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# On pseudoconvexity of Reinhardt domains (**)

## 1 - Introduction

Let $D$ be an open connected bounded domain in $\boldsymbol{C}^2$ and let us denote by

$$\log |D| = \{(x_1, x_2) | (e^{x_1}, e^{x_2}) \in D\} \subset \boldsymbol{R}^2$$

its logarithmic image.

When $D$ is Reinhardt, that is invariant under the maps

$$(z_1, z_2) \to (e^{i\theta_1} z_1, e^{i\theta_2} z_2), \qquad \forall \theta_1, \theta_2 \in \boldsymbol{R}$$

the following is a classical statement.

Theorem 1. *Assume* $(0, 0) \in D$. *Then* $D$ *is a domain of holomorphy if and only if it is complete (that is* $(z_1^0, z_2^0) \in D$ *implies that the bidisc* $\{|z_1| < |z_1^0|, |z_2| < |z_2^0|\}$ *is contained in* $D$*) and is log-convex (that is* $\log |D|$ *is a convex set).*

The aim of this note is to show that, in such a theorem, the hypothesis of log-convexity is the essential point, while the requirement of completeness may be considerably weakened, being almost (but of course not entirely) a by-product of the other hypotheses. In other words, we prove several properties enjoyed by any log-convex Reinhardt domain (even not containing the origin): such properties imply, in particular, that a Reinhardt domain $D$, whose closure $\overline{D}$ has non empty intersection with all coordinate hyperplanes $z_i = 0$, is log-convex if and only if $\overline{D}$ is complete.

It this note, we show that the properties of logarithmically convex Reinhardt domains bring to an interesting pair of conclusions:

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**a.** the hypothesis of completeness in Theorem 1 may be replaced by a much weaker one, i.e. an hypothesis which we call of *weak completeness* (see Definition 1 and the observations below)

**b.** any Reinhardt domain, even not containing the origin, is pseudoconvex if and only if it is log-convex and satisfies some suitable hypothesis on its intersections with the coordinate hyperplanes (see Theorem 1).

However, we have to remark that the characterization of arbitrary pseudoconvex Reinhardt domains is not new and it has been reached with a different approach by R. Carmignani in [1]. But our main object was to reach a proof that is based only on elementary tools and on geometric properties of convex sets. For this reason, in order to be clear and easily followed, we expose all arguments dealing with Reinhardt domains in $\boldsymbol{C}^2$, being aware that, *mutatis mutandis*, they can be easily repeated in any other dimension.

In order to state precisely our results, we need

Definition 1. Let $D$ be a Reinhardt domain in $\boldsymbol{C}^2$ and let us define

$$a_1(D) = \min_{(z_1,\,0)\in \overline{D}} |z_1| \qquad a_2(D) = \min_{(0,\,z_2)\in \overline{D}} |z_2| \,.$$

Then $D$ is called *weakly complete* if the following two conditions are satisfied:

$(0, 0)$ is not an isolated point of $\boldsymbol{C}^2 \backslash D$;

for any $(p, 0)$, $p \neq 0$ and $(0, q)$, $q \neq 0$, in $\overline{D}$, we have:

$$\{(z_1, 0),\ |z_1| \in (a_1(D),\ |p|)\} \subseteq D$$
$$\{(0, z_1),\ |z_2| \in (a_2(D),\ |q|)\} \subseteq D\,.$$

Furthermore, we will denote by $\widetilde{D}$ the smallest Reinhardt domain which is weakly complete and contains $D$.

Remark 1. Note that, in practice, given a Reinhardt domain $D$, it is possible to determine $\widetilde{D}$ by *filling in* all the possible *holes* of the sets $D \cap \{z_1 = 0\}$ and $D \cap \{z_2 = 0\}$. Also note that if $D$ is complete, then $a_1(D) = a_2(D) = 0$.

