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Characterization of the morphisms between Jacobians induced by morphisms between Riemann surfaces (**)

Introduction

Let X' and X be two compact connected Riemann surfaces of genus g' and g respectively, with $g' \geq g \geq 1$. If f: $X' \to X$ is a holomorphic surjective map, obviously f induces a map between the corresponding Jacobians: F: $J(X') \to J(X)$; F is a group homomorphism and is holomorphic.

One can wonder which may be necessary and sufficient conditions for a map F: $J(X') \to J(X)$ to be induced by a holomorphic map from X' onto X. This problem was suggested to me by F. Bardelli, to whom I am greatly indebted also for some helpful discussions. In this paper we give two answers to this question (respectively Theorem 1 in Sec. 1 and Theorem 2 in Sec. 2). Both imitate Torelli's theorem, the first one in a set-combinatorial sense, the second one in a topological sense.

The hypotheses of Theorem 2 can be easily and almost completely translated in polynomial equations in the entries of the matrix representing F and of the period matrices of X' and X. Then perhaps the second characterization of the morphisms between Jacobians induced by morphisms between Riemann surfaces may have some utilizations.

For instance, perhaps it may be useful to give an upper bound on the number of the morphisms between two fixed Riemann surfaces and also on the number of the morphisms between a fixed Riemann surface and an other Riemann surface varying among the ones of a same fixed genus.

Besides, Theorem 2 (together with equations resolving Schottky problem)

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may be useful to see

$$\mathrm{Hol}(g', g) = \{(A, f, B): A \in \mathcal{M}_{g'}, B \in \mathcal{M}_g, f: A \to B \text{ holomorphic surjective map}\}$$

as an open of some components of the set of zeroes of some equations in $(\mathcal{H}_{g'} \times R^{4g'g} \times \mathcal{H}_g)/\sim$, where $\mathcal{M}_{g'}$ and $\mathcal{M}_g$ are the moduli spaces of Riemann surfaces respectively of genus g' and g and $\mathcal{H}_{g'}$ and $\mathcal{H}_g$ are the Siegel upper half-spaces and $\sim$ is the equivalence relation given by the action of the modular groups.

From the following remark we see that for $g = 2, 3$ the maps between Jacobians induced by maps between Riemann surfaces are few.

Remark 1. Let $D^g_{g'}$ be the subset of $\mathcal{M}_{g'}$ corresponding to the set of the Riemann surfaces C such that there exists a map ϕ: $C \to B_g$ with B_g abelian variety of dimension g and such that $\phi(C)$ generates B_g as group. In [**1**] Colombo and Pirola proved that for $g = 1, 2, 3$ the set $D^g_{g'}$ is dense in $\mathcal{M}_{g'}$. Then $D^g_{g'}$ is a countable union of analytic subvarieties in $\mathcal{M}_{g'}$.

Besides we know that, for $g = 1, 2, 3$, almost all abelian varieties of dimension g are Jacobians of Riemann surfaces.

Fixed g' and g with $g \geq 2$, by the Hurwitz formula we know that there are only a finite number of natural numbers, that may be the degree of a map from a Riemann surface of genus g' to a Riemann surface of genus g. Thus the space $\mathcal{R}_{g', g}$ parametrizing the holomorphic surjective maps from Riemann surfaces of genus g' to Riemann surfaces of genus g has a finite number of connected components.

Let $g = 2, 3$. Consider the map $\mathcal{R}_{g', g} \to \mathcal{M}_{g'}$ associating to every element of $\mathcal{R}_{g', g}$, f: $X' \to X$ the first Riemann surface X'. By the previous remarks, the image of such map is contained in a finite number of components of $D^g_{g'}$. Since $D^g_{g'}$ is a countable union of analytic subvarieties, there is a not finite number of components of $D^g_{g'}$ such that for everyone of their elements, which represents a Riemann surface S, any map from $J(S)$ onto an Abelian variety of dimension g (which is generically a Jacobian) is not induced by a map between Riemann surfaces.

