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About positive operators on vector bundles (**)

Introduction

The study of a structure formed by a vector bundle endowed with a field of cones is suggested by the existence of time-oriented Lorentzian manifolds and justified by the wish for a globalisation of the results obtained till now in the study of positive operators (see the monographs of M. A. Krasnosel'skij [4] and [5]).

The first note, where a vector bundle endowed with a field of cones was considered, is probably D. Sullivan [12] of 1976. From 1988, the geometry of a differentiable manifold endowed with a field of tangent cones was studied in D. I. Papuc [6], [7] and [8]. The same author studied the general case of a regular vector bundle endowed with a field of cones ([9], [10]) and considered the case of time-oriented Lorentzian manifolds ([11]). In 1995 L. David studied opearators on a vector bundle endowed with an homogeneous n-hedral cone-field and some spectral properties of positive linear operators on a vector bundle endowed with an arbitrary cone-field ([1], [2]).

1 - Recall of some fundamental results (see [9])

1. Definition

A *regular vector bundle* is a vector bundle (E, p, M) for which M is a real topological paracompact connected without boundary manifold, $\dim M = n$, $\dim E = n + m$.

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A *field of cones* on a regular vector bundle (E, p, M) is a map

$$K: x \in M \to K(x) \subset E_x \subset E, \qquad E_x = p^{-1}(x)$$

such that the following axioms are satisfied:

A$_1$. For every $x \in M$ the set $K(x) \subset E_x$ is a convex pointed closed cone having interior points (in the topological space E_x).

A$_2$. The sets $\bigcup_{x \in M} \operatorname{Int} K(x)$ and $\bigcup_{x \in M} (E_x - K(x))$ are open subsets of E.

The structure formed by a *regular bundle* (E, p, M) endowed with a *field of cones* K will be denoted by $[(E, p, M); K]$.

Remark. A regular vector bundle (E, p, M) has a field of cones if and only if there is a continuous global section of non-zero vectors of (E, p, M). A good example of $[(E, p, M); K]$ is $[(TM, p, M); K]$ where M is a time-oriented Lorentzian manifold and, for any x of M, $K(x)$ is the quadratic cone of non spacelike time-oriented tangent vectors of M.

2. The geometry of a local fibre of the structure $[(E, p, M); K]$.

The pair $(E_x; K(x))$, $\forall x \in M$, is *a Krein space* ([4]). Hence it follows:

a. There is an *ordering relation* on E_x:

$$X \leqslant Y \Leftrightarrow Y - X \in K(x) \qquad X, Y \in E_x .$$

The pair $(E_x, \leqslant)$ is *an ordered topological vector space*, directed on both sides.

b. For every $Z \in \operatorname{Int} K(x)$, E_x is *Z-measurable* i.e.

$$\forall X \in E_x \quad \exists \lambda \in \boldsymbol{R}^+ \mid -\lambda Z \leqslant X \leqslant \lambda Z \qquad (\boldsymbol{R}^+ = \{\lambda \in \boldsymbol{R} \mid \lambda \geqslant 0\}).$$

E_x has *a norm* determined by $K(x)$ and a fixed $Z \in \operatorname{Int} K(x)$:

$$|\cdot|_Z : E_x \to \boldsymbol{R}^+ \quad \text{where} \quad |X|_Z = \min\{\lambda \in \boldsymbol{R}^+ \mid -\lambda Z \leqslant X \leqslant \lambda Z\}.$$

c. *The open-balls of E_x in the norm $|\cdot|_Z$* and *the open ordered intervals in $\leqslant$*, both determined by the same $X_0 \in E_x$, $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$ and $Z \in \operatorname{Int} K(x)$, coincide, i.e. $B(X_0, Z, \varepsilon) = (X_0 - \varepsilon Z, X_0 + \varepsilon Z\}$ where:

$$B(X_0, Z, \varepsilon) = \{X : |X - X_0|_Z < \varepsilon\}$$

$$(X_0 - \varepsilon Z, X_0 + \varepsilon Z) = \{X \in E_x \mid \exists \varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon,\ X_0 - \varepsilon_1 Z \leqslant X \leqslant X_0 + \varepsilon_1 Z\}.$$

The three topologies of E_x, the first determined by the structure (E, p, M), the second by the norm $|\cdot|_Z$ and the third by the open ordered intervals, coincide.

