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Bounded oscillations of difference equations (**)

1 - Introduction

Let $\boldsymbol{R}$ be the set of real numbers, $\boldsymbol{Z}$ denotes the set of integers and $\boldsymbol{N} = \{0, 1, 2, \ldots\}$. As usual, for any function $u : \boldsymbol{N} \to \boldsymbol{R}$ we define the forward difference operators as follows

$$\Delta u(n) = u(n+1) - u(n) \qquad \Delta^k u(n) = \Delta(\Delta^{k-1} u(n)) \qquad k \geq 1$$

$$\Delta^0 u(n) = u(n).$$

For all $k \in \boldsymbol{N}$ we use the usual factorial notation

$$(s)^{(k)} = s(s-1) \cdot \ldots \cdot (s-k+1) \text{ with } (s)^{(0)} = 1.$$

In this paper we are concerned with the oscillatory behavior of solutions of the nonlinear difference equation

$$\boldsymbol{E}(\delta) \qquad \Delta^m u(n) + \delta\ a(n)\ f(u(r(n))) = 0 \qquad m \geq 2,\ n \in \boldsymbol{N}$$

where $\delta = \pm 1$, $a : \boldsymbol{N} \to [0, \infty)$ $(a(n) \not\equiv 0$ eventually), $r : \boldsymbol{N} \to \boldsymbol{Z}$, $\lim_{n \to \infty} r(n) = \infty$, $r(n) \leq n$ for $n \geq n_0 \in \boldsymbol{N}$, $f : \boldsymbol{R} \to \boldsymbol{R}$, $uf(u) > 0$ for $u \neq 0$.

By a solution of $\boldsymbol{E}(\delta)$ we mean a real sequence u which is defined for $n \geq \min_{i \geq 0} r(i)$ and satisfies $\boldsymbol{E}(\delta)$ for n sufficiently large. We consider only such solutions which are nontrivial for large n. A nontrivial solution u of $\boldsymbol{E}(\delta)$ is said to be oscillatory if for every $n_0 \in \boldsymbol{N}$ there exists an $n \geq n_0$ such that $u(n)u(n+1) \leq 0$. Otherwise it is called nonoscillatory.

Recently some results concerning the oscillatory behavior of solutions of difference equations of higher order have been established in papers [1]-[3], [5], [7], [9], [10]. For the general theory of difference equations one can refer to e.g. [4], [6].

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Our aim in this paper is to obtain sufficient conditions for the oscillation of all bounded solutions of $\boldsymbol{E}(-1)$ when m is even as well as for $\boldsymbol{E}(1)$ when m is odd. These results extend some criteria that have been obtained for $\boldsymbol{E}(\delta)$ in case $m=2$ [8].

2 - Main results

The following theorem characterizes the oscillatory behavior of bounded solutions of $\boldsymbol{E}(\delta)$ (cf. [9], Theorem 1 and Theorem 2).

Theorem 1. *If the following conditions hold:*

I. $|f(u)|$ *is bounded away from zero if* $|u|$ *is bounded away from zero*

II. $\sum_{0}^{\infty} n^{m-1} a(n) = \infty$

then for m *even (resp. odd) all bounded solutions of* $\boldsymbol{E}(1)$ *(resp.* $\boldsymbol{E}(-1)$*) are oscillatory while for* m *odd (resp. even) all bounded solutions of* $\boldsymbol{E}(1)$ *(resp.* $\boldsymbol{E}(-1)$*) are either oscillatory or tending monotonically to zero as* $n \to \infty$.