The characterization of pseudoconvex Reinhardt domains in $\mathbf{C}^2$ which one can obtain from the properties of log-convex domains is

Theorem 2 (Main Theorem). *If $D$ is a bounded connected Reinhardt domain in $\boldsymbol{C}^2$ then:*

1. *$\widetilde{D}$ is pseudoconvex if and only if it is log-convex*

2. *$D$ is pseudoconvex if and only if $\widetilde{D}$ is pseudoconvex and one of the following identities occur*

a. $D = \widetilde{D}$      b. $D = \widetilde{D} \backslash \{z_1 = 0\}$

c. $D = \widetilde{D} \backslash \{z_2 = 0\}$      d. $D = \widetilde{D} \backslash \{z_1 z_2 = 0\}$.

Remark 2. As already mentioned, Theorem 1 is only for Reinhardt domains in $\boldsymbol{C}^2$, but the proof can be directly repeated for domains in $\boldsymbol{C}^n$, for any $n$. The only difference is that the number of possible identities to be satisfied by $D$ and $\widetilde{D}$ is in general $2^n$ instead of just 4.

Remark 3. As announced, for what concerns the Reinhardt domains which do contain the origin, Theorem 1 implies that weakly completeness plus log-convexity is equivalent to completeness plus log-convexity (see also Lemmata 1, 2 and Corollary 1).

## 2 - Pseudoconvexity of weakly complete Reinhardt domains

The goal of this section is to show that a weakly complete Reinhardt domain is pseudoconvex if and only if it is log-convex (Corollary 2 and Proposition 1). This, in particular, will prove the part 1 of the Main Theorem.

In the following statements, unless differently stated, $D$ is a log-convex bounded Reinhardt domain in $\boldsymbol{C}^2$ and $\overline{D}$ is its closure.

Lemma 1. *Let $P = (0, b)$ and $R = (\alpha, \beta)$ be two points of $\overline{D}$, with $b$, $\alpha$ and $\beta$ strictly positive real numbers. Then $\overline{D}$ contains all points with coordinates $(\gamma, \beta)$ with $\gamma \in [0, \alpha]$.*

Proof. $P = (0, b)$ is a limit point of a sequence of points $P_n = (a_n, b_n) \in D$, with $a_n$ real positive and different from 0. Being $D$ log-convex, any segment with endpoints $\widetilde{P}_n = (\log a_n, \log b_n)$ and $\widetilde{Q} = (\log \alpha, \log \beta)$ is entirely included in $\log |\overline{D}|$. Since $\log a_n$ tends to $-\infty$ and $\log b_n$ tends to $\log b$, all such segments tend to the unbounded half-line

$$l = \{(t, \log \beta) \,|\, t \in (-\infty, \log \alpha]\} \subset \log |\overline{D}| .$$

This implies that any point with coordinates $(\gamma, \beta)$ with $\gamma \in [0, \alpha]$ is in the closure of $\exp l$, i.e. in $\overline{D}$.

Lemma 2. *Let* $P = (0, b)$, $Q = (c, b)$ *and* $R = (\alpha, \beta)$ *be three points of* $\overline{D}$, *with* $b$, $c$, $\alpha$ *and* $\beta$ *strictly positive real numbers and* $\beta > b$. *Then,* $\overline{D}$ *contains the set* $\{(0, \gamma), \gamma \in [b, \beta]\}$.

Proof. Being $D$ log-convex, $\log |\overline{D}|$ contains the segment

$$\{\tilde{P}(t) = (t \log c + (1-t) \log \alpha, t \log b + (1-t) \log \beta), t \in [0, 1]\}.$$

Thus, $\overline{D}$ contains the points $P(t) = (c^t \alpha^{1-t}, b^t \beta^{1-t})$. Lemma 1 implies, in particular, that $P_\gamma = (0, \gamma) \in \overline{D}$ with $\gamma = b^t \beta^{1-t}$, for any $t$.

Corollary 1. *Suppose that* $(0, 0)$ *and* $P = (0, \beta)$ *are in* $\overline{D}$ *(*$\beta$ *positive real number). Then* $\overline{D}$ *contains any point* $P_\gamma = (0, \gamma)$ *with* $\gamma \in [0, \beta] \subset \boldsymbol{R}$.

