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R. CAVAZZONI and G. SPIGA (*)

# A note on Grad's 13 moment expansion in extended kinetic theory (**)

## 1 - Introduction

The role played by the expansion of the unknown distribution function in tensor Hermite polynomials, proposed by Grad [3] for the solution of the Boltzmann equation, is well known in kinetic theory [4]. In particular, Grad's 13 moment approximation has allowed a considerable improvement with respect to classical thermodynamics and its traditional local Maxwellian approximation of the distribution function, and has shed new light on the resolution of several historical paradoxes [1].

More precisely, the 13 moment method has constituted the basis for the introduction of a new significant theory of Mathematical Physics in the scientific literature, the so called Extended Thermodynamics, which is widely and successfully studied and applied nowadays [5]. In addition, in the hydrodynamic limit of collisionally dominated fluid, when one is interested in bulk correction terms up to the first order in the stiffness parameter, the 13 moment equations allow a very easy asymptotic analysis versus such a small parameter (the Knudsen number). The analysis yields straightforwardly [4], to first order, Newton's law for the deviatoric stress tensor and Fourier's law for the heat flux, as constitutive equations; the Navier-Stokes equations of fluid dynamics follow then immediately, without going through the Chapmann-Enskog procedure [2].

In the spirit of the last remark above, the present note is a first step towards a possible employment of the 13 moment approach in the frame of Extended

(*) Dip. di Matem., Univ. Parma, Via M. D'Azeglio 85, 43100 Parma, Italia.
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Kinetic Theory. In such a theory, recently introduced in the literature, interaction phenomena different from elastic scattering, like chemical reactions, absorptions, and generating processes, are taken into account and the combined effects of nonlinearity and nonconservativity are studied [6].

The first generalization needed is considering a mixture of several participating gases in the presence of one or more background species. Confining ourselves here to elastic scattering only, the relevant 13 moment equations are derived in Sec. 2. The important particular case of a one species gas of test particles (t.p.), interacting between themselves and with a fixed background of field particles (f.p.), in which they are embedded, is analyzed in Sec. 3. Linear collision terms appear thus together with the usual nonlinear ones, and t.p. momentum and kinetic energy are not conserved any longer.

Particular emphasis is given to the small mean free path asymptotics, according to the presence of two different mean free paths, and then of the different possible values of their ratio. It is shown that quite different asymptotic scenarios arise, depending on the relative importance, in the phase space balance, of t.p.-t.p. collisions on one side, and t.p-f.p. collisions on the other. Possible outputs include diffusion approximation and generalized Navier-Stokes equations.

## 2 - Moment equations

The starting point is the set of $N$ *Boltzmann equations* for the distribution functions $f^\alpha(\boldsymbol{x}, \boldsymbol{v}, t)$ of a rarefied gas mixture, which, in standard notation, reads as

$$(1) \qquad \begin{gathered} \frac{\partial f^\alpha}{\partial t} + v_i \frac{\partial f^\alpha}{\partial x_i} = \sum_{\beta=1}^{N} I^{\alpha\beta}[f^\alpha, f^\beta] \qquad \alpha = 1, 2, \dots, N \\ I^{\alpha\beta}[f^\alpha, f^\beta] = \int\int g\, \sigma^{\alpha\beta}(g, \chi)[f^\alpha(\boldsymbol{v}')f^\beta(\boldsymbol{w}') - f^\alpha(\boldsymbol{v})\, f^\beta(\boldsymbol{w})]\, \mathrm{d}\boldsymbol{w}\, \mathrm{d}\boldsymbol{n}' \end{gathered}$$

where summation over repeated arabic indices is implicitly understood and $g$ stands for $|\boldsymbol{v} - \boldsymbol{w}|$.

*Postcollisional velocities* are given by

$$\boldsymbol{v}' = r_{\alpha\beta}\boldsymbol{v} + r_{\beta\alpha}(\boldsymbol{w} + g\boldsymbol{n}') \qquad\qquad \boldsymbol{w}' = r_{\alpha\beta}(\boldsymbol{v} - g\boldsymbol{n}') + r_{\beta\alpha}\boldsymbol{w}$$

with $r_{\alpha\beta} = m^\alpha(m^\alpha + m^\beta)^{-1}$.