In view of Theorem 1, the problem of establishing conditions under which the bounded and nonoscillatory solutions vanish, makes sense only for difference equation $\boldsymbol{E}(-1)$ when m is even, as well as for $\boldsymbol{E}(1)$ when m is odd. These equations can be unified in the following form

$$\boldsymbol{E} \qquad \Delta^m u(n) + (-1)^{m+1} a(n) f(u(r(n))) = 0 \qquad m \geqslant 2,\ n \in \boldsymbol{N}.$$

To obtain sufficient conditions under which all bounded solutions of $\boldsymbol{E}$ are oscillatory we need the following

Lemma. *Let* $v: \boldsymbol{N} \to (0, \infty)$ *be a bounded sequence and for some* $m \geqslant 2$ $(-1)^m \Delta^m v(n) \geqslant 0$ *for every* $n \in \boldsymbol{N}$ *and* $\Delta^m v(n)$ *is not identically zero for large* n. *Then for every* $n \in \boldsymbol{N}$ *and* $i = 1, \ldots, m-1$

$$(1) \qquad (-1)^i \Delta^i v(n) > 0$$

and for all $n, q \in \boldsymbol{N}$ *with* $n \geqslant q$

$$(2) \qquad v(q) \geqslant (-1)^{m-1} \frac{(n-q+m-1)^{(m-1)}}{(m-1)!} \Delta^{m-1} v(n).$$

Proof. By the assumptions, we see that $\Delta^i v$ $(i = 1, \ldots, m-1)$ is of constant sign for all large n and $(-1)^m \Delta^{m-1} v$ is a nondecreasing sequence. We show that $(-1)^{m-1} \Delta^{m-1} v(n) > 0$ for $n \in \boldsymbol{N}$. In fact, if there exists

$n_1 \in N$ such that $(-1)^{m-1}\Delta^{m-1}v(n_1) = c < 0$, then $(-1)^{m-1}\Delta^{m-1}v(n) \leqslant c$ for $n \geqslant n_1$, which leads to the contradictory conclusion that $\lim_{n\to\infty} v(n) = \pm\infty$. Also, if for some $n_1 \in N$ $\Delta^{m-1}v(n_1) = 0$, then there is $n_2 \geqslant n_1$ such that $(-1)^{m-1}\Delta^{m-1}v(n_2) < 0$ or $\Delta^{m-1}v(n) = 0$ for all $n \geqslant n_1$ which is impossible.

Further, it is easy to see that if for some i, $0 < i < m-1$, $\Delta^i v(n)\Delta^{i+1}v(n) > 0$ for all large n, then $\lim_{n\to\infty} v(n) = \pm\infty$, which contradicts our assumption. This proves (1).

Next, by using the formula (cf. [4], p. 41 or [2])

$$v(q) = \sum_{i=0}^{m-2}(-1)^i \frac{(n-q+i)^{(i)}}{i!}\Delta^i v(n+1) + \frac{(-1)^{m-1}}{(m-2)!}\sum_{k=q}^{n}(k-q+m+2)^{(m-2)}\Delta^{m-1}v(k)$$

for every $n, q \in N$ with $n \geqslant q$, and (1), we get

$$v(q) \geqslant \frac{(-1)^{m-1}\Delta^{m-1}v(n)}{(m-2)!}\sum_{k=q}^{n}(k-q+m-2)^{(m-2)}$$

from which we obtain (2).

Theorem 2. *Assume that*

III. f *is a nondecreasing function*

VI. $\int_0^{\pm\alpha}\frac{du}{f(u)} < \infty, \quad \alpha > 0$

V. $\sum_0^{\infty}[n - r(n) + 1]^{m-1}a(n) = \infty$.

Then all bounded solutions of E *are oscillatory.*

Proof. Assume, for the sake of contradiction, that E has a bounded non-oscillatory solution u and without loss of generality, we may suppose that u is eventually positive. Then there is $n_1 \in N$ such that $u(r(n)) > 0$ for every $n \geqslant n_1$. Thus from E it follows that $(-1)^m\Delta^m u(n) \geqslant 0$ for $n \geqslant n_1$. Then, by Lemma, for every $n \geqslant n_1$ we have

$$(3) \qquad (-1)^i\Delta^i u(n) > 0 \qquad i = 1, \ldots, m-1.$$

In addition, since condition **V** implies **II** and also **I** is satisfied so, by Theorem 1, we must have $\lim_{n\to\infty} u(n) = 0$.

