varrho}$ is a well-ordered collection of LOTS, we denote by $\prod_{\lambda<\varrho} X_\lambda$ the lexicographically ordered product of the X_λ; we write X^ϱ when $X_\lambda = X$ for every λ, and $X_0 \times X_1$ when $\varrho = 2$.

If ϱ is an ordinal number and $\{X_\lambda\}_{\lambda<\varrho}$ is a well-ordered collection of pairwise disjont LOTS, we denote by $\sum_{\lambda<\varrho} X_\lambda$ the ordered sum of the X_λ, i.e. the set union of the X_λ equipped with the following order:

$$x < y \quad \begin{array}{l} \text{if } x \in X_\lambda\,, \quad y \in X_\mu \text{ and } \lambda < \mu \\ \text{if } x \in X_\lambda\,, \quad y \in X_\lambda \text{ and } x < y \text{ in } X_\lambda\,. \end{array}$$

It can be easily proved that the lexicographically ordered product of arcs is an arc. The same is not generally true for smooth arcs.

In [1] R. Arens proved that the lexicographically ordered product X^ω of a smooth arc X is a smooth arc. Our aim is to generalize such result.

For a smooth arc X, the following properties were proved [3]:

1. $|\omega| < |X| \leqslant 2^{|\omega|}$

2. X is first countable

3. $d(X) = w(X) \leqslant 2^{|\omega|}$.

Lemma 1. *If* $\{X_\lambda\}_{\lambda<\varrho}$ *is a well-ordered collection of pairwise disjoint* LOTS *and* Y *is a* LOTS *then* $\sum_{\lambda<\varrho}(X_\lambda \times Y) \approx (\sum_{\lambda<\varrho} X_\lambda) \times Y$.

Proof. It follows immediately from the definitions.

Lemma 2. *If* $X \approx [x^0, x^1]$ *is a smooth arc and* $\{X_\lambda\}_{\lambda<\varrho}$ *is a well-ordered countable collection of pairwise disjoint* LOTS *such that* $X_\lambda \approx [x^0, x^1[$, *then* $\sum_{\lambda<\varrho} X_\lambda \approx [x^0, x^1[$.

Proof. Since ϱ is countable, we can consider in X an increasing transfinite sequence $(a_\lambda)_{\lambda<\varrho}$ of type ϱ. Since X is a smooth arc, we have:

$$X_\lambda \approx [x^0, x^1[\approx [a_\lambda, a_{\lambda+1}[.$$

Let $k = \sup\{a_\lambda\}_{\lambda<\varrho}$. We obtain

$$\sum_{\lambda<\varrho} X_\lambda \approx \sum_{\lambda<\varrho} [a_\lambda, a_{\lambda+1}[= [a_0, k[\approx [x^0, x^1[.$$

Lemma 3. *Let α, δ, γ be ordinal numbers. If $\delta, \gamma < \omega^\alpha$, then $\delta + \gamma < \omega^\alpha$.*

Proof. It can be easily proved by transfinite induction with respect to α.

Lemma. 4. *Let α, δ be ordinal numbers. If $\delta < \omega^\alpha$, then $[\delta, \omega^\alpha[$ has the same order type as ω^α.*

Proof. From Lemma 3 it follows that the map: $f\colon [0, \omega^\alpha[\to [\delta, \omega^\alpha[$ defined by $f(\lambda) = \delta + \lambda$ is an isomorphism.

Lemma 5. *Let X be a smooth arc and let $\{X_\lambda\}_{\lambda<\omega^\alpha}$ be a well-ordered collection of arcs such that $X_\lambda \approx X$. If $\delta < \omega^\alpha$, then*

$$X^{\omega^\alpha} \approx \prod_{\lambda<\omega^\alpha} X_\lambda \approx \prod_{\delta\leq\lambda<\omega^\alpha} X_\lambda .$$

Proof. If follows directly from Lemma 4.

We can now prove

Theorem 1. *If X is a smooth arc and α is a countable ordinal number, then X^{ω^α} is a smooth arc.*

Proof. Put $X = [x^0, x^1]$, $X^{\omega^\alpha} = [y^0, y^1]$. We first prove that if $q = (q_\lambda)_\lambda$ is a point of X^{ω^α} different from y^0, then $[y^0, q]$ is isomorphic to $[y^0, y^1]$.

Let $D = \{\delta < \omega^\alpha \mid x^0 < q_\delta\}$. For every $\delta \in D$, we consider the set

$$V_\delta = \{(y_\lambda)_\lambda \in X^{\omega^\alpha} \mid y_\delta < q_\delta,\ y_\lambda = q_\lambda \text{ for every } \lambda < \delta\}.$$

The following relations can be easily verified:

1. $V_\delta < \{q\}$
2. $\delta < \delta' \Rightarrow V_\delta < V_{\delta'}$
3. $[y^0, q[= \bigcup_{\delta\in D} V_\delta \approx \sum_{\delta\in D} V_\delta .$

Moreover, it holds $V_\delta \approx [x^0, q_\delta[\times (\prod_{\delta+1 \leq \lambda < \omega^\alpha} X_\lambda)$ where $X_\lambda = X$. From Lemma 5 it follows $(\prod_{\delta+1 \leq \lambda < \omega^\alpha} X_\lambda) \approx X^{\omega^\alpha}$ and then $V_\delta \approx [x^0, q_\delta[\times X^{\omega^\alpha}$.

Put $Z_\delta = [x^0, q_\delta[$. From (3) it follows $[y^0, q[\approx \sum_{\delta \in D} V_\delta \approx \sum_{\delta \in D} (Z_\delta \times X^{\omega^\alpha})$ and by Lemma 1 $\sum_{\delta \in D} (Z_\delta \times X^{\omega^\alpha}) \approx (\sum_{\delta \in D} Z_\delta) \times X^{\omega^\alpha}$.

Since $[x^0, x^1]$ is a smooth arc, Z_δ is isomorphic to $[x^0, x^1[$. From Lemma 2 we obtain $\sum_{\delta \in D} Z_\delta \approx [x^0, x^1[$, and then $[y^0, q[\approx [x^0, x^1[\times X^{\omega^\alpha}$. All intervals $[y^0, q[$ $(y^0 < q \leq y^1)$ are also isomorphic to the same LOTS. It follows that $[y^0, q[\approx [y^0, y^1[$. If we add the points q and y^1 to $[y^0, q[$ and $[y^0, y^1[$, we obtain $[y^0, q] \approx [y^0, y^1]$.

In the same way it can be proved that if p is a point of X^{ω^α} different from y^1, then $[p, y^1]$ is isomorphic to $[y^0, y^1]$.

Let now p, q $(p < q)$ be points of X^{ω^α}. $[p, q]$ can be considered as an initial interval of $[p, y^1]$. Then we have $[y^0, y^1] \approx [p, y^1] \approx [p, q]$. This proves that X^{ω^α} is a smooth arc.

Example. If I denotes the real interval $[0, 1]$, the most natural examples of smooth arcs are given by $I^{\omega^\alpha} (\alpha < \omega_1)$. It could be proved (but the proof is not immediate and we do not give it here) that for $\alpha \neq \beta$, I^{ω^α} is not isomorphic to I^{ω^β}. We obtain also, for $0 < \alpha < \omega_1$, an uncountable family of non-real smooth arcs such that no two of them are isomorphic.

References

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Sommario

Nell'articolo si generalizza un procedimento di R. Arens per la costruzione di archi ordinatamente omogenei. Più precisamente, si fa vedere che opportune potenze lessicografiche di un arco ordinatamente omogeneo sono ancora archi ordinatamente omogenei

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