Proof. Let $Q_n = (c_n, b_n)$ be a sequence in $D$ which tends to $(0, 0)$ and let $R_m = (\alpha_m, \beta_m)$ a sequence which tends to $P$ (all coordinates of the points may be assumed to be positive real numbers and with $b_n < \beta_n$). By Lemma 1, the points $P_n = (0, b_n)$ are in $\overline{D}$; from Lemma 2 applied to the triple of points $P_n$, $Q_n$ and $R_n$, we may infer that $\overline{D}$ contains the points of the form $(0, \gamma)$ with $\gamma \in [b_n, \beta_m]$. Since $b_n$ tends to 0 and $\beta_m$ tends to $\beta$, we get the conclusion.

Lemma 3. *Let* $P = (a, b) \in \partial D$, *with* $a$ *and* $b$ *positive real numbers. Then* $P$ *admits a neighborhood* $U_P$ *so that* $U_P \cap D$ *is pseudoconvex.*

Proof. By hypothesis there exists a straight line

$$l: f(t_1, t_2) = \alpha t_1 + \beta t_2 + \log |c| = 0$$

such that:

1. $f(\log |P|) = 0$ 2. $f(\log |z|) > 0$ for any $z \in D \backslash \{z_1 z_2 = 0\}$.

The function $h(z) = c z_1^\alpha z_2^\beta - 1$ is holomorphic on a suitable ball $B_P$ centered at $P$, vanishes at $P$ by **1**, and does not vanish at any point $Q \in B_P \cap D$ since, by **2**, $\log |c z_1^\alpha z_2^\beta| > 0$ for any $z \in D$.

Note that, for any other point $Q$ of $\partial D \cap B_P$ a function which is holomorphic on $D \cap B_p$ and which vanishes at $Q$ exists for the same reasons. So, all points of $\partial D \cap B_p$ are essential for $B_P \cap D$, as well as all points which are in $\partial B_P$. This implies that $B_P \cap D$ is pseudoconvex.

Lemma 4. *Let us suppose that $D$ is log-convex and weakly complete. If $(0, 0) \in \partial D$, then either $\{z_1 = 0\} \cap \overline{D} = \{(0, 0)\}$ or $\{z_2 = 0\} \cap \overline{D} = \{(0, 0)\}$.*

Proof. Suppose not and let $(0, b)$ and $(c, 0)$ be two points in $\overline{D}$, with $b$ and $c$ positive real numbers. By Lemma 1, for any point $(\alpha, \beta)$ in $D$ with $\alpha$ and $\beta$ positive real numbers, all points with coordinates $(\gamma, \beta)$, with $\gamma \in [0, \alpha]$ are in $\overline{D}$.

Then, we apply Lemma 1 (with the reversed order of the coordinates) in order to show that, for any $\gamma \in [0, \alpha]$, the points $(\gamma, \delta)$, with $\delta \in [0, \beta]$, are in $\overline{D}$ too. Since $D$ is a Reinhardt domain, the whole polydisc

$$\mathcal{P} = \{(z_1, z_2) \mid |z_1| \leqslant \alpha, \ |z_2| \leqslant \beta\}$$

is included in $\overline{D}$. So $a_1(D) = a_2(D) = 0$ and, as $D$ is weakly complete, the point $(0, 0)$ may belong to $\partial D$ only if it is an isolated point of $C^2 \setminus D$. Contradiction.

Lemma 5. *Let $D$ be log-convex and weakly complete and let $P = (0, b)$ (or $P = (b, 0)$) be a boundary point of $D$, with $b$ non negative real number. Then, $P$ is an essential point for $D$.*

Proof. When $b = 0$, the statement is a direct consequence of Lemma 4. Assume then that $b > 0$: we want to show that there is no point $Q = (a, b)$ in $D$ with $a \neq 0$ complex number. From this, we will deduce that the function $(z_2 - b)^{-1}$ is holomorphic in $D$ and not extendible around $P$.

Suppose that $Q = (a, b) \in D$, with $a \neq 0$. Since $D$ is Reinhardt, we may suppose that $a$ is positive real. $D$ is open and hence it contains, for a convenient $\varepsilon > 0$, the set

$$A_\varepsilon = \{(a, \beta) \mid \beta \in [b - \varepsilon, b + \varepsilon]\}.$$

From Lemmata 1 and 2, this implies that

$$B_\varepsilon = \{(\gamma, \beta) \mid \beta \in [b - \varepsilon, b + \varepsilon] \quad \text{and} \quad \gamma \in [0, a]\}$$

is a subset $\overline{D}$. This implies also that $a_1(D) < b - \varepsilon$, and, by weakly completeness, that $P = (0, b)$ is inner to $D$. Contradiction.