Next, from the equality (comp. [4], p. 41)

$$\tag{4} \begin{aligned} \Delta^k u(n) &= \sum_{i=k}^{m-1} (-1)^{i-k} \frac{(p-n+i-k)^{(i-k)}}{(i-k)!} \Delta^i u(p+1) \\ &+ (-1)^{m-k} \frac{1}{(m-k-1)!} \sum_{j=n}^{p} (j-n+m-k-1)^{(m-k-1)} \Delta^m u(j) \end{aligned}$$

where $p \geqslant n \geqslant n_1$, $0 \leqslant k < m$, for $k = 1$ with regard to $\boldsymbol{E}$ we obtain

$$\tag{5} \begin{aligned} \Delta u(n) &= \sum_{i=1}^{m-1} (-1)^{i-1} \frac{(p-n+i-1)^{(i-1)}}{(i-1)!} \Delta^i u(p+1) \\ &- \frac{1}{(m-2)!} \sum_{j=n}^{p} (j-n+m-2)^{(m-2)} a(j) f(u(r(j))) . \end{aligned}$$

Choose $n_2 > n_1$ such that $r(n) \geqslant n_1$ for all $n \geqslant n_2$ and let $k > n_2$ be fixed. So, by (3), from (5) we have

$$\tag{6} -\Delta u(n) \geqslant \frac{1}{(m-2)!} \sum_{j=n}^{k} (j-n+m-2)^{(m-2)} a(j) f(u(r(j)))$$

where $n_1 \leqslant n \leqslant k$.

Dividing (6) by $f(u(n))$ and summing from n_1 to k, we get

$$\tag{7} \begin{aligned} &\sum_{n=n_1}^{k} \frac{-\Delta u(n)}{f(u(n))} \\ &\geqslant \frac{1}{(m-2)!} \sum_{n=n_1}^{k} \frac{1}{f(u(n))} \sum_{j=n}^{k} (j-n+m-2)^{(m-2)} a(j) f(u(r(j))) \\ &\geqslant \frac{1}{(m-1)!} \sum_{n=n_2}^{k} a(j) \sum_{n=r(j)}^{j} (j-n+m-2)^{(m-2)} \frac{f(u(r(j)))}{f(u(n))} . \end{aligned}$$

By the assumptions we see that

$$\frac{-\Delta u(n)}{f(u(n))} \leqslant \int_{u(n+1)}^{u(n)} \frac{du}{f(u)} \qquad n \geqslant n_1 .$$

Thus noting that $f(u(r(j))) \geqslant f(u(n))$ for $r(j) \leqslant n \leqslant j$, $n_2 \leqslant j \leqslant k$, we conclude from (7) that

$$\sum_{j=n_2}^{\infty} (j-r(j)+m-1)^{(m-1)} a(j) \leqslant (m-1)! \int_{0}^{u(n_1)} \frac{du}{f(u)} < \infty$$

which contradicts **V**. This completes the proof.

Theorem 3. *If condition* **I** *holds and*

VI. *r is a nondecreasing sequence*

VII. $\limsup_{n \to \infty} \sum_{k=r(n)}^{n} (k - r(n) + m - 1)^{(m-1)} a(k) > (m-1)!\, L_f$

where $L_f = \limsup_{u \to 0} \dfrac{u}{f(u)} < \infty$

then all bounded solutions of $\boldsymbol{E}$ *are oscillatory.*

Proof. Let u be a bounded nonoscillatory solution of $\boldsymbol{E}$ which can be supposed eventually positive. We note that condition **VII** implies **II**.

