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OGNIAN KASABOV (*)

A characterization of the extrinsic spheres in a Riemannian manifold (**)

1 - Introduction

In [3] K. Ogiue and R. Takagi propose different criteria for a surface in E^3 to be a sphere and give the following useful and practical condition

Theorem A. *Let M be a surface in E^3. Suppose that, through each point $p \in M$ there exist two circles of E^3 such that:*
they are contained in M in a neighbourhood of p
they are tangent to each other at p.
Then M is locally a plane or a sphere.

The so-called extrinsic spheres are a natural generalization of the ordinary sphere in E^m. It is interesting to have a characterization of an n-sphere, or more generally of an extrinsic sphere, similar to Theorem A. K. Ogiue and R. Takagi give such a criterion in [3] by means of n^2 circles through each point of the submanifold. Here we propose an analogue of Theorem A for extrinsic spheres. Namely we prove

Theorem B. *Let M be an n-dimensional ($n > 2$) submanifold of a Riemannian manifold $\tilde{M}$. Then M is either a totally geodesic submanifold of $\tilde{M}$ or*

(*) Higher Transport School «T. Kableschkov», Section of Mathematics, Slatina, 1574 Sofia, Bulgaria.
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an extrinsic sphere of $\widetilde{M}$, *if through each point* p *of* M *there exist two* $(n-1)$*-dimensional extrinsic spheres of* $\widetilde{M}$, *such that:*

they are contained in M *in a neighbourhood of* p

they are tangent to each other at p.

Corollary. *Let* M *be an* n*-dimensional* $(n > 2)$ *submanifold of the Euclidean space* E^m. *Suppose that through each point* p *of* M *there exist two* $(n-1)$*-spheres of* E^m, *such that:*

they are contained in M *in a neighbourhood of* p

they are tangent to each other at p.

Then M *is locally an* n*-plane or an* n*-sphere in* E^m.

2 - Preliminaries

Let $\widetilde{M}$ be a Riemannian manifold with metric tensor g and let M be an n-dimensional submanifold of $\widetilde{M}$. Denote by $\widetilde{\nabla}$ and ∇ the Riemannian connections of $\widetilde{M}$ and M, respectively. Then the *Gauss formula* is

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$$

for all vector fields X, Y on M, where σ is the *second fundamental form* of M in $\widetilde{M}$. Let ξ be a normal vector field. Then the *Weingarten formula* is

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

where $-A_\xi X$ and $D_X \xi$ are the tangential and the normal components of $\widetilde{\nabla}_X \xi$ respectively. Usually A_ξ is called the *shape operator*, corresponding to ξ and D the *connection in the normal bundle.* Also $g(A_\xi X, Y) = g(\sigma(X, Y), \xi)$ holds good. Recall then that the covariant derivative of σ with respect to the *connection* $\overline{\nabla}$ *of van der Waerden-Bortolotti* is given by

$$(\overline{\nabla}_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

The *mean curvature vector* H of M in $\widetilde{M}$ is defined by $H = n^{-1}$ trace σ.

The submanifold M is called *totally umbilical,* if $\sigma(x, y) = g(x, y) H$ for all $x, y \in T_p M$, $p \in M$ or equivalently $A_\xi x = g(\xi, H) x$ for all $x \in T_p M$, $\xi \in (T_p M)^\perp$, $p \in M$. In particular, if σ vanishes identically, M is said to be a *totally geodesic* submanifold of $\widetilde{M}$. A normal vector field ξ is called *parallel,* if $D_X \xi = 0$ for any vector field X on M.

A regular curve $\tau = (x_s)$ parametrized by arc length s is called a *circle* in $\tilde{M}$, if there exists a field Y_s of unit vectors along τ and a positive constant k, such that

$$\tilde{\nabla}_{X_s} X_s = kY_s \qquad \tilde{\nabla}_{X_s} Y_s = -kX_s$$

where X_s denotes the tangent vector of τ [2]. The number k is called the *radius* and Y_s the *main normal* of τ.

