Riv. Mat. Univ. Parma (5) 5 (1996), 23-31

S. Ivanov and I. Petrova (*)

Curvature operator with parallel Jordanian basis on circles (**)

1 - Introduction

Curvature is a fundamental notion of the differential geometry. A useful technique to describe the curvature along geodesics in a Riemannian manifold is the use of the Jacobi operator. In the present paper we describe the curvature along a unit circle using the curvature operator.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of the metric g. If $p \in M$ is a point, we denote by $T_p M$ the tangent space at p. The curvature operator is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ for every smooth vector fields X, Y on M.

A smooth curve $c(t)$: $I_\varepsilon \to M$, $I_\varepsilon = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, with tangent vector field $\dot{c}$, parametrized by the arc length, is said to be a *circle* of curvature k_1 if its first curvature k_1 is a constant different from zero and all other curvatures are zeros (see [4]). A unit circle is a circle with curvature $k_1 = 1$. For a unit circle we have

$$(1) \qquad \nabla_{\dot{c}} \dot{c} = \nu \qquad \nabla_{\dot{c}} \nu = -\dot{c} \, .$$

The unit vector field ν is the *first normal* of $c(t)$ and all other normals are paral-

---

(*) Faculty of Math. and Inf., Univ. Sofia, Bul. J. Bouchier 5, 1126 Sofia, Bulgaria.

(**) Received February 23, 1996. AMS classification 53 C 21. The authors are partially supported by Contracts MM 423, 413/1994 with the Ministry of Science and Education of Bulgaria and the first author by Contract 219/1994 with the University of Sofia «St. Kl. Ohridcki».

lel along $c(t)$. The differential equations for a unit circle are equivalent to

$$\nabla_{\dot c}^2 \dot c + \dot c = 0 . \tag{2}$$

If dim $M \geq 2$ then for every point $p \in M$ and every two orthonormal vectors $u, v \in T_pM$ there exists locally a unique unit circle $c(t)$, parametrized by the arc length, satisfying the initial conditions

$$c(0) = p \qquad \dot c(0) = u \qquad (\nabla_{\dot c} \dot c)(0) = v . \tag{3}$$

Let $c(t)$ be a unit circle on M. We consider the families of smooth skew-symmetric linear operators along $c(t)$ defined by

$$\kappa_c(X) = R(\dot c, \nabla_{\dot c} \dot c)X \qquad \kappa'_c(X) = (\nabla_{\dot c} R)(\dot c, \nabla_{\dot c} \dot c)X \tag{4}$$

for every vector field X along $c(t)$.

From (1) and (4) it follows that

$$\kappa'_c(X) = \nabla_{\dot c} \kappa_c(X) - \kappa_c(\nabla_{\dot c} X) . \tag{5}$$

We have the following characterization of *locally symmetric spaces.*

Proposition 1. *For an n-dimensional* ($n \geq 2$) *Riemannian manifold* (M, g) *the following conditions are equivalent:*

i. (M, g) *is a Riemannian locally symmetric space.*

ii. *For every unit circle c on M we have* $\kappa'_c = 0$.

iii. *For every unit circle c on M we have* $\kappa_c \circ \nabla_{\dot c} = \nabla_{\dot c} \circ \kappa_c$.

iv. *For every unit circle c on M the operator* κ_c *transforms a vector field parallel along c to a vector field parallel along c.*

v. *For every unit circle c on M the matrix of* κ_c *with respect to a basis of vector fields parallel along c is invariant i.e. the matrix has constant entries.*

Proof. The implication **i** ⇒ **ii** is clear. To prove the converse, let p be a point of M and u, v two orthonormal vectors at p. Let $c(t)$ be the unit circle on M determined by the initial conditions (3). For every $w \in T_pM$, from $\kappa'_c = 0$ we get

$$(\nabla_u R)(u, v) w = 0 . \tag{6}$$

By the well known properties of the Riemannian curvature operator we can conclude that (6) is valid for arbitrary $u, v, w \in T_pM$.