3. Global properties of a structure $[(E, p, M); K]$

A main tool in order to study a structure $[(E, p, M); K]$ is the *function* $\nu\colon \bigcup_{x \in M} (\operatorname{Int} K(x) \times E_x) \to \boldsymbol{R}^2$ defined by $\nu(Z, X) = (\alpha(Z, X), \beta(Z, X))$ where:

$$\alpha(Z, X) = \min\{\lambda \in \boldsymbol{R} \mid X \leqslant \lambda Z\} \qquad \beta(Z, X) = \max\{\lambda \in \boldsymbol{R} \mid \lambda Z \leqslant X\}.$$

The function ν is continuous and $\forall (Z, X), (Z, Y) \in \bigcup_{x \in M} (\operatorname{Int} K(x) \times E_x)$ has the *fundamental properties*:

I. $\nu(Z, \lambda X) = \lambda \cdot \nu(Z, X)$, $\nu(\lambda Z, X) = \lambda^{-1} \cdot \nu(Z, X)$ for any $\lambda \in \boldsymbol{R}^+ \setminus \{0\}$.

II. $\beta(Z, X) + \beta(Z, Y) \leqslant \beta(Z, X + Y) \leqslant \alpha(Z, X + Y) \leqslant \alpha(Z, X) + \alpha(Z, Y)$.

III. $\nu(Z, Z) = (1, 1) \qquad \nu(Z, X) = (0, 0) \Rightarrow X = 0$.

IV. $\alpha(Z, \lambda Z - X) = \lambda - \beta(Z, X)$, $\beta(Z, \lambda Z - X) = \lambda - \alpha(Z, X)$ for any $\lambda \in \boldsymbol{R}$.

Next theorem proves that a structure $[(E, p, M); K]$ can be determined by three *classical* elements: a vector bundle (E, p, M), a continuous global section of non-zero vectors of (E, p, M) and a function defined on E with values in $\boldsymbol{R}^2$.

Fundamental theorem. *Let be given a regular vector bundle (E, p, M), a continuous global section ζ of (E, p, M) such that $\zeta(x) \neq 0$ for any $x \in M$, a continuous function $\nu_\zeta : E \to \boldsymbol{R}^2$ satisfying conditions* I-IV *where $\nu(\zeta(p(X)), X)$ is replaced by $\nu_\zeta(X)$.*

The three elements (E, p, M), ζ, ν_ζ, uniquely determine a structure $[(E, p, M); K]$ where, for any x of M, $K(x) = \{X \in E_x \mid \beta_\zeta(X) \geqslant 0\}$ If in E_x we consider the ordering $X \leqslant Y$ defined by $Y - X \in K(x)$, then

$$\alpha_\zeta(X) = \min\{\lambda \in \boldsymbol{R} \mid X \leqslant \lambda\zeta(x)\} \qquad \beta_\zeta(X) = \max\{\lambda \in \boldsymbol{R} \mid \lambda\zeta(z) \leqslant X\}.$$

We list now some important *relations*:

a$_1$. For any $X \in E_x$ and any $Z, Z_1, Z_2 \in \operatorname{Int} K(x)$ we have:

$$X \in K(x) \Leftrightarrow \beta(Z, X) \geqslant 0 \qquad X - \beta(Z, X) Z \in \operatorname{Fr} K(x) \qquad \alpha(Z, X) Z - X \in \operatorname{Fr} K(x).$$

a$_2$. If $X \in B(X_0, Z, \varepsilon) = (X_0 - \varepsilon Z, X_0 + \varepsilon Z)$, then there exists $\varepsilon_1 \in \boldsymbol{R}$, $0 \leqslant \varepsilon_1 < \varepsilon$ such that

$$\alpha(Z, X_0) - \varepsilon_1 \leqslant \alpha(Z, X) \leqslant \alpha(Z, X_0) + \varepsilon_1 \quad \beta(Z, X_0) - \varepsilon_1 \leqslant \beta(Z, X) \leqslant \beta(Z, X_0) + \varepsilon_1$$

and conversely.

a$_3$. $|X|_Z = \max\{|\alpha(Z, X)|, |\beta(Z, X)|\}$.