A submanifold M of $\tilde{M}$ is said to be an *extrinsic sphere*, if it is totally umbilical and has non-zero parallel mean curvature vector. For an extrinsic sphere M we have

$$\tilde{\nabla}_X \tilde{\nabla}_X X = \nabla_X \nabla_X X - g(H, H) X$$

and hence

$$g(\tilde{\nabla}_X \tilde{\nabla}_X X, \xi) = 0 \tag{2.1}$$

for any unit vector field X on M and any vector field ξ normal to M.

3 - General lemmas

In this section we prepare two lemmas (Lemma 2 and Lemma 3) which will be useful in the proof of our theorem. The first of them shows that an extrinsic sphere is determined (locally) by its second fundamental form at one point. The second gives a condition for the second fundamental forms of two hypersurfaces to coincide.

Lemma 1. *Let S be an extrinsic sphere in a Riemannian manifold $\tilde{M}$ and denote by H its mean curvature vector. Then every geodesic of S through a point p of S is a circle in $\tilde{M}$ of radius $k = (g(H, H))^{\frac{1}{2}}$ and its main normal vector at p is $k^{-1}H_p$.*

Proof. Note that $k = (g(H, H))^{\frac{1}{2}}$ is a non-zero constant since H is parallel and non-zero. Let $\tau = (x_s)$ be a geodesic of S parametrized by arc length. Denote by X_s its tangent vector and define Y_s by $H_{x(s)} = kY_s$. Then by the Gauss and Weingarten formulas

$$\tilde{\nabla}_{X_s} X_s = kY_s \qquad \tilde{\nabla}_{X_s} Y_s = -kX_s$$

hold good and the lemma is proved.

Lemma 2. *Let $\widetilde{M}$ be a Riemannian manifold. Suppose that through a point p of $\widetilde{M}$ there exist two extrinsic spheres S_1 and S_2 of $\widetilde{M}$, which are tangent to each other at p. If their second fundamental forms at p coincide, then S_1 and S_2 coincide in a neighbourhood of p.*

Proof. Denote by H_i the mean curvature vector of S_i in $\widetilde{M}$. Note that $H_1 = H_2$ at p. Let r be a positive number, such that $\exp_i$ is a diffeomorphism of a neighbourhood $N_i(p, r)$ of the origin of $T_pS_1 = T_pS_2$ and a neighbourhood $U_i(p, r)$ of p in S_i, $i = 1, 2$, as in Proposition 3.4, Chapter IV of [1].

For a point $q \in U_1(p, r)$ let $\tau = (x_s)$, $s \in [0, s_0]$ be the only geodesic of S_1 in $U_1(p, r)$ joining p and q and parametrized by arc length. Let $\overline{\tau} = (\overline{x}_s)$ be defined by $\overline{x}_s = \exp_2(\exp_1^{-1} x_s)$. Then $\overline{\tau}$ is a geodesic in S_2 through p.

According to Lemma 1, τ and $\overline{\tau}$ are circles in $\widetilde{M}$ through p of radius $k = (g(H_1, H_1))^{\frac{1}{2}}$ and its main normal at p is $k^{-1}(H_1)_p = k^{-1}(H_2)_p$. But such a circle is unique [2]. Then τ and $\overline{\tau}$ coincide locally.

Let $\overline{q} = \exp_2(\exp_1^{-1} q)$. Since $d(p, q) = s_0 = d(p, \overline{q})$ it follows $q = \overline{q}$. So $q \in S_2$, thus proving the lemma.

Lemma 3. *Let S_1 and S_2 be two hypersurfaces through a point p of a Riemannian manifold M. Suppose also that S_1 and S_2 are tangent to each other and that there exists a smooth unit vector field N, defined in a neighborhood of p in M and such that N, restricted to S_i, is normal to S_i, for $i = 1, 2$. Then the second fundamental forms of S_1 and S_2 coincide at p.*

Proof. Let $x \in T_pS_1 = T_pS_2$. Since S_1 and S_2 are hypersurfaces the Weingarten formulas for S_1 and S_2 in M imply $\nabla_x N|_{S_i} = -A^i_{N_p} x$ for $i = 1, 2$, where $A^i_{N_p}$ is the shape operator of S_i in M and $N|_{S_i}$ denotes the restriction of N to S_i. But $\nabla_x N = \nabla_x N|_{S_1} = \nabla_x N|_{S_2}$. Hence $A^1_{N_p} = A^2_{N_p}$ and consequently the second fundamental forms of S_1 and S_2 coincide at the point p.