If $z \in T_pM$, replacing u by $u + z$ in (6) and using the second Bianchi identity we get $\nabla R = 0$ which proves **i**.

The equivalence of **ii**, **iii**, **iv** and **v** follows from (5).

We recall now some basic properties of the skew-symmetric linear operators.

Let A be a skew-symmetric linear operator on an Euclidean vector space E. If A has a real eigenvalue then it has to be zero. If A possesses a non-zero eigenvalue of multiplicity m; then it must have the form ia, where i is the imaginary unity and $a \in \boldsymbol{R}$. Then $-ia$ is also an eigenvalue of A and of the same multiplicity. Therefore A has eigenvalues $0, ia_1, -ia_1, \ldots, ia_l, -ia_l$ of multiplicities $m, m_1, \ldots, m_l, m_l$, respectively. Then there exists an orthonormal basis

$$Z_1, \ldots, Z_m, X_1^{(1)}, Y_1^{(1)}, \ldots, X_{m_1}^{(1)}, Y_{m_1}^{(1)}, \ldots, X_1^{(l)}, Y_1^{(l)}, \ldots, X_{m_l}^{(l)}, Y_{m_l}^{(l)}$$

of E such that:

$$\begin{aligned} &A(Z_p) = 0 && p = 1, \ldots, m \\ &A(X_s^{(j)}) = a_j Y_s^{(j)} \quad A(Y_s^{(j)}) = -a_j X_s^{(j)} && j = 1, \ldots, l; \;\; s = 1, \ldots, m_j . \end{aligned} \tag{7}$$

Such a basis of E is known as a *Jordanian basis* for A.

Definition 1. A Riemannian manifold (M, g) is said to be an *O-space* if for any unit circle $c(t)$ the operator κ_c has constant eigenvalues along $c(t)$.

Definition 2. A Riemannian manifold (M, g) is said to be *a T-space* if for any unit circle $c(t)$ the operator κ_c has a Jordanian basis, parallel along $c(t)$ (at least locally around almost every point of $c(t)$).

From Proposition 1 we have

Theorem 1. *Let (M, g) be a smooth Riemannian manifold. Then M is a locally symmetric Riemannian manifold iff M is an O-space and a T-space simultaneously.*

Due to this theorem we are led to study O-spaces and T-spaces separately. This leads to two natural generalizations of locally symmetric Riemannian manifolds.

In this paper we consider T-spaces. O-spaces will be studied in [3].

2 - Characterizations of T-spaces

Let E be an n-dimensional ($n \geq 2$) Euclidean vector space and let $A(t)$ be a C^k-family of skew-symmetric endomorphisms on E depending on a real parameter $t \in I$, where $k \in N \cup \{\infty, \omega\}$ and I is an open interval in $\boldsymbol{R}$. Let $0 \geq \mu_1(t) \geq \mu_2(t) \geq, \dots \geq \mu_n(t)$ be the uniquely defined functions on I that represent the eigenvalues of $A^2(t)$. Then $\lambda_k(t) = (-1)^k i\sqrt{-\mu_k(t)}$, $k = 1, \dots, n$ represent the eigenvalues of $A(t)$ (counted with their multiplicities).

Let U be the everywhere dense open subset of I where the multiplicity of every eigenvalue λ_k of $A(t)$ is constant. We need the following results (details can be taken from [5], p. 537).

Lemma. 1. *The functions $\lambda_1, \dots, \lambda_n$ are continuous on I and of class C^k on U.*

Lemma 2. *The family $A(t)$ has a C^k-Jordan decomposition around every point of U, that is, there exist C^k-maps $Z_1, \dots, Z_p, X^{(1)}_{m_1}, Y^{(1)}_{m_1}, \dots, X^{(l)}_{m_l}, Y^{(l)}_{m_l}$ such that*

$$Z_1(t), \dots, Z_p(t), X^{(1)}_1(t), Y^{(1)}_1(t), \dots, X^{(l)}_{m_l}(t), Y^{(l)}_{m_l}(t)$$

is an orthonormal basis of E which is a Jordanian basis for $A(t)$.