Finally if for the structure $[(E, p, M); K]$ it is given a continuous global section ζ of (E, p, M) such that $\zeta(x) \in \operatorname{Int} K(x)$ for any x of M, then by **a$_3$** we can consider $|\cdot|_\zeta : E_x \to \boldsymbol{R}^+$ defined by

$$|X_x|_\zeta = |X_x|_{\zeta(x)} = \max\{|\alpha(\zeta(x), X_x)|, \beta(\zeta(x), X_x)|\}.$$

2 - Operators. The cone of positive operators. The structure $(\Omega/\Omega_0; K)$

Definition 1. An *operator* from a vector bundle (E, p, M) to a vector bundle (E', p', M) is a continuous map $A: E \to E'$ such that we have $A(p^{-1}(x)) \subset p'^{-1}(x)$ for any x of M.

We shall denote the set of all these operators by Ω. The set Ω is *a real vector space* and a module over the ring of real continuous functions defined on M or on E.

As important particular subsets of Ω, we shall consider:

the linear subspace Ω_L of *linear operators* (morphisms from the vector bundle (E, p, M) to the vector bundle (E', p', M)),

the linear subspace $\Omega_{\Sigma'}$ of *section-operators*. Every element of $\Omega_{\Sigma'}$ will be defined by a global section of (E', p', M). If $\sigma' : M \to E'$, $p' \circ \sigma' = \mathrm{id}_M$, then the section-operator $A_{\sigma'}$ determined by σ' will be $A_{\sigma'}: E \to E'$, $A_{\sigma'} = \sigma' \circ p$.

Obviously $\Omega_L \cap \Omega_{\Sigma'} = \{A_0\}$, where A_0 is the constant operator determined by the zero global section of (E', p', M).

We shall consider now two structures $[(E, p, M); K]$ and $[(E', p', M); K']$

Definition 2. A *positive operator* from the structure $[(E, p, M); K]$ to the structure $[(E', p', M); K']$ is an operator $A: E \to E'$ such that $A(K(x)) \subset K'(x)$, $\forall x \in M$. The subset of all positive operators of Ω will be denoted by K_Ω.

Proposition 1. *The set K_Ω of all positive operators from $[(E, p, M); K]$ to $[(E', p', M); K']$ is a convex cone generating Ω.*

Proof. Indeed, K_Ω is a *cone* (if $A \in K_\Omega$ then $\varrho A \in K_\Omega$, $\forall \varrho \in \boldsymbol{R}^+$). K_Ω is a *convex set* (if $A_1, A_2 \in K_\Omega$ then $(1-\lambda)A_1 + \lambda A_2 \in K_\Omega$, $\forall \lambda \in [0, 1] \subset \boldsymbol{R}$). K_Ω is a *generating set* for Ω. This means that, for every $A \in \Omega$, there are two positive operators $A_1, A_2 \in K_\Omega$ such that $A = A_2 - A_1$. In order to prove this assertion we shall use the function ν (Sec. 1, 3) this function being defined for the structure $[(E', p', M); K']$. We shall put:

$$A_1(X) = |\alpha(\sigma'(p(X)), A(X))| \cdot \sigma'(p(X)) - A(X)$$

$$A_2(X) = \alpha(\sigma'(p(X)), A(X))| \cdot \sigma'(p(X))$$

where $\sigma' \in \mathrm{Int}(K')_{\Sigma}$. By virtue of relation $\mathbf{a}_1$ of Sec. 1,3 it follows that A_1, A_2 are positive operators. Obviously $A = A_2 - A_1$.

Among the positive operators from $[(E, p, M); K]$ to $[(E', p', M); K']$ we shall consider the set

(1) $$\Omega_0 = \{A \in \Omega \mid A(K(x)) = 0 \in K'(x), \ \forall x \in M\}.$$

Ω_0 is a linear subspace of Ω and we have $\Omega_0 \subset K_\Omega$, $\Omega_L \cap \Omega_0 = \Omega_{\Sigma'} \cap \Omega_0 = \{A_0\}$.

Then, we can consider the *quotient linear space* $\Omega/\Omega_0 = \{A + \Omega_0 \mid \forall A \in \Omega\}$.