The main macroscopic fields of practical interest are *number* and *mass density, drift velocity*, and *temperature*

$$(3)\quad n^{\alpha} = \int f^{\alpha}\,\mathrm{d}\boldsymbol{v}\,,\quad \varrho^{\alpha} = m^{\alpha} n^{\alpha}\,,\quad \boldsymbol{u}^{\alpha} = \frac{1}{n^{\alpha}} \int \boldsymbol{v} f^{\alpha}\,\mathrm{d}\boldsymbol{v}\,,\quad T^{\alpha} = \frac{1}{n^{\alpha}}\,\frac{m^{\alpha}}{3K} \int (\boldsymbol{c}^{\alpha})^2 f^{\alpha}\,\mathrm{d}\boldsymbol{v}$$

where $\boldsymbol{c}^{\alpha} = \boldsymbol{v} - \boldsymbol{u}^{\alpha}$.

It is well known that the conservation equations are not selfconsistent in the above moments, since they involve also *pressure tensor* and *heat flux*

$$(4)\qquad P_{ij}^{\alpha} = m^{\alpha} \int c_i^{\alpha} c_j^{\alpha} f^{\alpha}\,\mathrm{d}\boldsymbol{v} \qquad q_i^{\alpha} = \tfrac{1}{2}\, m^{\alpha} \int c_i^{\alpha} (\boldsymbol{c}^{\alpha})^2 f^{\alpha}\,\mathrm{d}\boldsymbol{v}\,.$$

It is worth introducing the auxiliary quantities

$$(5)\qquad p^{\alpha} = \frac{1}{3}\,\mathrm{tr}\, P_{ij}^{\alpha} = n^{\alpha} K T^{\alpha} \qquad U^{\alpha} = \frac{3KT^{\alpha}}{2m^{\alpha}} \qquad p_{ij}^{\alpha} = P_{ij}^{\alpha} - p^{\alpha}\delta_{ij}$$

where $K$ denotes *Boltzmann's constant* and $\delta_{ij}$ Kronecker's symbol. In particular, $p_{ij}^{\alpha}$ represents the (traceless) *deviatoric stress tensor* for species $\alpha$.

Later on, use will be made of the symbol

$$(6)\qquad \langle \tau_{ij} \rangle = \tfrac{1}{2}\,\tau_{ij} + \tfrac{1}{2}\,\tau_{ji} - \tfrac{1}{3}\,\tau_{kk}\delta_{ij}$$

for a general tensor $\boldsymbol{\tau}$.

Grad's procedure consists now in representing each *distribution function* as

$$(7)\qquad \begin{aligned} f^{\alpha}(\boldsymbol{x}, \boldsymbol{v}, t) &= n^{\alpha} \Big(\frac{m^{\alpha}}{2\pi K T^{\alpha}}\Big)^{\frac{3}{2}} e^{-\frac{m^{\alpha}}{2KT^{\alpha}}(\boldsymbol{c}^{\alpha})^2} \\ &\cdot \Big\{1 + \frac{1}{2p^{\alpha}}\, p_{ij}^{\alpha}\, \frac{m^{\alpha}}{KT^{\alpha}}\, c_i^{\alpha} c_j^{\alpha} + \frac{1}{5p^{\alpha}}\, \frac{m^{\alpha}}{KT^{\alpha}}\, q_i^{\alpha} c_i^{\alpha} \Big[\frac{m^{\alpha}}{KT^{\alpha}} (\boldsymbol{c}^{\alpha})^2 - 5\Big]\Big\} \end{aligned}$$

namely as a truncated Hermite polynomials expansion in which only the moments of interest are retained. In this approximation, a closed set of moment equations results from equations (1) by integration with respect to the velocity variable, after multiplication by $m^{\alpha}$, $m^{\alpha} v_i$, $\frac{1}{2} m^{\alpha} (\boldsymbol{c}^{\alpha})^2$, $m^{\alpha} c_i^{\alpha} c_j^{\alpha}$, $\frac{1}{2} m^{\alpha} c_i^{\alpha} (\boldsymbol{c}^{\alpha})^2$, respectively.