Moreover, we have

Lemma 3. *A family of skew-symmetric C^∞ operators $A(t)$ has a C^∞-Jordan decomposition of E not depending on t locally almost everywhere on I, iff $A \circ A' = A' \circ A$.*

The proof of Lemma 3 is analogous to the proof of Lemma 5 at p. 61 of [1].

Further we have

Theorem 2. *Let M be an n-dimensional ($n \geq 2$) connected smooth Riemannian manifold. Then the following conditions are equivalent:*

i. *M is a T-space*

ii. *$\kappa'_c \circ \kappa_c = \kappa_c \circ \kappa'_c$ for every unit circle c on M.*

Proof. Let p be a point of M and $c(t)$ a unit circle through the point p. Let $A(t)$ be the skew-symmetric endomorphism on T_pM obtained by a parallel translation of κ_c along the circle from $c(t)$ to p. We apply Lemma 3 and get the assertion.

As an immediate consequence of Theorem 2 we get

Theorem 3. *Let M be an n-dimensional ($n \geqslant 2$) connected smooth Riemannian manifold. Then the following conditions are equivalent:*

i. *M is a T-space*

ii. $(\nabla_X R)(X, Y) \circ R(X, Y) = R(X, Y) \circ (\nabla_X R)(X; Y)$

for every tangent vectors X, Y at any point of M.

Corollary 1. *Let $M = M_1 \times \ldots \times M_r$ be a locally reducible smooth Riemannian manifold and each M_i be smooth. Then M is a T-space, iff each of M_i is a T-space.*

3 - Classification of T-spaces

In this section we treat the classification problem for T-spaces of dimension two and three.

For the two dimensional case, we have

Theorem 4. *Every two-dimensional smooth Riemannian manifold (M, g) is a T-space.*

Proof. The curvature tensor of a 2-dimensional Riemannian manifold has the following expression

$$R(X, Y)Z = \frac{1}{2} s[g(Y, Z)X - g(X, Z)Y] \tag{8}$$

where s is the scalar curvature of g and $X, Y, Z \in T_pM$, $p \in M$.

Let $c(t)$ be a unit circle through p. For every vector field X parallel along $c(t)$ we consider the vector field $Y = g(X, \nu)\dot{c} - g(X, \dot{c})\nu$, where ν is the first normal of c. Using (1) it is easy to see that Y is a vector field parallel along c. Taking into account (8) we get $\kappa_c(X) = \frac{1}{2} sY$, $\kappa_c(Y) = -\frac{1}{2} sX$, which proves the assertion.

For the three dimensional case, we have

Theorem 5.

i. *Let (M, g) be a three-dimensional T-space of class C^∞. Then M is almost everywhere (that is, on an open and dense subset of M) locally isometric to one of the following spaces:*

a. *a space of constant Riemannian sectional curvature*

b. *a Riemannian product of the form $M_1 \times M_2$, where M_1 is an 1-dimensional Riemannian manifold and M_2 is a 2-dimensional Riemannian manifold.*

ii. *Any three-dimensional Riemannian manifold of type* **a** *and* **b** *is a T-space.*

Proof. (Part **i**). From Theorem 3 it follows that it is sufficient to study a Riemannian manifold satisfying condition

$$L_{v_1, v_2} = \kappa'_{v_1, v_2} \circ \kappa_{v_1, v_2} - \kappa_{v_1, v_2} \circ \kappa'_{v_1, v_2} = 0, \qquad v_1, v_2 \in T_pM \tag{9}$$

where $\kappa_{u,v}$, $\kappa'_{u,v}$ are skew-symmetryc endomorphisms of T_pM defined by

$$\kappa_{u,v} = R(u, v) \qquad \kappa'_{u,v} = (\nabla_u R)(u, v).$$

Since L_{v_1, v_2} is a skew-symmetric endomorphism of T_pM condition (9) is equivalent to the identities:

$$g(L_{v_1, v_2} v_1, v_2) = 0 \qquad g(L_{v_1, v_2} v_1, v_3) = 0 \qquad g(L_{v_1, v_2} v_2, v_3) = 0 \tag{10}$$

for each orthonormal basis v_1, v_2, v_3 of T_pM, $p \in M$.