Let $\pi: \Omega \to \Omega/\Omega_0$ be the natural projection and denote $\pi(A)$ by $[A]$. Then the maps $\pi|_{\Omega_L}, \pi|_{\Omega_{\Sigma'}}$ are *injective*. Also, $[A_1] = [A_2]$ if and only if $A_1(X) = A_2(X)$ for any X of $K(p(X))$.

The last remark permit us to consider the following important subset of Ω/Ω_0

(2) $$\boldsymbol{K} = \{[A] \mid A \in K_\Omega\}.$$

Obviously, $[A] \in \boldsymbol{K}$ if and only if $A(K(x)) \subset K'(x)$ $(\forall x \in M)$.

Proposition 2. *The set $\boldsymbol{K}$ is a convex pointed cone generating the linear space Ω/Ω_0.*

Proof. $\boldsymbol{K}$ is a *cone* in the linear space Ω/Ω_0. Indeed, if $[A] \in \boldsymbol{K}$ (i.e. $A \in K_\Omega$) and $\varrho \in \boldsymbol{R}^+$, then $\varrho[A] \in \boldsymbol{K}$ (since $\varrho A \in K_\Omega$). $\boldsymbol{K}$ is a *convex cone.* If $[A_1]$ and $[A_2]$ belong to $\boldsymbol{K}$, then $(1-\lambda)[A_1] + \lambda[A_2]$ belongs to $\boldsymbol{K}$ for every $\lambda \in [0, 1] \subset \boldsymbol{R}$. $\boldsymbol{K}$ is a *pointed cone.* If $[A]$ and $-[A]$ belong to $\boldsymbol{K}$, then A and $-A$ belong to K_Ω. This implies $A(X) = 0$ for any X of $K(p(X))$, consequently $A \in \Omega_0$ and so $[A] = 0$. $\boldsymbol{K}$ is a *generating set* for the linear space Ω/Ω_0. Let $[A]$ be an arbitrary element of Ω/Ω_0. For any operator A of Ω we proved that there are two positive operators. A_1, A_2, so that $A = A_1 - A_2$ (see Definition 2 and Proposition 1). From the last equality it follows that $[A] = [A_1] - [A_2]$.

By virtue of a previous remark we can give

Definition 3. We call *interior of the cone* $\boldsymbol{K}$ and we denote it by Int $\boldsymbol{K}$ the subset of Ω/Ω_0 defined by

(3) $$\mathrm{Int}\,\boldsymbol{K} = \{[B] \in \Omega/\Omega_0 \mid B(\mathrm{Int}\,K(x)) \subset \mathrm{Int}\,K'(x), \ \forall x \in M\}.$$

Obviously, $\mathrm{Int}\,\boldsymbol{K} \subset \boldsymbol{K}$.

3 - Ordering relation for the structure $(\Omega/\Omega_0; \boldsymbol{K})$

For the linear space Ω/Ω_0, we can define, by means of the cone $\boldsymbol{K}$, an *ordering relation* (see [4]) namely

$$[A_1] \leqslant [A_2] \Leftrightarrow [A_2] - [A_1] \in \boldsymbol{K} \qquad \text{for any } [A_1], [A_2] \text{ of } \Omega/\Omega_0. \tag{4}$$

Remark. Obviously $[A_2] - [A_1] \in \boldsymbol{K} \Leftrightarrow [A_2 - A_1] \in \boldsymbol{K}$ which is equivalent to

$$(A_2 - A_1)(X) \in K'(x) \Leftrightarrow A_1(X) \leqslant A_2(X) \qquad \text{for any } X \text{ of } K(x)$$

(in the ordering defined by K' for the structure $[(E', p', M); K']$).

Proposition 3. *The pair $(\Omega/\Omega_0; \leqslant)$ is an ordered vector space, directed on both sides.*

Proof. Obviously relation (4) is an ordering relation for the set Ω/Ω_0 and the pair $(\Omega/\Omega_0; \leqslant)$ is an ordered linear space [4].