As well known from the classical theory, the derivation requires quite detailed and cumbersome manipulations, with some additional care for cross interactions of different species. It will only be sketched below, under the simplifying assumption of Maxwell molecule cross sections, i.e.

$$(8)\qquad \sigma^{\alpha\beta}(g, \chi) = \frac{1}{g}\, B^{\alpha\beta}(\chi)$$

where $\cos\chi = \boldsymbol{n}'\cdot\boldsymbol{n}$, with $\boldsymbol{n} = \dfrac{1}{g}(\boldsymbol{v}-\boldsymbol{w})$. Integrals of collision terms can be cast as

$$(9)\qquad \int \varphi(\boldsymbol{v}) I^{\alpha\beta}[f^\alpha, f^\beta]\,\mathrm{d}\boldsymbol{v} = \int\int f^\alpha(\boldsymbol{v}) f^\beta(\boldsymbol{w})\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{w} \int B^{\alpha\beta}(\chi)[\varphi(\boldsymbol{v}') - \varphi(\boldsymbol{v})]\,\mathrm{d}\boldsymbol{n}'$$

so that the angular integrations can be explicitly performed in terms of

$$(10)\qquad \begin{aligned} &\int B^{\alpha\beta}(\chi)\,\mathrm{d}\boldsymbol{n}' = B_0^{\alpha\beta} \qquad \int n_i' B^{\alpha\beta}(\chi)\,\mathrm{d}\boldsymbol{n}' = B_1^{\alpha\beta} n_i \\ &\int n_i' n_j' B^{\alpha\beta}(\chi)\,\mathrm{d}\boldsymbol{n}' = \tfrac{1}{2} B_2^{\alpha\beta}\delta_{ij} + (B_0^{\alpha\beta} - \tfrac{3}{2} B_2^{\alpha\beta}) n_i n_j \end{aligned}$$

where

$$(11)\qquad B_k^{\alpha\beta} = 2\pi\int_0^\pi (\cos\chi)^k B^{\alpha\beta}(\chi)\,\mathrm{sen}\,\chi\,\mathrm{d}\chi \quad k = 0{,}1 \qquad B_2^{\alpha\beta} = 2\pi\int_0^\pi B^{\alpha\beta}(\chi)\,\mathrm{sen}^3\chi\,\mathrm{d}\chi\,.$$

Now, omitting all details, the sought macroscopic set of PDE turns out to be given, after some algebra, as follows. *Mass conservation* takes the standard form

$$(12)_a\qquad \frac{\partial\varrho^\alpha}{\partial t} + \frac{\partial}{\partial x_i}(\varrho^\alpha u_i^\alpha) = 0\,.$$

*Momentum conservation* is of course valid only for the mixture as a whole, but for each species we get ($i = 1, 2, 3$)

$$(12)_b\qquad \varrho^\alpha\Big(\frac{\partial u_i^\alpha}{\partial t} + u_j^\alpha\frac{\partial u_i^\alpha}{\partial x_j}\Big) + \frac{\partial p^\alpha}{\partial x_i} + \frac{\partial p_{ij}^\alpha}{\partial x_j} = -\varrho^\alpha\sum_{\beta=1}^N r_{\beta\alpha} n^\beta (B_0^{\alpha\beta} - B_1^{\alpha\beta})(u_i^\alpha - u_i^\beta)\,.$$

