Riv. Mat. Univ. Parma (5) **5** (1996), 51-54

B. A. Saleemi (*)

Curvatures of conformally equivalent manifolds (**)

1 - Preliminary results

Two Riemannian manifolds $(M, \langle\,,\rangle)$ and $(M', \langle\,,\rangle')$ are said to be *conformally equivalent* if there exists a diffeomorphism $\phi: M \to M'$ and a smooth map $\sigma: M \to \boldsymbol{R}$ such that

$$\langle \phi_*(X), \phi_*(Y)\rangle' = e^{2\sigma}\langle X, Y\rangle \circ \phi^{-1} \tag{1}$$

for all vector fields X and Y on M.

Let D be the Riemannian connection on M. Since we have $[X, Y] = D_X Y - D_Y X$, then for all vector fields X, Y and Z on M we get

$$\begin{aligned} 2\langle D_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\quad + \langle Y, [Z, X]\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle . \end{aligned} \tag{2}$$

A routine calculation involving (2) and relation $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$ leads to

Lemma 1. *If D and D' are the Riemannian connections on the conformally equivalent manifolds M and M', then we have*

$$D'_{\phi_*(X)}\phi_*(Y) = \phi_*(D_X Y + X(\sigma) Y + Y(\sigma) X - \langle X, Y\rangle \operatorname{grad}\sigma) \tag{3}$$

where $\operatorname{grad}\sigma$ *denotes the gradient of σ in the contravariant form.*

Denoting the curvature tensors of M and M' by R and R' respectively, we have

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(*) Dept. of Math., King Abdulaziz Univ., P.O. Box 9028, Jeddah, Saudi Arabia.
(**) Received March 20, 1996. AMS classification 53 A 10.

Theorem 1. *For all vector fields X, Y and Z on M we have*

$$
\begin{aligned}
(4)\qquad & R'(\phi_*(X), \phi_*(Y))\phi_*(Z) = \phi_*(R(X, Y)Z) \\
& +\phi_*(\langle Z, D_X \operatorname{grad}\sigma\rangle Y - \langle Z, D_Y \operatorname{grad}\sigma\rangle X) \\
& -\phi_*(\langle Y, Z\rangle D_X \operatorname{grad}\sigma - \langle X, Z\rangle D_Y \operatorname{grad}\sigma) \\
& +\phi_*(\langle Y, Z\rangle X(\sigma) \operatorname{grad}(\sigma) - \langle X, Z\rangle Y(\sigma) \operatorname{grad}(\sigma)) \\
& -\phi_*(\langle Y, Z\rangle \,|\operatorname{grad}(\sigma)|^2 X - \langle X, Z\rangle |\operatorname{grad}(\sigma)|^2 Y) \\
& +\phi_*(Z(\sigma)Y(\sigma)X - Z(\sigma)X(\sigma)Y).
\end{aligned}
$$

Proof. Using Lemma 1, we may write

$$
\begin{aligned}
(5)\qquad & D'_{\phi_*(X)}D'_{\phi_*(Y)}\phi_*(Z) = \phi_*(D_X D_Y Z + X(Y(\sigma))Z + X(Z(\sigma))Y) \\
& +\phi_*(Y(\sigma)D_X Z + Z(\sigma)D_X Y) \\
& -\phi_*(X\langle Y, Z\rangle \operatorname{grad}\sigma + \langle Y, Z\rangle D_X \operatorname{grad}\sigma) \\
& +\phi_*(X(\sigma)D_Y Z + X(\sigma)Z(\sigma)Y + X(\sigma)Y(\sigma)Z) \\
& +\phi_*(D_Y Z(\sigma)X + 2Z(\sigma)Y(\sigma)X - \langle Y, Z\rangle \,|\operatorname{grad}\sigma|^2 X) \\
& -\phi_*(\langle X, D_Y Z\rangle \operatorname{grad}\sigma + Z(\sigma)\langle X, Y\rangle \operatorname{grad}\sigma + Y(\sigma)\langle Z, X\rangle \operatorname{grad}\sigma).
\end{aligned}
$$

On the other hand, from (3) we derive

$$
\begin{aligned}
(6)\qquad D'_{\phi_*[X, Y]}\phi_*(Z) &= \phi_*\{D_{[X, Y]}Z + Z(\sigma)[X, Y]\} \\
&\quad +\phi_*\{[X, Y](\sigma)Z - \langle Z, [X, Y]\rangle \operatorname{grad}\sigma\}.
\end{aligned}
$$

Since by definition we have

$$R'(\phi_*(X), \phi_*(Y))\phi_*(Z) = D'_{\phi_*(X)}D'_{\phi_*(Y)}\phi_*(Z) - D'_{\phi_*(Y)}D'_{\phi_*(X)}\phi_*(Z) - D'_{\phi_*[X, Y]}\phi_*(Z)$$

taking into account (5) and (6), we are now able to obtain relation (4).