In order to prove the last part of Proposition 3, we shall fix an interior section σ' of $[(E', p', M); K']$, i.e. an element $\sigma'(x) \in \operatorname{Int} K'(x)$ for any x of M. For an arbitrary element $[A] \in \Omega/\Omega_0$, by means of the function ν defined for the structure $[(E', p', M); K']$ (Sec. 1,3) we can associate to the element $[A]$ two elements of Ω/Ω_0, namely $[\alpha(\sigma' \circ p, A) \cdot (\sigma' \circ p)]$ and $[\beta(\sigma' \circ p, A) \cdot (\sigma' \circ p)]$, such that

$$[\beta(\sigma' \circ p, A) \cdot (\sigma' \circ p)] \leqslant [A] \leqslant [\alpha(\sigma' \circ p, A) \cdot (\sigma' \circ p)] \tag{5}$$

where p is the projection of (E, p, M) and $\sigma' \circ p$ is the section-operator defined by σ'.

For two arbitrary elements $[A_1], [A_2] \in \Omega/\Omega_0$ we shall consider the elements

$$[A_3] = [\min\{\beta(\sigma' \circ p, A_1), \beta(\sigma' \circ p, A_2)\} \cdot (\sigma' \circ p)]$$

$$[A_4] = [\max\{\alpha(\sigma' \circ p, A_1), \alpha(\sigma' \circ p, A_2)\} \cdot (\sigma' \circ p)].$$

By virtue of (5) we have $[A_3] \leqslant [A_1]$, $[A_2] \leqslant [A_4]$.

Corollary. *If* $[B_1]$, $[B_2] \in \operatorname{Int} \boldsymbol{K}$, *then there are two elements* $[B_3]$, $[B_4]$ *of* $\operatorname{Int} \boldsymbol{K}$, *such that* $[B_3] \leqslant [B_1]$, $[B_2] \leqslant [B_4]$.

The Corollary is an immediate consequence of relation (5) if we remark that for any $[B]$ of $\operatorname{Int} \boldsymbol{K}$ we shall have $\beta(\sigma \circ p, B) > 0$ on $\operatorname{Int} \boldsymbol{K}(x)$.

The ordering relation (4) defined on the linear space Ω/Ω_0 permits us to determine *topological structures* of Ω/Ω_0.

Definition 4. An *ordered open interval centred in* $[A^*] \in \Omega/\Omega_0$, determined by an element $[B] \in \operatorname{Int} \boldsymbol{K}$, is a set

$$\begin{aligned} &«[A^*] - \varepsilon[B], [A^*] + \varepsilon[B]» \\ (6)\quad &= \{[A] \in \Omega/\Omega_0 \mid \exists \varepsilon, \varepsilon_1 \in \boldsymbol{R}^+, \varepsilon_1 < \varepsilon : [A^*] - \varepsilon_1[B] \leqslant [A] \leqslant [A^*] + \varepsilon_1[B]\}. \end{aligned}$$

Proposition 4. *The set of all ordered open intervals determined by the same element* $[B] \in \operatorname{Int} \boldsymbol{K}$ *is a base of a topology of* Ω/Ω_0, *denoted* $\tau_{[B]}$.

Proof.

a. Obviously, for every element of $[A] \in \Omega/\Omega_0$ there is an open ordered interval to which the element $[A]$ belongs ($[A] \in «[A] + \varepsilon[B], [A] + \varepsilon[B]»$ for an $\varepsilon \in \boldsymbol{R}^+ \setminus \{0\}$, arbitrarily fixed).

b. If an element $[A]$ belongs to two open ordered intervals, then it belongs to an open ordered interval included in both these intervals.

First we shall prove that if $[A] \in «[A^*] - \varepsilon[B], [A^*] + \varepsilon[B]»$ then there is an open ordered interval centred in $[A]$, included in $«[A^*] - \varepsilon[B], [A^*] + \varepsilon[B]»$.

The assumption implies that there exist ε, $\varepsilon_1 \in \boldsymbol{R}$, $0 < \varepsilon_1 < \varepsilon$ such that

$$[A^*] - \varepsilon_1[B] \leqslant [A] \leqslant [A^*] + \varepsilon_1[B].$$

Consequently for any ε_1' satisfing $\varepsilon > \varepsilon_1' > \varepsilon_1$ we have

$$[A^*] - \varepsilon_1'[B] \leqslant [A^*] - \varepsilon_1[B] \leqslant [A] \leqslant [A^*] + \varepsilon_1[B] \leqslant [A^*] + \varepsilon_1'[B].$$