The same occurs to *energy conservation*, for which one ends up with

$$(12)_c\qquad \begin{aligned} &\varrho^\alpha\Big(\frac{\partial U^\alpha}{\partial t} + u_i^\alpha\frac{\partial U^\alpha}{\partial x_i}\Big) + p^\alpha\frac{\partial u_k^\alpha}{\partial x_k} + p_{ij}^\alpha\frac{\partial u_j^\alpha}{\partial x_i} + \frac{\partial q_k^\alpha}{\partial x_k} \\ &= -n^\alpha\sum_{\beta=1}^N n^\beta (B_0^{\alpha\beta} - B_1^{\alpha\beta})[3r_{\alpha\beta}r_{\beta\alpha}K(T^\alpha - T^\beta) - r_{\beta\alpha}^2 m^\alpha(\boldsymbol{u}^\alpha - \boldsymbol{u}^\beta)^2]\,. \end{aligned}$$

The equations for the *deviatoric stress* $(i, j = 1, 2, 3)$ read as

$$
\begin{aligned}
&\frac{\partial p_{ij}^{\alpha}}{\partial t} + \frac{\partial}{\partial x_k}(u_k^{\alpha} p_{ij}^{\alpha}) + p_{ik}^{\alpha} \frac{\partial u_j^{\alpha}}{\partial x_k} + p_{jk}^{\alpha} \frac{\partial u_i^{\alpha}}{\partial x_k} \\
&- \frac{2}{3} \delta_{ij} p_{kl}^{\alpha} \frac{\partial u_l^{\alpha}}{\partial x_k} + 2p^{\alpha} \langle \frac{\partial u_i^{\alpha}}{\partial x_j} \rangle + \frac{4}{5} \langle \frac{\partial q_i^{\alpha}}{\partial x_j} \rangle \\
&= -2 \sum_{\beta=1}^{N} (B_0^{\alpha\beta} - B_1^{\alpha\beta})[r_{\alpha\beta} r_{\beta\alpha} (n^{\beta} p_{ij}^{\alpha} - n^{\alpha} p_{ij}^{\beta}) - r_{\beta\alpha}{}^2 n^{\alpha} n^{\beta} m^{\alpha} \langle (u_i^{\alpha} - u_i^{\beta})(u_j^{\alpha} - u_j^{\beta}) \rangle] \\
&- \frac{3}{2} \sum_{\beta=1}^{N} B_2^{\alpha\beta} (r_{\beta\alpha}{}^2 n^{\beta} p_{ij}^{\alpha} + r_{\alpha\beta} r_{\beta\alpha} n^{\alpha} p_{ij}^{\beta} + r_{\beta\alpha}{}^2 n^{\alpha} n^{\beta} m^{\alpha} \langle (u_i^{\alpha} - u_i^{\beta})(u_j^{\alpha} - u_j^{\beta}) \rangle).
\end{aligned}
\tag{12$_d$}
$$

Only five of them, out of six, are independent, for each $\alpha$, since the sum of the diagonal ones $(i = j)$ yields the identity $0 = 0$.