Corollary 2. *If $D$ is log-convex and weakly complete, then it is pseudoconvex.*

Proof. By Lemmata 3 and 5, any boundary point $P$ of $D$, with nonnegative real coordinates, admits a neighborhood $U_P$ such that $U_P \cap D$ is pseudoconvex. Any the other point of $\partial D$ fullfils this property because the map

$$(z_1, z_2) \to (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$$

is an automorphism of $D$ for any pair $(\theta_1, \theta_2)$. So $D$ is pseudoconvex (see Proposition 3.3.10 in [2]).

To show the reverse implication, let us first recall that:

$(x_1, x_2) \in \partial \log |D|$ implies that $(e^{x_1}, e^{x_2}) \in \partial D \backslash \{z_1 z_2 = 0\}$

$(z_1, z_2) \in \partial D \backslash \{z_1 z_2 = 0\}$ implies that $(\log |z_1|, \log |z_1\}) \in \partial \log |D|$.

From this, we get

Proposition 1. *If $D$ is pseudoconvex then it is log-convex.*

Proof. Let $D$ be pseudoconvex and let $T_{\log |D|}$ be the tube over $\log |D|$, i.e. the set defined as follows

$$T_{\log|D|} = \{(z_1, z_2) \in \boldsymbol{C}^2 \,|\, (\mathrm{Re}\, z_1, \mathrm{Re}\, z_2) \in \log |D|\}$$

$$= \{(\log |\zeta_1| + i\alpha_1, \log |\zeta_2| + i\alpha_2) \forall \alpha_1, \alpha_2 \in \boldsymbol{R}^2, (\zeta_1, \zeta_2) \in D, \zeta_1 \zeta_2 \neq 0\}.$$

By definitions, the exponential map

$$\exp : \boldsymbol{C}^2 \to \boldsymbol{C}^2 \qquad \exp(z_1, z_2) = (e^{z_1}, e^{z_2})$$

is such that $\exp(T_{\log |D|}) \subset D \backslash \{\zeta_1 \zeta_2 = 0\}$. As previously noted, we have also

$$\exp(\partial T_{\log |D|}) \subset \partial D \backslash \{\zeta_1 \zeta_2 = 0\}.$$

Being $D$ pseudoconvex, for any $p \in T_{\log |D|}$ there exists an open neighbourhood $B$ of $\exp p \in \partial D$ such that $B \cap D$ is pseudoconvex. Thus

$$\exp^{-1}(B \cap D) = \exp^{-1}(B) \cap T_{\log |D|}$$

is still pseudoconvex (since exp is a local biholomorphism). Hence $T_{\log |D|}$ is pseudoconvex (see Prop. 3.3.10 in [2]): Theorem 3.5.1 in [2] implies that $\log |D|$ is convex and that $D$ is log-convex.

### 3 - Proof of the Main Theorem

The proof of the Main Theorem (Theorem 2) will be complete when part **2** of the statement is proved. Let us assume again that $D$ is a connected Reinhardt domain (not necessarily log-convex) and let $\widetilde{D}$ as in Definition 1.

Lemma 6. *Suppose* $(a, 0) \in \tilde{D} \backslash D$ *(resp.* $(0, b) \in \tilde{D} \backslash D$*). Then there exists* $a' \neq 0$ *(resp.* $b' \neq 0$*) such that* $(a, a') \in D$ *(resp.* $(b', b) \in D$*).*

Proof. We may assume that $a$ is positive. If $\{z_1 = a\} \cap \partial D$ were empty, and hence $\{|z_1| = a\} \cap \partial D = \emptyset$, we would have that either

$$\{|z_1| < a\} \supset D \qquad \text{or} \qquad \{|z_1| > a\} \supset D$$

because of the connectness of $D$ and the fact that $\{|z_1| = a\}$ disconnects $\boldsymbol{C}^2$.