2 - Two-dimensional Riemannian manifolds

We assume that M and M' are both 2-dimensional Riemannian manifolds. Let p, e_1, e_2 be an orthonormal frame field on M. We denote the dual coframe by p, ω_1, ω_2. Primes will denote quantities pertaining to M'.

Then Theorem 1 enables us to prove

Theorem 2. *If M and M' are conformally equivalent, then their Gaussian*

(Riemannian) curvatures satisfy relation

$$K' = e^{-2\sigma}(K - \Delta\sigma) \circ \phi^{-1} \tag{7}$$

where Δ is the Beltrami-Laplace operator.

Proof. In Theorem 1 we set $X = Z = e_1$, $Y = e_2$. We also denote by σ_i and σ_{ij} the first and second order partial derivatives of σ.

Taking into account that we have

$$e_j(\sigma) = \sigma_j \qquad \operatorname{grad}\sigma = \sigma_1 e_1 + \sigma_2 e_2 \qquad j = 1, 2$$

relation (4) reduces to

$$\begin{aligned} R'(\phi_*(e_1), \phi_*(e_2))\phi_*(e_1) &= \phi_*(R(e_1, e_2)e_1 + D_{e_2}\operatorname{grad}\sigma) \\ &+ \phi_*(\langle e_1, D_{e_1}\operatorname{grad}\sigma\rangle e_2 - \langle e_1, D_{e_2}\operatorname{grad}\sigma\rangle e_1). \end{aligned} \tag{8}$$

Now:

$$D_{e_j}\operatorname{grad}\sigma = \sigma_{j1}e_1 + \sigma_{j2}e_2\sigma_r + \omega_r^1(e_j)e_1 + \sigma_r\omega_r^2(e_j)e_2 \tag{9}$$

$$\Delta\sigma = \sigma_{11} + \sigma_{22} - \sigma_r\omega_1^r(e_1) - \sigma_r\omega_2^r(e_2) \tag{10}$$

where the ω_i^k are the connection forms and a sum with respect to r is understood.

Putting these values in (8), we get

$$R'(\phi_*(e_1), \phi_*(e_2))\phi_*(e_1) = \phi_*(R(e_1, e_2)e_1 + (\Delta\sigma)e_2) \tag{11}$$

and then

$$\langle R'(\phi_*(e_1), \phi_*(e_2))\phi_*(e_1), \phi_*(e_2)\rangle' = \langle\phi_*(R(e_1, e_2)e_1 + (\Delta\sigma)e_2), \phi_*(e_2)\rangle. \tag{12}$$

Since by (1) we have

$$\langle\phi_*(e_j), \phi_*(e_j)\rangle = e^{2\sigma}\circ\phi^{-1} \qquad \langle\phi_*(e_1), \phi_*(e_2)\rangle = 0\circ\phi^{-1} \qquad j = 1, 2$$

then

$$\langle R'(\phi_*(e_1), \phi_*(e_2))\phi_*(e_1), \phi_*(e_2)\rangle = -K'(e^{4\sigma}\circ\phi^{-1}). \tag{13}$$

Finally, using (13) and again (1), from (12) we obtain (7).

Note. An unexpected pay-off from equation (7) is the known result that the *conformally equivalent closed, oriented 2-manifolds have the same Euler characteristic.*

In effect, starting from (7) and writing $f = e^{-2\sigma}(K - \Delta(\sigma))$ in equation (6) of [2], p. 165, we get

$$\int_M f\phi^*(\omega_1' \wedge \omega_2') = \int_{M'} K' \omega_1' \wedge \omega_2'. \tag{14}$$

Using Green's theorem ([3], p. 281), we have

$$\int_M K\omega_1 \wedge \omega_2 = \int_{M'} K' \omega_1' \wedge \omega_2' \qquad \text{i.e.} \qquad \chi(M) = \chi(M').$$

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Sommario

Viene stabilita una relazione tra i tensori di curvatura di due varietà riemanniane n-dimensionali conformemente equivalenti. Nel caso particolare $n = 2$ *(superfici) si ottiene un interessante risultato per le curvature gaussiane. Da questo, utilizzando i teoremi di Gauss-Bonnet e di Green, è possibile dedurre la nota proprietà che le due superfici hanno la stessa caratteristica di Eulero-Poincaré.*

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