Finally, the equations for *heat flux* may be written as $(i = 1, 2, 3)$

$$
\begin{aligned}
&\frac{\partial q_i^{\alpha}}{\partial t} + \frac{\partial}{\partial x_k}(u_k^{\alpha} q_i^{\alpha}) + \frac{2}{5} q_j^{\alpha} \frac{\partial u_j^{\alpha}}{\partial x_i} + \frac{7}{5} q_j^{\alpha} \frac{\partial u_i^{\alpha}}{\partial x_j} + \frac{2}{5} q_i^{\alpha} \frac{\partial u_k^{\alpha}}{\partial x_k} \\
&+ \frac{5}{2} p^{\alpha} \frac{\partial}{\partial x_i}(\frac{KT^{\alpha}}{m^{\alpha}}) + \frac{7}{2} p_{ij}^{\alpha} \frac{\partial}{\partial x_j}(\frac{KT^{\alpha}}{m^{\alpha}}) + \frac{KT^{\alpha}}{m^{\alpha}} \frac{\partial p_{ij}^{\alpha}}{\partial x_j} \\
&- \frac{1}{\varrho^{\alpha}} p_{ij}^{\alpha} \frac{\partial p^{\alpha}}{\partial x_j} - \frac{1}{\varrho^{\alpha}} p_{ij}^{\alpha} \frac{\partial p_{jk}^{\alpha}}{\partial x_k} \\
&= - \sum_{\beta=1}^{N} (B_0^{\alpha\beta} - B_1^{\alpha\beta})[(3r_{\alpha\beta}{}^2 r_{\beta\alpha} + r_{\beta\alpha}{}^3) n^{\beta} q_i^{\alpha} - 4r_{\alpha\beta} r_{\beta\alpha}{}^2 n^{\alpha} q_i^{\beta} \\
&-10 r_{\alpha\beta} r_{\beta\alpha}{}^2 (n^{\beta} p^{\alpha} - n^{\alpha} p^{\beta})(u_i^{\alpha} - u_i^{\beta}) - 4r_{\alpha\beta} r_{\beta\alpha}{}^2 (n^{\beta} p_{ij}^{\alpha} - n^{\alpha} p_{ij}^{\beta})(u_j^{\alpha} - u_j^{\beta}) \\
&+ 2r_{\beta\alpha}{}^3 n^{\alpha} n^{\beta} m^{\alpha} (u_i^{\alpha} - u_i^{\beta})(\boldsymbol{u}^{\alpha} - \boldsymbol{u}^{\beta})^2] \\
&- \sum_{\beta=1}^{N} B_2^{\alpha\beta} [2r_{\alpha\beta} r_{\beta\alpha}{}^2 (n^{\beta} q_i^{\alpha} + n^{\alpha} q_i^{\beta}) + 5r_{\alpha\beta} r_{\beta\alpha}{}^2 n^{\alpha} n^{\beta} K(T^{\alpha} - T^{\beta})(u_i^{\alpha} - u_i^{\beta}) \\
&- 2r_{\alpha\beta} r_{\beta\alpha}{}^2 n^{\alpha} p_{ij}^{\beta} (u_j^{\alpha} - u_j^{\beta}) - (\frac{3}{2} r_{\beta\alpha}{}^3 - \frac{1}{2} r_{\alpha\beta} r_{\beta\alpha}{}^2) n^{\beta} p_{ij}^{\alpha} (u_j^{\alpha} - u_j^{\beta}) \\
&- r_{\beta\alpha}{}^3 n^{\alpha} n^{\beta} m^{\alpha} (u_i^{\alpha} - u_i^{\beta})(\boldsymbol{u}^{\alpha} - \boldsymbol{u}^{\beta})^2].
\end{aligned}
\tag{12$_e$}
$$

In the case $N = 1$, the right hand sides vanish in $(12)_a$, $(12)_b$, $(12)_c$ and collapse to the known expressions $-\frac{3}{4}B_2 n p_{ij}$ in $(12)_d$ and $-\frac{1}{2}B_2 n q_i$ in $(12)_e$. Collision contributions all vanish, as physically clear, in the limiting case $r_{\alpha\beta} \to 1$ (i.e. $r_{\beta\alpha} \to 0$). In the opposite limiting case, $r_{\alpha\beta} \to 0$, one can notice the disappearance of the damping term proportional to $T^\alpha - T^\beta$ in $(12)_c$: energy conservation applies in fact to each species in all $\alpha\beta$ collisions. It is also worth remarking that the microscopic details of each collision are represented solely by the integrated microscopic collision frequencies [4]

$$(13) \qquad C_k^{\alpha\beta} = 2\pi \int_0^\pi (1 - \cos^k \chi) B^{\alpha\beta}(\chi) \operatorname{sen} \chi \, d\chi \qquad k = 1, 2 .$$

Finally, the presence of one or more background species can be accounted for by imposing the relevant distribution function to be fixed independently from the process going on, and by dropping the equations relevant to that species from the set (12) [6].

## 3 - Single species in a background host medium

We shall consider here the physical problem sketched in the Introduction of a single gas, whose molecules may interact between themselves, as well as with the background particles.