It follows

$$[A^*] - \varepsilon_1'[B] \leqslant [A] - (\varepsilon_1' - \varepsilon_1)[B] \leqslant [A] \leqslant [A] + (\varepsilon_1' - \varepsilon_1)[B] \leqslant [A^*] + \varepsilon_1'[B].$$

Therefore for any ε^* satisfying $0 < \varepsilon^* < (\varepsilon_1' - \varepsilon_1)$ we get

$$[A^*] - \varepsilon_1'[B] \leqslant [A] - \varepsilon^*[B] \leqslant [A] \leqslant [A] + \varepsilon^*[B] \leqslant [A^*] + \varepsilon_1'[B]$$

and this implies

$$«[A] - (\varepsilon_1' - \varepsilon_1)[B], [A] + (\varepsilon_1' - \varepsilon_1)[B]» \subset «[A^*] - \varepsilon[B], [A^*] + \varepsilon[B]».$$

By virtue of this last result if

$$[A] \in «[A_1] - \varepsilon_1[B], [A_1] + \varepsilon_1[B]» \cap «[A_2] - \varepsilon_2[B], [A_2] + \varepsilon_2[B]»$$

then, for an $\varepsilon^* < \min\{(\varepsilon_1' - \varepsilon_1), (\varepsilon_2' - \varepsilon_2)\}$, we have

$$«[A] - \varepsilon^*[B], [A] + \varepsilon^*[B]» \subset «[A_1] - \varepsilon_1[B], [A_1] + \varepsilon_1[B]» \cap «[A_2] - \varepsilon_2[B], [A_2] + \varepsilon_2[B]».$$

The propositions **a** and **b** prove that the set of all open ordered intervals determined by the same element $[B] \in \operatorname{Int} \boldsymbol{K}$ is a *base of a topology* of Ω/Ω_0, denoted by $\tau_{[B]}$.

4 - Norms and distances for the structures $(\Omega/\Omega_0; \boldsymbol{K})$

Definition 5. An element $[A] \in \Omega/\Omega_0$ is called $[B]$*-measurable*, $[B] \in \operatorname{Int} \boldsymbol{K}$, if there is $\lambda \in \boldsymbol{R}^+ \setminus \{0\}$, such that $-\lambda[B] \leqslant [A] \leqslant \lambda[B]$.

There are elements $[A] \in \Omega/\Omega_0$ which are not $[B]$*-measurable*. The set of all elements of Ω/Ω_0 which are $[B]$-measurable will be denoted by $\Delta_{[B]}$. This set is a real linear subspace of Ω/Ω_0.

Proposition 5. *The set* $\Delta = \{\Delta_{[B]} \,|\, [B] \in \operatorname{Int} \boldsymbol{K}\}$, *is a covering of* Ω/Ω_0.

Proof. Let $[A]$ be an arbitrary element of Ω/Ω_0. We must determine an element $[B] \in \operatorname{Int} \boldsymbol{K}$ such that $[A]$ is $[B]$-measurable (Definition 5).

By means of a positive global section σ' of $[(E', p', M); K']$, (that is $\sigma'(x) \in \operatorname{Int} K'(x)$, $\forall x \in M$), we consider the positive section-operator $B = \sigma' \circ p$. By means of the function ν defined for the structure $[(E', p', M): K']$ we have (Sec. 1,3)

$$\beta(\sigma' \circ p, A) \cdot \sigma' \circ p \leqslant A \leqslant \alpha(\sigma' \circ p, A) \cdot \sigma' \circ p\,. \tag{7}$$

We consider an open local-finite covering of E, $U = \{U_a \,|\, a \in J\}$, such that $\forall a \in J$ the closure of U_a will be a compact set. We consider also a partition of unity $\{f_a : E \to \boldsymbol{R},\ a \in J\}$, subordinated to the convering U. For every $\operatorname{cl} U_a$ $(a \in J)$ we shall consider the numbers:

$$\alpha_a = \max\{\alpha(\sigma' \circ p(X), A(X)) \,|\, \forall X \in \operatorname{cl} U_a\},\ \beta_a = \min\{\beta(\sigma' \circ p(X), A(X)) \,|\, \forall X \in \operatorname{cl} U_a\}$$

and $\lambda_a \in \boldsymbol{R}^+ \setminus \{0\}$ such that $-\lambda_a \leqslant \beta_a \leqslant \alpha_a \leqslant \lambda_a$. The positive operator we are looking for will be $B = \Sigma f_a \lambda_a \sigma' \circ p$. By virtue of the relation (7), for every open set $U_a (a \in J)$, and for any X of U_a, we have

$$-\lambda_a \sigma' \circ p \leqslant \beta(\sigma' \circ p(X), A(x)) \cdot \sigma' \circ p \leqslant A(X) \leqslant \alpha(\sigma' \circ p(X), A(X)) \cdot \sigma' \circ p \leqslant \lambda_a \sigma' \circ p\,.$$