Notations. In all the paper X' and X will be two compact connected Riemann surfaces of genus g' and g respectively, with $g' \geq g \geq 1$. In all the paper $J(X')$ means a fixed representative of the class of the Jacobian of X' in the set of principally polarized abelian varieties up to isomorphisms. Analogously for $J(X)$.

Fix a point Q' of X'; in all the paper let ν be Abel's map of X' from Q'. The image in $J(X')$ under ν of the positive divisors on X' of degree $\bar{d} \leq d$ will be denoted by V^d; its translate by an element $a \in J(X')$ will be denoted by V^d_a. Analogously fix a point Q of X and in all the paper let μ be Abel's map of X from Q. We define $W^d = \mu(X^{(d)})$ and $W^d_b = W^d + b$. Set $(W^d_b)^* = -W^d_b + \mu(Z)$, where Z is a canonical divisor on X.

In general, if A is a subset of a Jacobian and t an element of this Jacobian, we define $A_t = A + t$.

Let ω' be the polarization on $J(X')$, given by the intersection of 1-cycles on X', and let θ' be the dual polarization on $\mathrm{Pic}^\circ(X')$, which can be computed as follows: let $\{a_i, b_i\}_{i=1\dots g'}$ be a symplectic basis of $H_1(X', Z)$ and let $\{a_i^\vee, b_i^\vee\}_{i=1\dots g'}$ be the dual basis of $H^1(X', Z)$. In the universal covering of $\mathrm{Pic}^\circ(X')$ we consider the coordinates s_i, $s_{i+g'}$ dual to $\{a_i^\vee, b_i^\vee\}$; we have $\theta' = \sum_{i=1}^{g'} ds_i \wedge ds_{i+g'}$. We call ω and θ the analogous polarizations respectively on $J(X)$ and on $\mathrm{Pic}^\circ(X)$.

Remark 2. Let F be a holomorphic map $J(X') \to J(X)$ such that $F(0) = 0$, (thus F is a homomorphism). Obviously F is induced by a holomorphic map from X' onto X if and only if there exists $K \in J(X)$ such that $F(V^1) = W^1 + K$. In this case one can see easily that, if $g \geq 2$, there is only one map inducing F and evidently this map is $\mu^{-1} \circ (F|_{V^1} - K) \circ \nu$.

1 - A first answer to the problem

Generalizing Martens' proof of Torelli's theorem ([3]), we have a first not obvious answer to our problem.

Theorem 1. *Let F: $J(X') \to J(X)$ be a holomorphic map such that $F(0) = 0$. Assume also that there exists $K \in J(X')$ such that $F(V^{g-1}_K) = W^{g-1}$. Suppose that if $F(V^1) \not\subset W^{g-1}_b$, then $F(V^1) \cap W^{g-1}_b$ is equal to g points counted with multiplicity and their sum is b + const. Then there is a holomorphic surjective map f: $X' \to X$ inducing either F or $-F$.*

Remark 3. Evidently if F is a map: $J(X') \to J(X)$ induced by a holomorphic map from X' onto X, then F satisfies the hypotheses of Theorem 1.

Proof of Theorem 1. In the proof we use the lemmas in [3]. By Remark 2 in the Introduction, it suffices to prove that $F(V^1)$ is a translate of W^1.

Let r be the smallest integer such that $F(V^1) \subset W^{r+1}_a$ or $F(V^1) \subset (W^{r+1}_a)^*$ for some a. (We have $r < g - 1$ evidently). We want to show that $r = 0$.

First let's suppose that $F(V^1) \subset W_a^{r+1}$.
Let $x \in W^1$, $y \in W^{g-1-r}$; we set $b = a + x - y$.

Claim 1. *Unless* $W_a^{r+1} \subset W_b^{g-1}$, *we have*

$$F(V^1) \cap W_b^{g-1} = F(V^1) \cap W_b^{g-1} \cap W_a^{r+1} = (F(V^1) \cap W_{a+x}^r) \cup (F(V^1) \cap S)$$

where $S = W_a^{r+1} \cap (W_{y-a}^{g-2})^*$. *Observe that* $F(V^1) \cap W_{a+x}^r$ *depends only on* x *and* $F(V^1) \cap S$ *depends only on* y.