The latter will be supposed to be in an equilibrium state, with constant number density $\tilde{n}$ and temperature $\tilde{T}$. Drift velocity may always be assumed to be zero, by a proper choice of the reference frame, and in addition deviatoric stress and heat flux are bound to vanish. Unnecessary indices are dropped and a tilda is used to label quantities relevant to the background.

We will resort to the dimensionless form of equations (1) or (12), which leads to the apperance, in front of the collision terms, of inverse Knudsen numbers, namely of the dimensionless mean free paths $\bar{\varepsilon}$ and $\tilde{\varepsilon}$ for t.p.-t.p. and for t.p.-f.p. encounters, respectively. Attention will be focussed on a formal derivation of the asymptotic limit when $\bar{\varepsilon}$ and/or $\tilde{\varepsilon}$ tend to zero, along the same line which leads to the hydrodynamic limit represented by the Navier-Stokes equations for the classical Boltzmann equation, without background medium. It is clear that such a limit will crucially depend on the ratio $\bar{\varepsilon}/\tilde{\varepsilon}$, namely on the kind of collision which essentially drives the process.

The 13 *moment equations* read as

$$
\begin{aligned}
&D_a(\varrho) = 0 \qquad D_b(u_i) = -\frac{1}{\tilde{\varepsilon}}\,\tilde{n}\,\tilde{C}_1\tilde{r}\varrho u_i \\
&D_c(U) = -\frac{1}{\tilde{\varepsilon}}\,\tilde{n}\,\tilde{C}_1 n[3r\tilde{r}K(T-\tilde{T}) - \tilde{r}^2 m\boldsymbol{u}^2] \\
&D_d(p_{ij}) = -\frac{2}{\tilde{\varepsilon}}\,\tilde{n}\,\tilde{C}_1(r\tilde{r}p_{ij} - \tilde{r}^2 nm\langle u_i u_j\rangle) - \frac{3}{2\tilde{\varepsilon}}\,\tilde{n}\,\tilde{C}_2(\tilde{r}^2 p_{ij} + \tilde{r}^2 nm\langle u_i u_j\rangle) \\
&\qquad - \frac{3}{4\bar{\varepsilon}}\,C_2 n p_{ij} \\
(14)\quad &D_e(q_i) = -\frac{1}{\tilde{\varepsilon}}\,\tilde{n}\,\tilde{C}_1[(\tilde{r}^3 + 3r^2\tilde{r})q_i - 10r\tilde{r}^2 nK(T-\tilde{T})u_i] \\
&\qquad + \frac{1}{3}n\tilde{C}_1[4r\tilde{r}^2 p_{ij}u_j - 2\tilde{r}^3 nmu_i\boldsymbol{u}^2] \\
&\qquad - \frac{1}{\tilde{\varepsilon}}\,\tilde{n}\,\tilde{C}_2[2r\tilde{r}^2 q_i + 5r\tilde{r}^2 nK(T-\tilde{T})u_i] \\
&\qquad + \frac{1}{\tilde{\varepsilon}}\tilde{n}\tilde{C}_2\Big[\frac{3\tilde{r}^3 - r\tilde{r}^2}{2}\,p_{ij}u_j + \tilde{r}^3 nmu_i\boldsymbol{u}^2\Big] - \frac{1}{2\bar{\varepsilon}}\,C_2 n q_i
\end{aligned}
$$

where $D_p$ stands for the left hand side of equation $(12)_p$, $p = a, \ldots, e$.

Of course, (14) is a set of singularly perturbed PDE, but, if one is interested only in the asymptotic limit for the bulk region and after the initial transient, initial and boundary layer effects can be neglected (and resumed later to set up initial and boundary conditions), and macroscopic fields may be expanded in non-negative powers of the chosen small parameter $\varepsilon$ (the scaling factor), e.g.

$$
u_i = u_i^{(0)} + \varepsilon u_i^{(1)} + \ldots . \tag{15}
$$

The asymptotic analysis consists then in comparing equal powers of $\varepsilon$.

We examine below some meaningful particular cases for the ratio $\bar{\varepsilon}/\tilde{\varepsilon}$.