Multiplying last relations by f_a and adding these relations after the values of $a \in J$, we obtain $-[B] \leqslant [A] \leqslant [B]$.

Definition 6. On the set $\Delta = \{\Delta_{[B]} \mid [B] \in \operatorname{Int} \boldsymbol{K}\}$ we define *relation*

$$\Delta_{[B']} \leqslant \Delta_{[B'']} \Leftrightarrow \Delta_{[B']} \subset \Delta_{[B'']} \,. \tag{8}$$

Remark that $\Delta_{[B']} \subset \Delta_{[B'']} \Leftrightarrow [B'] \in \Delta_{[B'']} \Leftrightarrow \exists \lambda \in \boldsymbol{R}^+ \setminus \{0\}$ such that $[B'] \leqslant \lambda [B'']$.

Proposition 6. *Relation* (8) *is an ordering relation on* Δ, *directed on both sides.*

Proof. Obviously, the relation $\leqslant$ defined by (8) is an ordering relation. In order to prove that it is directed on both sides, we must prove that for any $\Delta_{[B']}$, $\Delta_{[B'']}$ there are two elements $\Delta_{[B^*]}$, $\Delta_{[B^{**}]}$ such that $\Delta_{[B^*]} \leqslant \Delta_{[B']}$, $\Delta_{[B'']} \leqslant \Delta_{[B^{**}]}$ ($[B^*], [B'], [B''], [B^{**}] \in \operatorname{Int} \boldsymbol{K}$). But this last relation follows from the relation

$$\forall [B'], [B''] \in \operatorname{Int} \boldsymbol{K},\ \exists [B^*], [B^{**}] \in \operatorname{Int} \boldsymbol{K} \Rightarrow [B^*] \leqslant [B'], [B''] \leqslant [B^{**}]$$

(see the above Remark and Corollary).

On the set $\Delta_{[B]}$, ($[B] \in \operatorname{Int} K(x)$) we consider the *map*

$$|\cdot|_{[B]} : \Delta_{[B]} \to \boldsymbol{R}^+ \tag{9}$$

defined by $|[A]|_{[B]} = \min\{\lambda \in \boldsymbol{R}^+ \mid -\lambda[B] \leqslant [A] \leqslant \lambda[B]\}$, for any $[A]$ of $\Delta_{[B]}$.

Proposition 7. *The map* (9) *is a monotone norm of the linear space* $\Delta_{[B]}$ ([4]).

For the linear subspace $\Delta_{[B]}$ of Ω/Ω_0 we consider the quotient space $(\Omega/\Omega_0)/\Delta_{[B]}$. Then, for every class $[A] + \Delta_{[B]} \in (\Omega/\Omega_0)/\Delta_{[B]}$ we consider the *map*

$$\begin{aligned} &d_{[B]} : ([A] + \Delta_{[B]}) \times ([A] + \Delta_{[B]}) \to \boldsymbol{R}^+ \\ &d_{[B]} : ([A] + [A'], [A] + [A'']) = |[A'] - [A'']|_{[B]} \in \boldsymbol{R}^+ \,. \end{aligned} \tag{10}$$

Proposition 8. *The function* $d_{[B]}$ *defined by* (10) *is a distance on* $(\Omega/\Omega_0)/\Delta_{[B]}$.

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Sommario

Siano (E, p, M), (E', p', M') *due fibrati vettoriali regolari dotati, rispettivamente, di campi di coni* K, K'. *Un operatore* $A: E \to E'$ *è una applicazione continua locale, che muta fibre in fibre. Viene studiato l'insieme* Ω *degli operatori in relazione ai campi di coni* K, K' *(cono degli operatori positivi, relazione d'ordine, norme e distanze).*

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