In fact, if $\sum_{0}^{\infty} n^{m-1} a(n) < \infty$, then

$$0 < \limsup_{n \to \infty} \sum_{k=r(n)}^{n} (k - r(n) + m - 1)^{(m-1)} a(k)$$

$$\leqslant \limsup_{n \to \infty} \sum_{k=r(n)}^{n} (k + m - 1)^{(m-1)} a(k)$$

$$\leqslant 2^{m-1} \limsup_{n \to \infty} \sum_{k=r(n)}^{\infty} k^{m-1} a(k) = 0$$

which is a contradiction. Thus, by Theorem 1, we must have $\lim_{n \to \infty} u(n) = 0$. Also, we see as previously that (3) holds. Further, by (4), one can write

$$u(q) = \sum_{i=0}^{m-1} (-1)^i \frac{(n-q+i)^{(i)}}{i!} \Delta^i u(n+1) + \frac{(-1)^m}{(m-1)!} \sum_{k=q}^{n} (k - q + m - 1)^{(m-1)} \Delta^m u(k)$$

for $n \geqslant q \geqslant n_1$, with regard to $\boldsymbol{E}$ and (1), we get

$$u(q) \geqslant \frac{1}{(m-1)!} \sum_{k=q}^{n} (k - q + m - 1)^{(m-1)} a(k) f(u(r(k))).$$

Now we choose $n_2 \geqslant n_1$ such that $r(n) \geqslant n_1$ for every $n \geqslant n_2$. Therefore

$$u(r(n)) \geqslant \frac{1}{(m-1)!} \sum_{k=r(n)}^{n} (k - r(n) + m - 1)^{(m-1)} a(k) f(u(r(k))), \quad n \geqslant n_2.$$

Since u is a decreasing sequence for $n \geqslant n_1$ we get

$$u(r(n)) \geqslant \frac{u(r(n))}{(m-1)!} \inf_{k \geqslant r(n)} \frac{f(u(r(k)))}{u(r(k))} \sum_{k=r(n)}^{n} (k - r(n) + m - 1)^{(m-1)} a(k)$$

that is

$$(m-1)! \geqslant \inf_{u \leqslant u(r(r(n)))} \frac{f(u)}{u} \sum_{k=r(n)}^{n} (k - r(n) + m - 1)^{(m-1)} a(k)$$

and so

$$\sum_{k=r(n)}^{n} (k - r(n) + m - 1)^{(m-1)} a(k) \leqslant (m-1)! \sup_{0 < u \leqslant u(r(r(n)))} \frac{u}{f(u)} .$$

But since $u(n) \to 0$, as $n \to \infty$ the last inequality contradicts condition **VII.** Thus the proof is complete.

Corollary 1. *Consider the linear difference equation of the form*

$$\boldsymbol{E}_1 \qquad \Delta^m u(n) + (-1)^{m+1} a(n) u(r(n)) = 0 \qquad m \geqslant 2,\ n \in \boldsymbol{N}$$

where a and r are defined as before with r satisfying **VI**. *If we have*

$$\limsup_{n \to \infty} \sum_{k=r(n)}^{n} (k - r(n) + m - 1)^{(m-1)} a(k) > (m-1)!$$

then every bounded solution of $\boldsymbol{E}_1$ is oscillatory.

Theorem 4. *Suppose that conditions* **I** *and* **VI** *are satisfied and*

$$\textbf{VIII.} \qquad \limsup_{n \to \infty} \sum_{k=r(n)}^{n} (r(n) - r(k) + m - 1)^{(m-1)} a(k) > (m-1)!\, L_f$$

where L_f is defined in **VII**.

Then all bounded solutions of $\boldsymbol{E}$ are oscillatory.