As it is well known, the curvature tensor R of a 3-dimensional Riemannian manifold can be expressed in terms of the Ricci tensor ric, Ricci operator Ric and the scalar curvature s of g, namely

$$\begin{aligned} R(X, Y)Z &= \mathrm{ric}(Y, Z)X - \mathrm{ric}(X, Z)Y \\ &+ g(Y, Z)\,\mathrm{Ric}(X) - g(X, Z)\,\mathrm{Ric}(Y) - \frac{1}{2}s[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{11}$$

For each orthonormal basis v_1, v_2, v_3 of T_pM we get by straightforward computations that conditions (10) are equivalent to the following equations:

$$\mathrm{ric}(v_2, v_3)(\nabla_{v_1}\mathrm{ric})(v_1, v_3) - \mathrm{ric}(v_1, v_3)(\nabla_{v_1}\mathrm{ric})(v_2, v_3) = 0 \tag{12}$$

$$\begin{aligned} &-\mathrm{ric}(v_j, v_3)[2(\nabla_{v_1}\mathrm{ric})(v_1, v_1) + 2(\nabla_{v_1}\mathrm{ric})(v_2, v_2) - v_1(s)] \\ &\qquad + [s - 2\,\mathrm{ric}(v_3, v_3)](\nabla_{v_1}\mathrm{ric})(v_j, v_3) = 0 \qquad j = 1, 2. \end{aligned} \tag{13}$$

At any point $p \in M$ we consider the Ricci operator Ric as a linear self-adjoint operator on the tangent space T_pM. Let Ω be the subset of M on which the number of distinct eigenvalues of Ric is locally constant. This set is open and dense in M.

On Ω we can choose C^∞ eigenfunctions of Ric, say r_1, r_2, r_3, such that at each point of Ω they form the spectrum of Ric (see for example [1]). We fix a point $p \in \Omega$. Then there exists a local orthonormal frame field E_1, E_2, E_3 on an open connected neighbourhood U of p, such that

$$\mathrm{Ric}(E_i) = r_i E_i \qquad i = 1, 2, 3\,. \tag{14}$$

We set $\omega_{ij,k} = g(\nabla_{E_i} E_j, E_k)$, $i, j, k = 1, 2, 3$. Using (14) we get

$$(\nabla_{E_i} \mathrm{ric})(E_j, E_k) = \delta_{jk} E_i(r_k) + (r_j - r_k)\,\omega_{ij,k} \tag{15}$$

where δ_{jk} is the Kronecker's symbol.

Lemma 4. *For distinct* $i, j, k \in \{1, 2, 3\}$ *the following formulae are valid:*

$$(r_i - r_j)(r_i - r_k)\,\omega_{jj,i} = 0 \qquad (r_i - r_j)\,\omega_{ki,j} = 0\,. \tag{16}$$

Proof of Lemma 4. We set in (12):

$$v_1 = E_1 \qquad v_2 = \frac{E_2 + E_3}{\sqrt{2}} \qquad v_3 = \frac{E_2 - E_3}{\sqrt{2}}$$

and then

$$v_1 = E_1 \qquad v_2 = \frac{E_2 + E_3}{\sqrt{2}} \qquad v_3 = \frac{E_2 - E_3}{\sqrt{2}}\,.$$

Subtracting the obtained equations from each other we get

$$(r_2 - r_3)(r_1 - r_2)\,\omega_{11,2} = 0 \qquad (r_2 - r_3)(r_1 - r_3)\,\omega_{11,3} = 0\,.$$

Using (13) instead of (12), in the exactly same way as above, we get $(r_2 - r_3)\,\omega_{12,3} = 0$. Similarly we get all other identities (16) which proves Lemma 4.

Further we shall investigate three cases, namely the cases in which the number d of distinct eigenvalues of the Ricci operator in U is one, two or three.