**Case A.** When t.p.-t.p. and t.p.-f.p. collisions are equally important (i.e. the two mean free paths are of the same order, both small on a macroscopic scale), we may set $\bar{\varepsilon} = \tilde{\varepsilon} = \varepsilon$.

Upon expanding $u_i$, $T$, $p_{ij}$, $q_i$ it is easily found that, to leading order, we have

$$
u_i^{(0)} = 0 \qquad T^{(0)} = \tilde{T} \qquad p_{ij}^{(0)} = 0 \qquad q_i^{(0)} = 0 . \tag{16}
$$

Going on to first order approximations, one gets

$$u_i^{(1)} = -\frac{1}{\tilde{r}\tilde{n}\tilde{C}_1}\frac{K\tilde{T}}{\varrho}\frac{\partial n}{\partial x_i} \qquad T^{(1)} = 0 \qquad p_{ij}^{(1)} = 0 \qquad q_i^{(1)} = 0\,. \tag{17}$$

Inserting this into the first of equations (14) yields for the density the *diffusion equation*

$$\frac{\partial n}{\partial t} = \frac{\varepsilon}{\tilde{r}\tilde{n}\tilde{C}_1}\frac{K\tilde{T}}{m}\nabla^2 n \tag{18}$$

with an $O(\varepsilon)$ diffusion coefficient, determined only by background properties.

Effects of t.p.-t.p. collisions arise only in the second order approximation

$$\begin{aligned} u_i^{(2)} &= \frac{K\tilde{T}}{(\tilde{r}\tilde{n}\tilde{C}_1)^2}\frac{\partial}{\partial t}\left(\frac{1}{\varrho}\frac{\partial n}{\partial x_i}\right) \qquad\qquad q_i^{(2)} = 0 \\ T^{(2)} &= \frac{K\tilde{T}^2}{3r\tilde{r}(\tilde{n}\tilde{C}_1)^2}\left[\frac{1}{\tilde{r}}\frac{\partial}{\partial x_i}\left(\frac{1}{\varrho}\frac{\partial n}{\partial x_i}\right) + \frac{m}{\varrho^2}\frac{\partial n}{\partial x_i}\frac{\partial n}{\partial x_i}\right] \\ p_{ij}^{(2)} &= \left(2r\tilde{r}\tilde{n}\tilde{C}_1 + \frac{3}{2}\tilde{r}^2\tilde{n}\tilde{C}_2 + \frac{3}{4}nC_2\right)^{-1} \\ &\cdot\left[\frac{2nK^2\tilde{T}^2}{\tilde{r}\tilde{n}\tilde{C}_1}\left\langle\frac{\partial}{\partial x_j}\left(\frac{1}{\varrho}\frac{\partial n}{\partial x_i}\right)\right\rangle + \left(2\tilde{C}_1 + \frac{3}{2}\tilde{C}_2\right)\frac{K^2\tilde{T}^2}{\tilde{n}\tilde{C}_1^2}\frac{1}{\varrho}\left\langle\frac{\partial n}{\partial x_i}\frac{\partial n}{\partial x_j}\right\rangle\right]. \end{aligned} \tag{19}$$

If the first of (17) and (19) are used for $u_i$ in the first of (14), a second order correction to (18) results, with third derivatives of $n$. In general, all fields except density are determined algebraically at each step in terms of $n$ and of the background, and a single PDE for $n$ follows. In particular, $u_i = O(\varepsilon)$, $T - \tilde{T} = O(\varepsilon^2)$, $p_{ij} = O(\varepsilon^2)$, and $q_i = O(\varepsilon^3)$.

The case in which instead the host medium has a local Maxwellian distribution could be treated in the same way with corresponding results. Gradients and time derivatives of $\tilde{n}$, $\tilde{u}_i$, $\tilde{T}$ would affect the expressions for $u_i$, $T$, $p_{ij}$, $q_i$ at each step, and the quantities $u_i - \tilde{u}_i$, $T - \tilde{T}$, $p_{ij}$, $q_i$ would all be $O(\varepsilon)$. Details are omitted for brevity.