The result follows by virtue of Lemma 4 in [3].

Claim 2. *Fixed* x *in* W^1, *we have that* $F(V^1) \not\subset W_b^{g-1}$ *for almost all choices of* y *in* W^{g-1-r}, *that is for almost all choices of* b *in* $-W_{-(a+x)}^{g-1-r}$ *(and therefore* $W_a^{r+1} \not\subset W_b^{g-1}$ *for at least the same* y *and* b*). Namely defining*

$$A(x) = \{b \in -W_{-(a+x)}^{g-1-r} \text{ such that } F(V^1) \not\subset W_b^{g-1}\}$$

we have that $-W_{-(a+x)}^{g-1-r} - A(x)$ *is a Zariski closed subset of* $-W_{-(a+x)}^{g-1-r}$ *and the dimension of each of its components is less than* $g - 1 - r$.

Proof. The set, $-W_{-(a+x)}^{g-1-r} - A(x)$ which is a Zariski closed set, is a proper subset of $-W_{-(a+x)}^{-1-r}$, because if $F(V^1) \subset W_b^{g-1}$ for any $b \in W_{-(a+x)}^{g-1-r}$, then by Lemma 3 in [3] we have that $F(V^1) \subset W_{a+x}^r$ and this is absurd for the assumption on r. Clearly each of the components of $-W_{-(a+x)}^{g-1-r} - A(x)$ has dimension less than $g - 1 - r$; if not it would be equal to all $-W_{-(a+x)}^{g-1-r}$ since $-W_{-(a+x)}^{g-1-r}$ is irreducible.

Claim 3. *For any* $m \leq g$ *we have* $\dim F(V^m) = m$.

Proof. If $m = g - 1$, it is obvious by the hypothesis of the theorem $F(V_K^{g-1}) = W^{g-1}$. Consider the case $m < g - 1$. We have

$$g - 1 = \dim F(V^{g-1}) = \dim[F(V^m) + F(V^{g-1-m})] \leq \dim F(V^m) + \dim F(V^{g-1-m}).$$

If by absurd $\dim F(V^m) < m$ then, since $\dim F(V^{g-1-m}) \leq g - 1 - m$, it would be inpossible that $\dim F(V^{g-1}) = g - 1$, so we have proved our claim in this case.

Finally, if $m = g$, for some t we have

$$F(V^g) = F(\bigcup_{s \in V^1} (V^{g-1} + s)) = \bigcup_{s \in V^1} (W_t^{g-1} + F(s)) = W_t^{g-1} + F(V^1).$$

$F(V^g)$ is irreducible; so if it had dimension $g - 1$, then $W_t^{g-1} + F(s) = W_t^{g-1}$ for every $s \in V^1$. But this implies $F(s) = 0$ for every $s \in V^1$, that is $V^1 \subset \operatorname{Ker} F$ and this is absurd.

Claim 4. *Fixed x in W^1, $F(V^1) \cap W^r_{a+x}$ contains at most one point.*

Proof. If not, as b varies in $A(x)$ (whose complement in $-W^{g-1-r}_{-(a+x)}$ is a Zariski closed subset such that the dimension of each of its components is less than $g-1-r$), the sum of the g points of $F(V^1) \cap W^{g-1}_b$ varies in a translate of $F(V^{g-2})$ (by Claim 1) and then, by the fact that the sum of the points of $F(V^1) \cap W^{g-1}_b$ is equal to b + const, we should have an inclusion of $(W^{g-1-r})^*$ in a translate of $F(V^{g-2})$, say $(W^{g-1-r})^* \subset F(V^{g-2}_q)$. Then we have:

$$F(V^1) \subset \bigcap\{F(\Theta^{g-1}_{-v}): v \in V^{g-2}\} = \text{translate of } \bigcap\{F(\Theta^{g-1}_{K-v}): v \in V^{g-2}_q\}$$

$$= \text{translate of } \bigcap\{(F(\Theta^{g-1}_K))_{-u}: u \in F(V^{g-2}_q)\} = \text{translate of } \bigcap\{(W^{g-1}_{-u}: u \in F(V^{g-2}_q)\}$$

$$\subset \text{translate of } \bigcap\{W^{g-1}_{-u}: u \in (W^{g-1-r})^*\} = \text{translate of } (W^r)^*$$

by Lemma 3 in [3]. But the fact that $F(V^1) \subset$ translate of $(W^r)^*$ is absurd for the assumption on r.