Proof. Suppose that $\boldsymbol{E}$ has a bounded nonoscillatory solution u and let $u(n) > 0$ eventually. Moreover, since for all large n and every k with $r(n) \leqslant k \leqslant n$, $r(k) \geqslant m - 1$ we have

$$0 \leqslant r(n) - r(k) + m - 1 \leqslant k .$$

Then from **VIII** we derive $\limsup_{n \to \infty} \sum_{k=r(n)}^{n} k^{m-1} a(k) > 0$, which, in view of $\lim_{n \to \infty} r(n) = \infty$, implies **II**. Thus, by Theorem 1, we must have $\lim_{n \to \infty} u(n) = 0$. In addition, as in the proof of Theorem 2 we see, by the Lemma, that there exists $n_1 \in \boldsymbol{N}$ such that (3) holds and

$$(8) \qquad u(q) \geqslant (-1)^{m-1} \frac{(n - q + m - 1)^{(m-1)}}{(m-1)!} \Delta^m u(n) \qquad \text{for } n \geqslant q \geqslant n_1 .$$

Thus for every k, n with $r(n) \leqslant k \leqslant n$ and $n \geqslant n_2 \geqslant n_1$ we have $r(n) \geqslant r(k) \geqslant n_1$, and therefore, by (8), we have

$$(9) \qquad u(r(k)) \geqslant (-1)^{m-1} \frac{(r(n) - r(k) + m - 1)^{(m-1)}}{(m-1)!} \Delta^{m-1} u(r(n)) .$$

Next, from $\boldsymbol{E}$ we get

$$(-1)^m[\Delta^{m-1}u(n+1)-\Delta^{m-1}u(r(n))]=\sum_{k=r(n)}^{n}a(k)f(u(r(k)))$$

for every $n \geqslant n_2$ and so, by (3) and (9), we have

$$(-1)^{m-1}\Delta^{m-1}u(r(n)) \geqslant \inf_{k \geqslant r(n)} \frac{f(u(r(k)))}{u(r(k))} \sum_{k=r(n)}^{n} a(k)u(r(k))$$

$$\geqslant \frac{1}{(m-1)!} \inf_{0<u \leqslant u(r(r(n)))} \frac{f(u)}{u}(-1)^{m-1}\Delta^{m-1}u(r(n))$$

$$\cdot \sum_{k=r(n)}^{n}(r(n)-r(k)+m-1)^{(m-1)}a(k)$$

and consequently

$$\sum_{k=r(n)}^{n}(r(n)-r(k)+m-1)^{(m-1)}a(k) \leqslant (m-1)! \sup_{0<u \leqslant u(r(r(n)))} \frac{u}{f(u)}, \; n \geqslant n_2 .$$

But, since $\lim_{n\to\infty} u(n)=0$ and $\lim_{n\to\infty} r(n)=\infty$ this inequality contradics **VIII**. Thus the proof is complete.

Corollary 2. *Every bounded solution of* $\boldsymbol{E}_1$ *is oscillatory if*

$$\limsup_{n\to\infty} \sum_{k=r(n)}^{n}(r(n)-r(k)+m-1)^{(m-1)}a(k) > (m-1)!$$

and **VI** *holds.*

From Theorems 3 and 4 we obtain the following

Corollary 3. *Consider the difference equation of the form*

$$\boldsymbol{E}_2 \qquad \Delta^m u(n) = (-1)^m a(n) f(u(n)) \qquad m \geqslant 2,\; n \in \boldsymbol{N}$$

where a *and* f *are defined as before.*

If condition **I** *holds and* $\limsup_{n\to\infty} a(n) > L_f$, *where* L_f *is defined in* **VII**, *then every bounded solution of* $\boldsymbol{E}_2$ *is oscillatory. In particular, every bounded solution of the equation*

$$\Delta^m u(n) = (-1)^m a(n) u(n) \qquad m \geqslant 2,\; n \in \boldsymbol{N},$$

is oscillatory if $\limsup_{n\to\infty} a(n) > 1$.

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Sommario

Il lavoro contiene alcune condizioni sufficienti perchè tutte le soluzioni limitate di certe equazioni alle differenze risultino oscillatorie.

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