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On strong commutativity preserving traces of biadditive maps (**)

Throughout the present paper R will denote an associative ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$. A ring R is called *prime* if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is called *semiprime* in case $aRa = \{0\}$ implies $a = 0$. By a *derivation* we mean a mapping $d: R \to R$ such that $d(x + y) = d(x) + d(y)$ and $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. A mapping $F: R \to R$ is called *commutativity preserving* on R if $[x, y] = 0$ implies that $[F(x), F(y)] = 0$, for all $x, y \in R$. The mapping F is called *strong commutativity preserving* (*scp*) on R if $[F(x), F(y)] = [x, y]$ for all $x, y \in R$. A mapping $B: R \times R \to R$ will be called *symmetric* if $B(x, y) = B(y, x)$ holds for all pairs $x, y \in R$. A mapping $f: R \to R$ defined by $f(x) = B(x, x)$, where B is a symmetric mapping will be called *trace of B*. It is obvious that, in case B is a symmetric mapping, which is also biadditive (i.e. additive in both the arguments), the trace of B satisfies the relation $f(x + y) = f(x) + f(y) + 2B(x, y)$, $x, y \in R$. A symmetric biadditive mapping $D: R \times R \to R$ is called *symmetric biderivation* if $D(xy, z) = D(x, z)y + xD(y, z)$ is fulfilled for all $x, y, z \in R$.

In the last two decades, there has been an increasing interest in derivation on rings, and many of the results have involved commutativity (see [1] for partial reference). The concept of symmetric biderivation was introduced by G. Maksa in [5] (see also [6] where an example can be found). It is shown in [6] that symmetric biderivations are related to general solutions of some functional equations. Some results concerning symmetric biderivation on prime and semiprime rings can be obtained in [7] and [8]. Recently a lot has been explored about commutativity preserving mapping (for reference see [2]). Very recently H. E. Bell and M. N. Daif [1] investigated commutativity of prime and semiprime rings ad-

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mitting a derivation which is *scp* on certain subset of R. Our aim in the present paper is to initiate the study of a more general concept than *scp* mapping are, that is we consider the situation when mappings F and G of a ring R satisfy relation $[F(x), G(y)] = [x, y]$ for all $x, y \in R$. We begin our discussion with the following theorem, when traces of biadditive mappings are *scp* on R.

Theorem 1. *Let R be a ring of characteristic different from two. Suppose that there exist symmetric biadditive mappings $B_1 : R \times R \to R$ and $B_2 : R \times R \to R$ such that $[B_1(x, x), B_2(y, y)] = [x, y]$ for all $x, y \in R$. Then R is commutative.*

Proof. If f_1 and f_2 are the traces of B_1 and B_2 respectively, then by our hypothesis, we have

$$[f_1(x), f_2(y)] = [x, y] \quad \text{for all } x, y \in R. \tag{1}$$

Replacing x with $x + y$ yields that

$$[f_1(x), f_2(y)] + [f_1(y), f_2(y)] + 2[B_1(x, y), f_2(y)] = [x, y]$$

$$\text{i.e. } 2[B_1(x, y), f_2(y)] = 0\,.$$

But since R is 2-torsion free, we have $[B_1(x, y), f_2(y)] = 0$. Again, replacing y with $x + y$ in the last equation gives that

$$[f_1(x), f_2(y)] + 2[f_1(x), B_2(x, y)] + 2[B_1(x, y), B_2(x, y)] = 0\,. \tag{2}$$

Now replace x with $-x$ in (2), to get

$$[f_1(x), f_2(y)] - 2[f_1(x), B_2(x, y)] + 2[B_1(x, y), B_2(x, y)] = 0\,. \tag{3}$$

Now, adding (2) and (3) we have $2[f_1(x), f_2(y)] + 4[B_1(x, y), B_2(x, y)] = 0$, which yields that $[f_1(x), f_2(y)] + 2[B_1(x, y), B_2(x, y)] = 0$. Compare the last equation with (1), to get

$$[x, y] + 2[B_1(x, y), B_2(x, y)] = 0 \qquad \text{for all } x, y \in R\,. \tag{4}$$

Again replace x with $-x$ in (4), to get

$$[x, y] - 2[B_1(x, y), B_2(x, y)] = 0 \qquad \text{for all } x, y \in R\,. \tag{5}$$

Comparing of (4) and (5) yields $2[x, y] = 0$ i.e. $[x, y] = 0$ for all $x, y \in R$, and hence R is commutative.