Claim 5. *There exist two distinct points of W^1, say x' and x'', such that $F(V^1) \cap W^r_{a+x'}$ and $F(V^1) \cap W^r_{a+x''}$ contain at least one point each (and then exactly one point each by the Claim 4).*

In fact, as x varies in W^1, W^r_{a+x} describes all W^{r+1}_a (which contains $F(V^1)$); but $F(V^1)$ cannot be contained in W^r_{a+x} for a certain x by the assumption on r. So we can find x' and x'' with $x' \neq x''$ such that $F(V^1) \cap W^r_{a+x'} \neq \emptyset$ and $F(V^1) \cap W^r_{a+x''} \neq \emptyset$.

Claim 6. *There is a translate of $F(V^1)$ intersecting W^1 in two distinct points.*

Proof. Let us consider x' and x'' found in Claim 5. We can take y in W^{g-1-r} such that $a + x' - y \in A(x')$ and $a + x'' - y \in A(x'')$. With this choice of y, we have:

$$F(V^1) \cap W^{g-1}_{a+x'-y} = (F(V^1) \cap W^r_{a+x'}) \cup (F(V^1) \cap S)$$

$$F(V^1) \cap W^{g-1}_{a+x''-y} = (F(V^1) \cap W^r_{a+x''}) \cup (F(V^1) \cap S)$$

where $F(V^1) \cap W^r_{a+x'}$ and $F(V^1) \cap W^r_{a+x''}$ consist only of one point, denoted by $Q(x')$ and $Q(x'')$, respectively.

Since, subtracting the sum of the points of $F(V^1) \cap W^{g-1}_{a+x'-y}$ from the sum of the points of $F(V^1) \cap W^{g-1}_{a+x''-y}$, we obtain $Q(x'') - Q(x')$, we derive $(a + x'' - y) - (a + x' - y) = Q(x'') - Q(x')$ and then $x'' - Q(x'') = x' - Q(x')$.

We remember that $x', x'' \in W^1$ and $Q(x'), Q(x'') \in F(V^1)$. Consequently, there exists h such that $F(V_h^1) \cap W^1 \supset \{x', x''\}$ and, since $x' \neq x''$, this concludes the proof of Claim 6.

At the beginning of the proof we have supposed $F(V^1) \subset W_a^{r+1}$. If $F(V^1)$ is contained in $(W_a^{r+1})^*$, we can prove in an analogous way that there exists a translate of $F(V^1)$ which intersects W^1 in two points.

Now we are ready to end the proof of Theorem 1.

In Claim 6 we have proved that $F(V_h^1) \cap W^1 \supset \{x', x''\}$ with $x' \neq x''$. By Lemma 4 in [3] we have: $W_{-x'}^{g-1} \cap W_{-x''}^{g-1} = W^{g-2} \cup (W_{x'+x''}^{g-2})^*$. Besides

$$W_{-x'}^{g-1} \cap W_{-x''}^{g-1} = (F(V_K^{g-1}))_{-x'} \cap (F(V_K^{g-1}))_{-x''} \supset F(V_{K-h}^{g-2}).$$

Therefore $F(V_{K-h}^{g-2})$, being irreducible, is contained either in W^{g-2} or in $(W_{x'+x''}^{g-2})^*$.

Let's suppose $F(V_{K-h}^{g-2})$ is contained in W^{g-2}; then $F(V_{K-h}^{g-2}) = W^{g-2}$ because, by Claim 3, dim $F(V_{K-h}^{g-2})$ is $g-2$.

Finally: $W^1 = \bigcap \{W_{-u}^{g-1} \colon u \in W^{g-2}\} = \bigcap \{F(V_K^{g-1})_{-u} \colon u \in F(V_{K-h}^{g-2})\} \supset F(V_h^1)$ (the first equality is Lemma 3 in [3]). So $F(V_h^1) = W^1$ (because, by Claim 3, they have the same dimension).