4 - Proof of Theorem B

For a point p of M let S_{1p} and S_{2p} be extrinsic spheres through p as in the statement of the theorem. Let X be a vector field on S_{1p}. Then a direct calculation gives

$$g(\widetilde{\nabla}_X \widetilde{\nabla}_X X, \overline{\xi}) = g((\overline{\nabla}_X \sigma)(X, X) + 3g(\sigma(\nabla_X X, X), \overline{\xi})$$

for any vector field $\bar{\xi}$, normal to M. Because of (2.1) this implies

$$(\overline{\nabla}_X \sigma)(X, X) + 3\sigma(\nabla_X X, X) = 0 \tag{4.1}$$

for any unit vector field X on S_{1p}.

Denote by ∇^1 the Riemannian connection of S_{1p}. Let ξ_1 be a local normal unit vector field for S_{1p} in M. Then for all vector fields X, Y on S_{1p} we have

$$\nabla_X Y = \nabla^1_X Y + h_1(X, Y)\,\xi_1$$

$h_1 \xi_1$ being the second fundamental form of S_{1p} in M. Hence, using the Gauss formula of M in $\tilde{M}$ we find

$$\tilde{\nabla}_X Y = \nabla^1_X Y + h_1(X, Y)\,\xi_1 + \sigma(X, Y).$$

Denote by H_1 the mean curvature vector of S_{1p} in $\tilde{M}$. Since S_{1p} is totally umbilical in $\tilde{M}$, we get

$$g(X, Y) H_1 = h_1(X, Y)\,\xi_1 + \sigma(X, Y). \tag{4.2}$$

Note that ξ_1 is orthogonal to $\sigma(X, Y)$ for all vector fields X, Y on S_{1p}. Then (4.2) shows that S_{1p} is totally umbilical in M. More explicitly, putting $\lambda_1 = g(H_1, \xi_1)$, from (4.2) we obtain

$$h_1(X, Y) = \lambda_1 g(X, Y) \tag{4.3}$$

$$\sigma(X, Y) = (H_1 - \lambda_1 \xi_1)\, g(X, Y). \tag{4.4}$$

Suppose now that X is a unit vector field on S_{1p}. Then X is orthogonal to $\nabla^1_X X$ and consequently from (4.1), (4.3) and (4.4) we derive

$$(\overline{\nabla}_X \sigma)(X, X) + 3\lambda_1 \sigma(X, \xi_1) = 0. \tag{4.5}$$

Put $\xi = \xi_1(p)$. Then by (4.5) we get

$$(\overline{\nabla}_x \sigma)(x, x) + 3\lambda_1(p)\,\sigma(x, \xi) = 0$$

for any unit vector x in $T_p S_{1p} = T_p S_{2p}$. It is easy to see that we have also

$$(\overline{\nabla}_x \sigma)(x, x) + 3\lambda_2(p)\,\sigma(x, \xi) = 0$$

where $\lambda_2(p)\,\xi\, g$ is the second fundamental form at p of S_{2p} in M. The last two equations and Lemma 2 imply

$$\sigma(x, \xi) = 0 \tag{4.6}$$

for any vector x in $T_p S_{1p} = T_p S_{2p}$.

According to (4.4) we may write $\sigma(x, y) = g(x, y)\eta$ for $x, y \in T_pS_{1p}$, where η is given by

$$\eta = H_1(p) - \lambda_1(p)\xi = H_2(p) - \lambda_2(p)\xi$$

H_2 being the mean curvature vector of S_{2p} in $\tilde{M}$. Denote $\zeta = \sigma(\xi, \xi)$. Then using (4.6) we find for any $x, y \in T_pM$

$$\sigma(x, y) = (g(x, y) - g(x, \xi)g(y, \xi))\eta + g(x, \xi)g(y, \xi)\zeta$$

and hence

$$A_{\operatorname{tr}\sigma}x = g(\eta, \operatorname{tr}\sigma)x + g(\zeta - \eta, \operatorname{tr}\sigma)g(x, \xi)\xi \tag{4.7}$$

for any $x \in T_pM$. We put $\mu_0 = g(\eta, \operatorname{tr}\sigma)$, $\nu_0 = g(\zeta, \operatorname{tr}\sigma)$. Then according to (4.7) ξ is an eigenvector of $A_{\operatorname{tr}\sigma}$ at p and the corresponding eigenvalue is ν_0. Analogously any unit vector x in $T_pS_{1p} = T_pS_{2p}$ is an eigenvector of $A_{\operatorname{tr}\sigma}$ with corresponding eigenvalue μ_0.