Theorem 2. *Let R be a ring of characteristic different from two. Suppose that there exist symmetric biadditive mappings $B_1 : R \times R \to R$ and $B_2 : R \times R \to R$ such that $[B_1(x, y), B_2(x, y)] = [x, y]$ for all $x, y \in R$. Then R is commutative.*

Proof. Let f_1 and f_2 be the traces of B_1 and B_2 respectively. Replacing x with $x + y$ in relation

$$[B_1(x, y), B_2(x, y)] = [x, y] \tag{6}$$

gives $[f_1(y), B_2(x, y)] + [f_1(y), f_2(y)] + [B_1(x, y), f_2(y)] = 0$, for all $x, y \in R$. But in view of (6), we see that $[f_1(y), f_2(y)] = 0$, and hence the above yields that

$$[f_1(y), B_2(x, y)] + [B_1(x, y), f_2(y)] = 0 \qquad \text{for all } x, y \in R. \tag{7}$$

Now replace y with $x + y$ in (7) and compare the relation so obtained with (6), to get

$$[f_1(y), f_2(x)] + [f_1(x), f_2(y)] + 4[x, y] = 0 \quad \text{for all } x, y \in R. \tag{8}$$

Putting $2x$ for x in (8) and using the fact that the characteristic of R is different from two, we have

$$[f_1(y), f_2(x)] + [f_1(x), f_2(y)] + 2[x, y] = 0 \quad \text{for all } x, y \in R. \tag{9}$$

From (8) and (9), it follows that $2[x, y] = 0$ i.e. $[x, y] = 0$, for all $x, y \in R$, and hence R is commutative.

In view of the above theorems a natural question arises: what can we say about the commutativity of the ring R if it satisfies either of the properties

$$[B_1(x, y), B_2(x, y)] - [x, y] \in Z(R) \text{ or } [B_1(x, x), B_2(y, y)] - [x, y] \in Z(R)$$

for all $x, y \in R$? Theorems 3 and 4 deal with commutativity of semiprime rings satisfying these properties.

Theorem 3. *Let R be a semiprime ring of characteristic different from two. Suppose that there exist two symmetric biadditive mappings $B_1, B_2 : R \times R \to R$ such that $[B_1(x, x), B_2(y, y)] - [x, y] \in Z(R)$, for all $x, y \in R$. Then R is commutative.*

Proof. If f_1 and f_2 are the trace of B_1 and B_2 respectively, then by our hypothesis we have

$$[f_1(x), f_2(y)] - [x, y] \in Z(R) \qquad \text{for all } x, y \in R. \tag{10}$$

Replacing x with $x+y$ in (10) yield that

$$[f_1(x), f_2(y)] + [f_1(y), f_2(y)] + 2[B_1(x, y), f_2(y)] - [x, y] \in Z(R)$$

which reduces to

$$[B_1(x, y), f_2(y)] \in Z(R) \quad \text{for all } x, y \in R. \tag{11}$$

Putting $x+y$ for y in (11) and comparing the relation so obtained with (10) and (11), we arrive to

$$[f_1(x), f_2(y)] + 2[f_1(x), B_2(x, y)] + 2[B_1(x, y), B_2(x, y)] \in Z(R). \tag{12}$$

Replace $-x$ for x in (12), to get

$$[f_1(x), f_2(y)] - 2[f_1(x), B_2(x, y)] + 2[B_1(x, y), B_2(x, y)] \in Z(R). \tag{13}$$

Combining (12) and (13), we have $2[f_1(x), f_2(y)] + 4[B_1(x, y), B_2(x, y)] \in Z(R)$, which implies that

$$[f_1(x), f_2(y)] + 2[B_1(x, y), B_2(x, y)] \in Z(R) \quad \text{for all } x, y \in R. \tag{14}$$

Subtract (10) from (14), to get

$$[x, y] + 2[B_1(x, y), B_2(x, y)] \in Z(R) \quad \text{for all } x, y \in R. \tag{15}$$

Now putting $-x$ for x in (15) and subtracting the relation so obtained from (15), we find that $2[x, y] \in Z(R)$ i.e. $[x, y] \in Z(R)$.