If we consider the second possibility, that is $F(V_{K-h}^{g-2}) \subset (W_{x'+x''}^{g-2})^*$, we can prove, in a completely analogous way, that $F(V^1)$ is contained in a translate of $-W^1$ (and hence equal).

Now the proof of Theorem 1 is complete.

In [2] Debarre shows that, if C is a Riemann surface of genus g and W^{g-d} the image of the symmetric product $C^{(g-d)}$) by the Abel-Jacobi map in $J(C)$, as usual, one has that *any effective algebraic cycle in* $J(C)$ *with class* $\omega^d/d!$ *is a translate of either* W^{g-d} *or* $-W^{g-d}$ (where ω is the polarization on $J(C)$, given by the intersection of 1-cycles on C). So we can weaken the hypotheses of Theorem 1, applying the theorem of Debarre to X. We obtain

Theorem 1′. *Let* $F \colon J(X') \to J(X)$ *be a holomorphic map such that* $F(0) = 0$ *and assume that* $F(V^{g-1})$ *has class* ω. *Suppose that, if* $F(V^1) \not\subset W_b^{g-1}$, *then* $F(V^1) \cap W_b^{g-1}$ *is equal to* g *points counted with multiplicity and their sum is* $b + const$. *Then there is a holomorphic surjective map* $f \colon X' \to X$ *inducing either* F *or* $-F$.

2 - A second answer to the problem

In order to enunciate the second theorem, which is the second answer to our problem, it is useful to make some remarks.

Remark 4. If F is a holomorphic not constant map: $J(X') \to J(X)$ such that $F(0)=0$, the number $\deg(F|_{V^1}: V^1 \to F(V^1))$ is well defined and doesn't depend on the base point from which we consider Abel's map ν (because F is an homomorphism).

Remark 5. Let f be a holomorphic map from X' onto X and let $F: J(X') \to J(X)$ and $P: \mathrm{Pic}^\circ(X) \to \mathrm{Pic}^\circ(X')$ the maps induced by f. We have:

1. $\deg F|_{V^1} = \deg f \stackrel{\mathrm{def}}{=} d$
2. $F_*: H_2(J(X'), Z) \to H_2(J(X), Z)$ is such that $F_*([V^1]) = d[W^1]$
3. $P^*: H^2(\mathrm{Pic}^\circ(X'), Z) \to H^2(\mathrm{Pic}^\circ(X), Z)$ is such that $P^*(\theta') = d\theta$.

Remark 6. Let $F: J(X') \to J(X)$ be a holomorphic map such that $F(0)=0$ and $P: \mathrm{Pic}^\circ(X) \to \mathrm{Pic}^\circ(X')$ its dual. Then, for any $n \in N$ we have

$$F_*([V^1]) = n[W^1] \quad \Leftrightarrow \quad P^*(\theta') = n\theta .$$

In fact for every Riemann surface C we have:

$$\begin{aligned} H_2(J(C), Z) &= \wedge^2 H_1(J(C), Z) &&= \wedge^2 H_1(C, Z) \\ H^2(\mathrm{Pic}^\circ(C), Z) &= \wedge^2 H^1(\mathrm{Pic}^\circ(C), Z) &&= \wedge^2 H_1(C, Z) \end{aligned}$$

and $[\mu(C)] \in H_2(J(C), Z)$ corresponds to $\theta \in H^2(\mathrm{Pic}^\circ(C), Z)$, because $[\mu(C)]$ is equal to $\sum a_i \wedge b_i$ in the isomorphism $H_2(J(C), Z) = \wedge^2 H_1(C, Z)$; then we have

$$\begin{array}{ccc} H_2(J(X'), Z) & = & H^2(\mathrm{Pic}^\circ(X'), Z) \\ \downarrow F_* & & \downarrow P^* \\ H_2(J(X), Z) & = & H^2(\mathrm{Pic}^\circ(X), Z) . \end{array}$$

The maps F_* and P^* are the same map, if we read them in the isomorphism $H_2(J(C), Z) = \wedge^2 H_1(C, Z) = H^2(\mathrm{Pic}^\circ(C), Z)$.