**Case B.** When only t.p.-t.p. encounters are dominant, whereas the t.p.-f.p. mean free path is of the same order of the macroscopic lengths, we may set $\bar{\varepsilon} = \varepsilon$ and $\tilde{\varepsilon} = 1$, and proceed in the same way as before. Expanding $p_{ij}$, $q_i$ one gets immediately, at the lowest order in $\varepsilon$

$$p_{ij}^{(0)} = 0 \qquad q_i^{(0)} = 0 . \tag{20}$$

Now $p_{ij}$, $q_i$ are both $O(\varepsilon)$ and the first correction is easily obtained at the next step

$$\begin{aligned} p_{ij}^{(1)} &= -\frac{8}{3}\frac{KT}{C_2}\langle\frac{\partial u_i}{\partial x_j}\rangle + \frac{2}{3}\tilde{r}^2\tilde{n}\,\frac{4\tilde{C}_1 - 3\tilde{C}_2}{C_2}\,m\langle u_i u_j\rangle \\ q_i^{(1)} &= -5\,\frac{K^2 T}{mC_2}\,\frac{\partial T}{\partial x_i} + 2\tilde{r}^2\tilde{n}\,\frac{2\tilde{C}_1 - \tilde{C}_2}{C_2}\,(5rK(T-\tilde{T}) - \tilde{r}mu^2)\,u_i . \end{aligned} \tag{21}$$

In the absence of background medium, only the first addend would be left in each of the right hand sides and the well known Newton's and Fourier's laws would be recovered. The additional terms are thus corrections to the deviatoric stress and to the heat flux due to the presence of collisions with f.p.

If equations (21) are inserted into the first three equations in (14) (with $\tilde{\varepsilon} = 1$), a set of generalized Navier-Stokes equations results, which again is not written down here for brevity.

It is worth remarking that the effects of background are present as $O(1)$ corrections in the right hand sides, due to non-conservation of momentum and energy for t.p., and as $O(\varepsilon)$ corrections in the left hand sides, due to f.p. contributions to stress tensor and heat flux.

Contrary to case A, the asymptotic limit yields now a generalized hydrodynamics. Burnett and higher order type of approximations would arise by pushing further the expansion in $\varepsilon$.

**Case C.** When t.p.-f.p. collisions alone are dominant, whereas the t.p.-t.p. mean free path is of the same order of the macroscopic lenghts, we may put $\tilde{\varepsilon} = \varepsilon$ and $\bar{\varepsilon} = 1$. This case includes the purely linear problem in which t.p.-t.p. encounters are negligible: it is sufficient to put formally $C_2 = 0$. The first two steps are the same as for case A, and again the diffusion equation (18) arises for the density. Differences appear only from the second order corrections on. For instance, equation (19) is modified only in that the addend $\frac{3}{4}C_2 n$ should be dropped (t.p.-t.p. collisions affect only the third order). The qualitative situation remains the same.

Other cases could easily be devised and analyzed. For instance, the one in which $\bar{\varepsilon} = \varepsilon^2$ and $\tilde{\varepsilon} = \varepsilon$ (t.p.-t.p. mean free path small versus t.p.-f.p. mean free path, in turn small on a macroscopic scale). By expanding $u_i$, $T$, $p_{ij}$, $q_i$, (16), (17) and then (18) are still in order, but the further development shows that we have $T - \widetilde{T} = O(\varepsilon^2)$, $p_{ij} = O(\varepsilon^3)$ and $q_i = O(\varepsilon^4)$.

A more detailed analysis of the several possible physical situations, consideration of inelastic scattering and investigation of the Extended Thermodynamics arising from Extended Kinetic Theory, will hopefully be matter of future research.

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## Sommario

*Nell'ambito della Teoria Cinetica Estesa vengono ricavate le equazioni dei 13 momenti di Grad per una miscela di gas e vengono studiati i possibili limiti asintotici, al tendere a zero dei numeri di Knudsen, per un gas di particelle immerse in un mezzo di supporto.*

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