Further replace y by yx, to get $[x, y]x \in Z(R)$. This implies that $[x, y][x, z] = 0$ for all $x, y, z \in R$. Putting in the last relation zy for z, we obtain $[x, y]z[x, y] = 0$, for all $x, y, z \in R$, whence it follows that $[x, y] = 0$ for all $x, y \in R$, and R is commutative.

Theorem 4. *Let R be a semiprime ring of characteristic different from two. Suppose that there exist two symmetric biadditive mappings $B_1, B_2 : R \times R \to R$ such that $[B_1(x, y), B_2(x, y)] - [x, y] \in Z(R)$, for all $x, y \in R$. Then R is commutative.*

Proof. By our hypothesis we have

$$[B_1(x, y), B_2(x, y)] - [x, y] \in Z(R) \quad \text{for all } x, y \in R. \tag{16}$$

Replace x with $x+y$ in (16), to get

$$[B_1(x, y), f_2(y)] + [f_1(y), B_2(x, y)] \in Z(R), \tag{17}$$

where f_1 and f_2 are the traces of B_1 and B_2 respectively. Further, replacing y

with $x+y$ in (17) and comparing the relation so obtained with (16) and (17) we arrive to

$$[f_1(x), f_2(y)] + [f_1(y), f_2(x)] + 4[B_1(x, y), B_2(x, y)] \in Z(R). \tag{18}$$

Combining (16) and (18), we obtain

$$[f_1(x), f_2(y)] + [f_1(y), f_2(x)] + 4[x, y] \in Z(R) \quad \text{for all } x, y \in R. \tag{19}$$

Putting $2x$ for x in (19), and using the fact that R is 2-torsion free, we find

$$[f_1(x), f_2(y)] + [f_1(y), f_2(x)] + 2[x, y] \in Z(R) \quad \text{for all } x, y \in R. \tag{20}$$

Now combine (19) and (20), to get $2[x, y] \in Z(R)$ i.e. $[x, y] \in Z(R)$, for all $x, y \in R$, which is known to imply commutativity of R.

Theorem 5. *Let R be a ring of characteristic different from two. Suppose that there exists a symmetric biadditive mapping $B: R \times R \to R$ such that $x - B(x, x) \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. Let f be the trace of B. Then, we have

$$[x, y] = [f(x), y] \quad \text{for all } x, y \in R. \tag{21}$$

Replacing x with $x+y$ in (21), and using (21), we get

$$[f(y), y] + 2[B(x, y), y] = 0 \quad \text{for all } x, y \in R. \tag{22}$$

Let us write $-x$ instead of x in (22) and add the relation so obtained to (22), to get

$$[f(y), y] = 0 \quad \text{for all } y \in R. \tag{23}$$

The linearization of (23) yields that

$$[f(x), x] + [f(y), x] + 2[B(x, y), x] + [f(x), y] + [f(y), y] + 2[B(x, y), y] = 0.$$

Combining the last relation with (22) and (23), we have

$$[f(x), y] + [f(y), x] = 0 \quad \text{for all } x, y \in R. \tag{24}$$

Again replacing x with $-x$ in (24) and adding the relation so obtained to (24), we get $[f(x), y] = 0$, and hence in view of (21), we get the required result.

References

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Sommario

In questo lavoro si esamina se un anello R sia commutativo quando la traccia di un'applicazione simmetrica e biadditiva di R abbia la proprietà di conservare la commutatività in senso forte.

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