Theorem 2. *Let F be a holomorphic not constant map $J(X') \to J(X)$ such that $F(0)=0$. Define $d = \deg F|_{V^1}$ and be P the dual map of F. Let also $P^*: H^2(\mathrm{Pic}^\circ(X'), Z) \to H^2(\mathrm{Pic}^\circ(X), Z)$ be such that $P^*\theta' = d\theta$ (i.e. $F_*: H_2(J(X'), Z) \to H_2(J(X), Z)$ is such that $F_*([V^1]) = d[W^1]$).*

Then there is a holomorphic surjective map $f: X' \to X$ inducing either F or $-F$.

Proof. Define $D = F(V^1)$. D is an irreducible analytic subvariety of dimension 1. If $[D]$ denotes the foundamental class of D, we have $F_*([V^1]) = d[D]$. On the other hand, by the hypothesis we have that $F_*([V^1]) = d[W^1]$. Therefore $d[W^1] = d[D]$ and then $[W^1] = [D]$.

Matsusaka's theorem ([5], [4]) tells us that, if in an abelian variety A of dimension g with principal polarization λ there is an irreducibile curve of minimal class (that is, it corresponds to $\lambda^{g-1}/(g-1)!$ by Poincaré duality), then A is its Jacobian, the curve is canonically embedded in A and λ is the canonical polarization.

Take $J(X)$ as A, ω as λ, D as the curve of the minimal class. Then calling $\widetilde{D}$ the Riemann surface associated to D, we have a biholomorphism $B\colon J(\widetilde{D}) \to J(X)$ such that:

$$B^* \omega = \textit{canonical polarization of } J(\widetilde{D})$$

$$B(\textit{image of Abel's map of } \widetilde{D} \textit{ in } J(\widetilde{D})) = D\,.$$

Then, by Martens' proof of Torelli's theorem ([3]) (that is by Theorem 1 of this paper in the case $g = g'$), we have that $D = F(V^1)$ is equal either to a translate of W^1 or to a translate of $-W^1$. This concludes the proof of Theorem 2 (by Remark 2 in the Introduction).

Remark 7. Both F and $-F$ are induced by maps from X' onto X if and only if X is hyperelliptic.

In fact for every Riemann surface C of genus $g \geqslant 2$ we have that $\mu(C)$ is equal to a translate of $-\mu(C)$ if and only if C is hyperelliptic.

Remark 8. The hypotheses of Theorem 2 can be easily and almost completely translated in polynomial equations in the entries of the matrix representing F and of the period matrices of X' and X.

Proof. As in the previous part of the paper, let X' and X be two Riemann surfaces and g' and g their respective genuses; let $g' \geqslant g > 2$. We will denote

$$\mathcal{F} = \{F\colon J(X') \to J(X) \text{ such that } F(0) = 0, \textit{ is holomorphic}, F_*[V^1] = \deg F|_{V^1}[W^1]\}$$

and $\mathcal{P} = \{P\colon P \textit{ dual } \text{of } F \in \mathcal{F}\}$.

If $F \in \mathcal{F}$ and P is its dual, let's consider the maps lifting F and P to the universal coverings; we call $\widetilde{F}$ and $\widetilde{P}$ the (unique) matrices expressing these maps in the bases on $\boldsymbol{R}\{a_1, b_1, \ldots\}$ and $\{a_1^\vee, b_1^\vee, \ldots\}$; $\widetilde{P}$ is the transpose of $\widetilde{F}$. We can regard the $\widetilde{P}$'s as points of $M(2g' \times 2g, \boldsymbol{R}) \simeq \boldsymbol{R}^{4g'g}$.

The $\widetilde{P}$'s are clearly matrices with entries in $\boldsymbol{Z}$, so they belong to the lattice $\boldsymbol{Z}^{4g'g}$ in $\boldsymbol{R}^{4g'g}$.