Case A. $d=1$. In this case (M, g) is an *Einstein manifold* on U. The three-dimensional Einstein manifolds are precisely the spaces of constant sectional curvature (see Proposition 1.120 in [2]) which are of course, T-spaces.

Case B. $d=2$. We may assume $r_2 = r_3 = r \neq r_1$. From (16) we have

$$\omega_{22,1} = \omega_{33,1} = \omega_{23,1} = \omega_{32,1} = 0\,. \tag{17}$$

We shall prove that

$$\omega_{11,2} = \omega_{11,3} = 0\,. \tag{18}$$

Let $A = (a^1, a^2, a^3)$, $B = (b^1, b^2, b^3)$, $C = (c^1, c^2, c^3)$ be three orthonormal vectors in $\boldsymbol{R}^3$. Then the vectors $v_1 = a^i E_i$, $v_2 = b^i E_i$, $v_3 = c^i E_i$ (the Einstein summation convention is assumed) are orthonormal vector fields on M. From (12) we get by straightforward computations

$$\sum_{i,j,l=1}^{3} [c^1 (r_1 - r)(a^1 b^j - b^1 a^j)\, a^i c^l (\nabla_{E_i} \mathrm{ric})(E_j, E_l)] = 0\,. \tag{19}$$

Taking into account (17), (15) and keeping in mind $r_1 - r \neq 0$, from (19) we obtain

$$a^1 (c^1)^2 (c^3 \omega_{11,2} - c^2 \omega_{11,3}) = 0\,. \tag{20}$$

We take A, B, C such that $a^1 \neq 0$ and $c^1 \neq 0$. Then (20) implies (18).

We define $V_1 = \mathrm{span}\{E_1\}$, $V_2 = V_1^{\perp} = \mathrm{span}\{E_2, E_3\}$. Because of (17) and (18) both distributions V_1 and $V_1^{\perp}$ are autoparallel. Hence, (M, g) is locally isometric to a Riemannian product of an 1-dimensional Riemannian manifold and a 2-dimensional Riemannian manifold.

Case C. ($d=3$). In this case (16) implies $\omega_{ij,k} = 0$ for all $i, j, k = 1, 2, 3$, which implies $r_1 = r_2 = r_3 = 0$ anywhere on U and this is included in the case A.

(Part **ii**) It is easy to see from Theorem 4 and Corollary 1 that the spaces of type **a** and **b** are T-spaces.

Remark. It is interesting to compare T-spaces with *C-spaces* (spaces, for which the Jacobi operator has constant eigenvalues along every geodesic) and with *P-spaces* (spaces, for which the eigenspaces of the Jacobi operator are parallel along every geodesic), defined and studied in [1]. In general T-spaces and

C-spaces are different. For 2-manifolds the class of T-spaces coincides with the class of P-spaces. From Theorem 5 and Theorem 7 of [1] it follows that a 3-dimensional T-space almost everywhere locally is a P-space, but there exist 3-dimensional P-spaces which locally are not T-spaces even almost everywhere.

The following question seems to be natural: For $n > 3$ is every n-dimensional T-space almost everywhere locally a P-space?

References

[1] J. BERNDT and L. VANHECKE, *Two natural generalizations of locally symmetric spaces*, Differential Geom. Appl. 2 (1992), 57-80.

[2] A. BESSE, *Einstein manifolds*, Springer, Berlin 1987.

[3] S. IVANOV and I. PETROVA, *Riemannian manifolds in which certain curvature operator has constant eigenvalues along each circle*, to appear.

[4] K. NOMIZU and K. YANO, *On circles and spheres in Riemannian geometry*, Math. Ann. **210** (1974), 163-170.

[5] Z. SHABO, *Structure theorems on Riemannian spaces satisfying* $R(X, Y) \circ R = 0$, **I**. *The local version*, J. Differential Geom. **17** (1982), 531-582.

Sommario

Sono considerate le varietà riemanniane, il cui operatore di curvatura ammette una base di Jordan sulle circonferenze. Si dà una classificazione locale, a meno di isometrie, nei casi di dimensione due e tre.

* * *