Assume, for instance, that the first case is true and consider the Reinhardt domain $A = \tilde{D} \cap \{|z_1| < a\} \supseteq D$. If $A$ is proved to be weakly complete the following contradiction appears: by definition of $\tilde{D}$, we should have $\tilde{D} \subseteq \{|z_1| < a\}$, while we know that $(a, 0) \in \tilde{D}$.

As $a_1(A) \geqslant a_1(\tilde{D})$ and $|p| \leqslant a$, if $(p, 0) \in \bar{A}$, then

$$\{(z_1, 0): |z_1| \in (a_1(A), |p|)\} \subseteq \{(z_1, 0): |z_1| \in (a_1(\tilde{D}), |p|)\} \subseteq A$$

that is $A$ is weakly complete.

In the second case the proof is completely analogous.

We are now able to prove the Main Theorem. It remains to prove only part 2 (see the beginning of Sec. 2).

Proof of Main Theorem. First, we will prove that, if $D$ is pseudoconvex, then $\tilde{D}$ is pseudoconvex and that either **a**, **b**, **c** or **d** occurs: in this case $D$ is necessarily log-convex (Proposition 1) and being $\log |D| = \log |\tilde{D}|$, $\tilde{D}$ is pseudoconvex (Corollary 2).

Assume now that **a**, **b**, **c** and **d** are false. Then (as $D$ differs from $\tilde{D}$ only on the coordinate hyperplanes) it should exist at least one point on the coordinate hyperplanes, say $P = (\alpha, 0)$, which would belong to $D$, and a point $(a, 0)$, which would belong to $\tilde{D} \backslash D$. By Lemma 6 there should exist $a'$ such that $Q = (a, a') \in D$. The simultaneous existence of $P$ and $Q$ in $D$ implies that the point $(a, 0)$ is in $\partial D$ (see Lemma 1). To get a contradiction it is sufficient to show that any $f$, holomorphic in D, has a holomorphic extension to a neighbourhood of $(a, 0)$.

Let

$$f = \sum_{\alpha_1, \alpha_2 \in \boldsymbol{Z}^2} f_{\alpha_1 \alpha_2} z_1^{\alpha_1} z_2^{\alpha_2}$$

be the Laurent expansion of $f$ (such expansion exists because $f$ is defined over a Reinhardt domain; see [3]). Now, observe that, since $(\alpha, 0) \in D$, we have that the Laurent expansion is of the form

$$f = \sum_{\alpha_1 \in Z} \sum_{\alpha_2 \in Z^+} f_{\alpha_1 \alpha_2} z_1^{\alpha_1} z_2^{\alpha_2} \qquad Z^+ = \{\alpha_2 \in Z \,|\, \alpha_2 \geq 0\}$$

with uniform convergence on compact sets. From $(a, a') \in D$, we also have that the above series converges in a neighbourhood of $(a, 0)$ (being all $\alpha_2 \geq 0$).

To show the converse, observe that, if **a** holds, the pseudoconvexity of $D$ is obvious. Assume now that **b** is true: as $\tilde{D}$ differs from $D$ on the coordinate hyperplane $\{z_1 = 0\}$, any boundary point of the form $(a, b)$, with $a \neq 0$, is essential (by Lemmata 3 and 5); on the other hand, any other boundary point in $\{z_1 = 0\}$ is essential because the holomorphic function $z_1^{-1}$ does not extend there. Analogous arguments prove the statement when **c** or **d** are true.

## References

[1] R. Carmignani, *Envelopes of holomorphy and holomorphic convexity*, Trans. Amer. Math. Soc. **179** (1973), 415-431.

[2] S. Krantz, *Function theory of several complex variables*, Wiley, New York 1982.

[3] R. Narasimhan, *Several complex variables*, Chicago Univ. Press, Chicago 1971.

## Sommario

*Le proprietà geometriche degli insiemi convessi in $\boldsymbol{R}^n$ sono utilizzate per determinare condizioni necessarie e sufficienti perché un dominio di Reinhardt limitato e convesso (non necessariamente contenente l'origine) risulti pseudoconvesso.*

*Viene ottenuta una nuova semplice dimostrazione di un classico risultato di R. Carmignani, che generalizza le condizioni di pseudoconvessità per i domini di Reinhardt contenenti l'origine.*

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