Being the P's and the F's holomorphic, the lifting maps of the P's and of the F's are $\boldsymbol{C}$-linear. So the $\widetilde{P}$'s with $P \in \mathcal{P}$ belong to the $2g'g$-subspace of the $\boldsymbol{C}$-linear maps; precisely:

we call A the matrix giving the change of bases from the basis of the lattice of $J(X')$ $\{a_1, b_1, \dots\}$ to the basis $\{e_1, ie_1, \dots\}$ where $\{e_1, \dots, e_{g'}\}$ is a complex basis of the universal covering of $J(X')$; we call B the analogous for $J(X)$; with this notation, the $\boldsymbol{C}$-linearity of the lifting map of F is equivalent to:

$$(B\widetilde{P}^t A^{-1})_{2i-1,2k-1} = (B\widetilde{P}^t A^{-1})_{2i,2k} \qquad (B\widetilde{P}^t A^{-1})_{2i-1,2k} = -(B\widetilde{P}^t A^{-1})_{2i,2k-1}$$

for $i = 1, \dots, g$ and $k = 1, \dots, g'$.

Clearly the condition $P^* \theta' = m\theta$ is the same as $\widetilde{P}^t H_{2g'} \widetilde{P} = mH_{2g}$, where we denote with H_{2s} the matrix $2s \times 2s$, whose entries are

$$(H_{2s})_{ij} = \begin{cases} 1 & \text{if } i = j+1 \text{ and } j \text{ odd} \\ -1 & \text{if } i = j-1 \text{ and } j \text{ even} \\ 0 & \text{otherwise}. \end{cases}$$

This concludes the proof of Remark 8.

Thus perhaps the characterization of the maps between Jacobians induced by maps between Riemann surfaces of Theorem 2 may be useful to give an upper bound on the number of the holomorphic surjective maps between two fixed Riemann surfaces.

Let us call

$$\mathrm{Hol}(X', X) = \{f\colon X' \to X \text{ holomorphic surjective}\}.$$

Since a map $F\colon J(X') \to J(X)$ may be induced at most by one map from X' onto X, in order to estimate the cardinality of $\mathrm{Hol}(X', X)$, it is sufficient to estimate $\#\{F\colon J(X') \to J(X)$ induced by holomorphic maps from X' onto $X\}$, which is equal to $\#\mathcal{F}$ if X is hyperelliptic, to $\frac{1}{2}\#\mathcal{F}$ if X is not hyperelliptic. As usually the symbol $\#$ denotes cardinality.

Indeed in finding this estimate we use only Remark 5 and not Theorem 2. But the fact that the hypotheses of Theorem 2 are sufficient (apart from a sign) for a morphism between Jacobians to be induced by a morphism between Riemann surfaces, tells us that if we succeed in using completely these conditions, we have a good estimate.

Perhaps, analogously, fixed X' of genus g' and fixed g with $g' \geqslant g \geqslant 2$, one may find an estimate of the cardinality of

$$\mathrm{Hol}(X', g) = \{f\colon X' \to X\colon f \text{ holomorphic surjective, } X \text{ Riemann surface of genus } g\}.$$

Besides, Theorem 2 (together with equations resolving Schottky problem) may be useful to see

$\mathrm{Hol}(g', g) = \{(A, f, B)\colon A \in \mathcal{M}_{g'}, B \in \mathcal{M}_g, f\colon A \to B$ holomorphic surjective map$\}$

as an open of some components of the set of zeroes of some equations in $(\mathcal{H}_{g'} \times R^{4g'g} \times \mathcal{H}_g)/\sim$, where $\mathcal{H}_{g'}$ and $\mathcal{H}_g$ are the Siegel upper half-spaces and $\sim$ is the equivalence relation given by the action of the modular groups.

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Sommario

Siano X' e X due superfici di Riemann (compatte e connesse) di genere g' e g rispettivamente, con $g' \geq g \geq 1$. Se $f\colon X' \to X$ è una mappa olomorfa surgettiva, ovviamente f induce una mappa fra le corrispondenti Jacobiane. Ci si può chiedere quali siano condizioni necessarie e sufficienti affinché un'applicazione $F\colon J(X') \to J(X)$ sia indotta da un'applicazione olomorfa da X' su X. In questo lavoro si danno due risposte a questa questione.

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