Suppose that M is not totally umbilical at p, i.e. $\mu_0 \neq \nu_0$. Let the continuous functions μ and ν be eigenvalues of $A_{\operatorname{tr}\sigma}$, such that $\mu(p) = \mu_0$ and $\nu(p) = \nu_0$. Then $\mu(q) \neq \nu(q)$ for any q in a sufficiently small neighbourhood U of p in M. It follows directly or by the implicit function theorem that ν is a smooth function on U. Then its corresponding field N of unit eigenvectors is also smooth on U.

We shall show that the restriction of N to S_{ip} is orthogonal to S_{ip} for $i = 1, 2$. Indeed according to the definitions of μ, ν and N

$$A_{\operatorname{tr}\sigma}x = \mu(q)x + (\nu(q) - \mu(q))g(x, N)N_q$$

for all $x \in T_qM$, $q \in U$. Hence

$$g(\sigma(x, y), \operatorname{tr}\sigma) = (\nu(q) - \mu(q))g(x, N)g(y, N)$$

for all orthogonal vectors $x, y \in T_qM$, $q \in U$. On the other hand by (4.4) we have $g(\sigma(x, y), \operatorname{tr}\sigma) = 0$ for all orthogonal vectors $x, y \in T_qS_{1p}$, $q \in S_{1p}$. Consequently, for arbitrary orthogonal vectors $x, y \in T_qS_{1p}$, $g(x, N)g(y, N) = 0$ holds good, i.e. at most one of any two orthogonal vectors in T_qS_{1p} is orthogonal to N_q. Hence it follows easily that in fact any vector in T_qS_{1p} is orthogonal to N_p. So the restriction of N to S_{1p} is orthogonal to S_{1p}. Analogously the restriction of N to S_{2p} is orthogonal to S_{2p}.

Then, according to Lemma 3 the second fundamental forms of S_{1p} and S_{2p} coincide at p. Now using Lemma 2 we conclude that S_{1p} and S_{2p} coincide in a neighbourhood of p, which is a contradiction. So $\mu_0 = \nu_0$ and M is totally umbilical at p. Since p is an arbitrary point of M, it follows that M is totally umbilical in $\tilde{M}$.

It remains to show that the mean curvature vector H of M is parallel. Since M is totally umbilical, by (4.1) we conclude that $D_x H = 0$ for any $x \in T_q S_{1p}$, $q \in S_{1p}$. Suppose that for the above defined vector ξ we have $D_\xi H \neq 0$. Let $\bar{\eta}$ be a normal vector field of M, defined in a neighbourhood of p, such that the differential form ω, given by $\omega(X) = g(D_X H, \bar{\eta})$ for a vector field X on M is not zero at any point of U. Define a vector field Y on U by $\omega(X) = g(X, Y)$. Then Y does not vanish in U. Denote $N = (g(Y, Y))^{-\frac{1}{2}} Y$. Note that the restriction of N to S_{ip} is normal to S_{ip} for $i = 1, 2$. Using again Lemmas 2 and 3 we conclude that S_{1p} and S_{2p} coincide in a neighbourhood of p, which is a contradiction. So H is parallel. Then M is totally geodesic or an extrinsic sphere in $\tilde{M}$, according to the length of H being zero or not. This proves our theorem.

References

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[2] K. Nomizu and K. Yano, *On circles and spheres in Riemannian geometry*, Math. Ann. **210** (1974), 163-170.

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Sommario

Nel 1984 K. Ogiue ed R. Takagi hanno dato una condizione perché una superficie dello spazio ordinario sia localmente un piano o una sfera. Viene qui ottenuta una condizione dello stesso tipo perché una sottovarietà M di una varietà Riemanniana $\tilde{M}$ sia totalmente geodetica oppure sia una sfera estrinseca.

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